A proof of Lemma 6.8

The natural number i gets defined twice in the proof of 6.8; the two definitions contradict each other, and the first one is a red herring.

We assume q to be $<^*$-least s.t. $q \in P^n_M \setminus R^n_M$, so $R^n_M \ni p = p_n(M) <^* q$. We aim to produce some $\bar{q} <^* q$, $\bar{q} \in P^n_M \setminus R^n_M$ to reach a contradiction.

Let $i \leq n$ be least with $p \restriction i <^* q \restriction i$. We have some $z \in [\rho_i(M)]^{<\omega}$ and some $j < \omega$ s.t.

(1) $M^{i-1}, p \restriction i-1 \models$ " $\exists p' <^* q(i-1) \; q(i-1) = h(j, (z, p'))$, "

as we may choose $p' = p(i-1)$. Letting $m \in \omega$ be the Gödel no. of the relevant Σ_1 formula displayed in (1), we have that

(2) $M^i, q \restriction i \models$ " $\exists z \in [OR]^{<\omega} \; (m, z) \in A^{i, q \restriction i}$ ",

so that we may pick $z \in [\rho_i(M)]^{<\omega}$ as in (1) in a way that

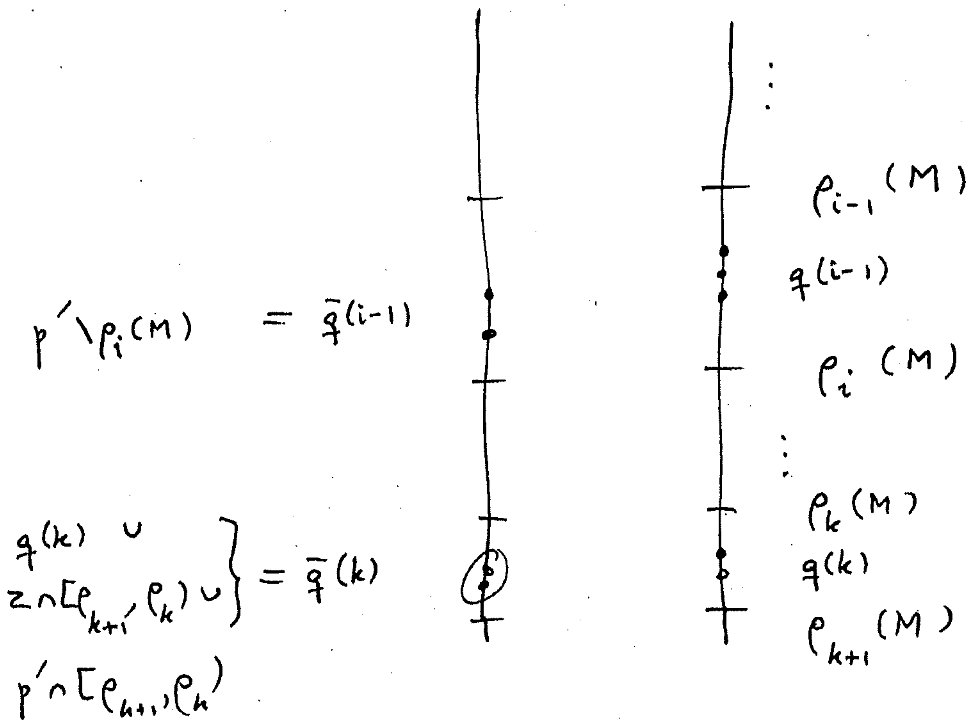
(3) z is lightface Σ_1 definable over $M^i, q \restriction i$.

We may then pick

$$(4) \qquad p' \in \mathrm{Hull}_{\Sigma_1}^{M^{i-1},\, p\restriction i-1}(\{z,\ q(i-1)\})$$

s.t.

$$(5) \qquad p' <^* q(i-1) \ \wedge \ q(i-1) = h_{M^{i-1},\, p\restriction i-1}(j, (z, p')).$$



We now define $\bar{q}$ by

$$\bar{q}(k) = \begin{cases} p(k) = q(k), & k < i-1 \\ p' \setminus \rho_i(M), & k = i-1 \\ q(k) \cup (z \cap [\rho_{k+1}(M), \rho_k(M))) \cup \\ \qquad p' \cap [\rho_{k+1}(M), \rho_k(M)), & k \geq i. \end{cases}$$

Clearly, $\bar{q} <^* q$.

We claim that $\bar{q} \in P^n_M \setminus R^n_M$. This immediately follows from

(6) for all $k < n$,

$$\mathrm{Hull}_{\Sigma_1}^{M^{k,\bar{q}\restriction k}}(\rho_{k+1}(M) \cup \{\bar{q}(k)\}) = \mathrm{Hull}_{\Sigma_1}^{M^{k,q\restriction k}}(\rho_{k+1}(M) \cup \{q(k)\}).$$

This is trivial for $k < i-1$. For $k = i-1$, $\subset$ follows from (4), and $\supset$ follows from (5). ~~[illegible]~~ We are left with having to show

(6) for $k \geq i$.

We first claim that (3) yields that

(7) $z \cap \rho_k(M)$ is lightface Σ_1 definable over $M^{k,q\restriction k}$, all $k \geq i$.

Namely, suppose that $\bar{z} \in [\rho_k(M)]^{<\omega}$ is lightface Σ_1 definable over $M^{k,q\restriction k}$. Then $\bar{z} \cap \rho_{k+1}(M)$ is also lightface Σ_1 definable over $M^{k,q\restriction k}$, i.e.,

$$\xi \in \bar{z} \cap \rho_{k+1}(M) \longleftrightarrow M^{k,q\restriction k} \models \varphi(\xi),$$

some Σ_1 fmla φ, so that for the right $m < \omega$,

$$\xi \in \bar{z} \cap \rho_{k+1}(M) \leftrightarrow (m,\xi) \in A^{k+1,q\restriction k+1},$$

which trivially gives that $\bar{z} \cap \rho_{k+1}(M)$ is lightface Σ_1 definable over $M^{k+1,q\restriction k+1}$.

Next we claim that this plus (4) implies that

(8) $p' \cap \rho_k(M)$ is lightface Σ_1 definable over $M^{k, q\restriction k}$, all $k \geq i$.

For $k = i$, $p' \cap \rho_i(M) \in \mathrm{Hull}_{\Sigma_1}^{M^{i-1, q\restriction i-1}}(\{z, q(i-1)\})$, so that for the right $m < \omega$,

$$\xi \in p' \cap \rho_k(M) \leftrightarrow (m, (\xi, z)) \in A^{i, q\restriction i},$$

so that by (3), $p' \cap \rho_i(M)$ is indeed lightface Σ_1 definable over $M^{i, q\restriction i}$. But then (8) follows by the same induction as for (7).

Let us now show (6) for $k = i$, for this we need to recall why (6) is true for $k = i-1$. By (6) for $k = i-1$, $|M^{i, \bar{q}\restriction i}| = |M^{i, q\restriction i}|$. Moreover, by (4), there is some recursive $e : \omega \longrightarrow \omega$ s.t. for $x \in |M^{i, q\restriction i}|$ and $l < \omega$,

(9) $(l, x) \in A^{i, \bar{q}\restriction i} \Leftrightarrow (e(l), (x, z)) \in A^{i, q\restriction i}$,

as for any Σ_1 formula φ, $M^{i-1, \bar{q}\restriction i-1} \models \varphi(x, \bar{q}(i-1))$
$\Leftrightarrow M^{i-1, q\restriction i-1} \models \varphi(x, h(m, (z, q(i-1))))$, some fixed $m < \omega$ witnessing (4).

But then if φ is Σ_1,

$$M^{i,\bar{q}\restriction i} \models \varphi(x, \bar{q}(i), A^{i,\bar{q}\restriction i}) \longleftrightarrow$$

$$M^{i,q\restriction i} \models \varphi'(x, q(i) \cup (z \cap [p_{i+1}, p_i)) \cup (p' \cap [p_{i+1}, p_i), A^{i,q\restriction i}),$$

where φ' results from φ by making replacements as given by (9). By (7) and (8) there is a uniform way to replace φ' by φ'' so that the above is in turn equivalent with

$$M^{i,q\restriction i} \models \varphi''(x, q(i), A^{i,q\restriction i}).$$

By repeating this argument with (5) instead of (4), we also get a recursive $\varphi \mapsto \varphi^*$ with

$$M^{i,q\restriction i} \models \varphi(x, q(i), A^{i,q\restriction i}) \longleftrightarrow$$

$$M^{i,\bar{q}\restriction i} \models \varphi^*(x, \bar{q}(i), A^{i,\bar{q}\restriction i}).$$

This gives (6) for $k = i$.

The argument just given may be repeated to show that actually for all $k \geq i$ there are recursive $\varphi \mapsto \varphi''$ and $\varphi \mapsto \varphi^*$ (all formulae being Σ_1) s.t.

$$(12)\quad \begin{cases} M^{k,\bar{q}\restriction k} \models \varphi(x, \bar{q}(k), A^{k,\bar{q}\restriction k}) \longleftrightarrow \\ M^{k,q\restriction k} \models \varphi''(x, q(k), A^{k,q\restriction k *}) \end{cases}$$

and

$$(13)\quad \begin{cases} M^{k,q\restriction k} \models \varphi(x, q(k), A^{k,q\restriction k}) \longleftrightarrow \\ M^{k,\bar{q}\restriction k} \models \varphi(x, \bar{q}(k), A^{k,\bar{q}\restriction k}). \end{cases}$$

This gives (6), as desired. $\dashv$