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INCOMPARABLE DEGREES IN $L[\mathcal{E}]$

RALF SCHINDLER AND XIANGHUI SHI

Abstract. We study the structures of generalized degrees at singular cardinals in pure extender models of the form $L[\mathcal{E}]$. It turns out that in such models, say at a singlue cardinal λ, there is a nice correspondence between the structural property for the generalized degrees at λ and the large cardinal strength λ carries. More precisely, assume that no transitive model of ZF^- has an inner model with one Woodin cardinal, in any fully iterable pure extender model, the generalized degree structures at countable cofinality strong limit cardinals λ has incomparable degrees above every degree if and only if λ is an ω-limit of measurable cardinals.

§1. Introduction. As natural generalizations of classical notion of Turing degree, the notions of hyperarithmetic degree (from Δ^0_1-reduction to Δ^1_1-reduction) and α-degree (an analogue of Turing degree at limit ordinals $> \omega$) were extensively studied (see Sacks' book [10]). There is even a notion of α-degree analogue of hyperarithmetic degree at ordinals greater than ω that have countable cofinality—generalized hyperarithmetic degree (see Sacks-Slaman [11]). These studies were mostly conducted within Gödel's constructible universe L. Particularly worth mentioning is Sy Friedman's seminal work [2] — inspired by Silver's work on the Singular Cardinal Hypothesis, he proved that the structure of $\aleph_{\omega_1}$-degrees in L is well-ordered above the $\aleph_{\omega_1}$-jump. This gives a negative answer to the Post problem for $\aleph_{\omega_1}$-degrees. Sy Friedman also mentioned in this paper that Harrington and Solovay proved independently that there are incomparable $\aleph_\omega$-degrees (over every $\aleph_\omega$-degree), therefore the structure of $\aleph_\omega$-degrees is not a linear ordering.

In 2015, the second author introduced (see [14] and [15]) the notion of Γ-degree, where Γ is a reasonable set of set theory axioms, as a generalization of the above-mentioned notions of degrees and promoted the higher degree theory project. To every set of ordinals $X \subset \mathsf{Ord}$, we associate a minimal Γ-model M_X. And for any cardinal λ and any two subsets $A, B \subset \lambda$, we say A *is* Γ*-reducible to* B, written as $A \leq_\Gamma B$, if $M_A \cap \mathscr{P}(\lambda) \subseteq M_B \cap \mathscr{P}(\lambda)$ (see Definition 2.1). The notion of Γ-degrees are the equivalence classes induced from this reduction. In [14] the

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second author investigated the structures of Γ-degrees at singular cardinals in various L-like canonical inner models and he used Zermelo set theory (i.e. $\Gamma = \mathsf{Z}$) as an example to illustrate the idea.

THEOREM 1.1 ([14]). *Assume $V = L$ or $L[\bar{\mu}]$ (where $\bar{\mu}$ is a sequence of normal ultrafilters). Assume that there is no inaccessible limit of measurable cardinals.*[1] *Then the partial order of Z-degrees at any singular cardinal λ is wellordered on a cone if and only if λ is not an ω-limit of measurable cardinals.*[2]

Thus in canonical models below ω many measurable cardinals, i.e. in L and $L[\bar{\mu}]$ where $\bar{\mu}$ is a finite sequence of normal ultrafilters, the partial order of Z-degrees at *every* singular cardinal is wellordered on a cone. Not until one gets into a model $L[\bar{\mu}]$ such that

$$L[\bar{\mu}] \models \text{“}\exists \lambda\, (\lambda \text{ is an } \omega\text{-limit of measurable cardinals)”},$$

one starts to see non-linear Z-degree structures appearing at singular cardinals which are ω-limits of measurable cardinals. These facts are proved by applying the corresponding covering lemmas within the minimal Γ-models and analyzing the combinatorics of the indiscernible sequences that generate the covering sets.

The (full) covering lemmas are assertions of the form that under a suitable anti-large cardinal assumption, every set $X \in V$ can be covered by a set Y in the core model (if exists) with size $|Y| \leq |X| \cdot \aleph_1$. The first of this sort is Jensen's covering lemma for L, then Dodd-Jensen's for one measurable cardinal, and Mitchell's for sequence of measures.

The covering argument in [14] does more than proving Theorem 1.1. It works up to $L[\bar{\mu}]$-models in which there is no inaccessible limit of measurable cardinals — the analysis relies on the existence of a system of indiscernibles $\langle C_\nu : \nu < \theta \rangle$, with each $C_\nu \subseteq \kappa_\nu$ is finite or a Prikry sequence with the property

(I.16)

$$\forall \langle x_\nu : \nu < \theta \rangle \in K \left(\big(\forall \nu < \theta\, (x_\nu \in U_\nu)\big) \longrightarrow \Big| \bigcup \{C_\nu - x_\nu : \nu < \theta\} \Big| < \omega \right),$$

where K denotes the core model and $\langle \kappa_\nu : \nu < \theta \rangle$ is an increasing enumeration of measurable cardinals in K and each U_ν is the full measure on κ_ν. We follow Mitchell [8] labelling this property as (I.16). Such system of indiscernibles if exists does not depend on the set x to be covered. Utilizing this property, one can identity the Γ-degrees as the $\equiv_\Gamma$-equivalence class of systems of indiscernibles, where $\equiv_\Gamma$ is the equivalence relation induced by the Γ-reduction. The following theorem gives a concise account of the partial order of Γ-degrees at singular cardinals in $L[\bar{\mu}]$, the canonical model for ω many measurable cardinals.

THEOREM 1.2 ([14], Theorem 2.13). *Assume $V = L[\bar{\mu}]$, where $\bar{\mu} = \langle \mu_n : n < \omega \rangle$, each μ_n is a measure on κ_n. Let $\kappa_\omega = \sup_n \kappa_n$. Suppose λ is a singular cardinal.*

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[1] This theorem is not explicitly stated in [14] — it summarizes several results in that paper. The reason for the second assumption on the non-existence of inaccessible limit of measurable cardinals can be found in §4 of Mitchell article [8]. One can remove the second assumption, the discussion will appear shortly after Theorem 1.2.

[2] λ is an *ω-limit of measurables* if $\mathrm{cf}(\lambda) = \omega$ and λ is a limit of measurable cardinals.

1. *If $\lambda \neq \kappa_\omega$, then the* Z*-degrees at λ are wellordered on a cone;*[3]
2. *If $\lambda = \kappa_\omega$, then above the* Z*-degrees of $\bar{\mu}$,*
   (a) Z*-degrees are prewellordered via their* Z*-ordinals (the heights of the minimal* Z*-models).*
   (b) *the* Z*-degrees in the η-th level (i.e. the degrees whose* Z*-ordinals equal to α_η, the η-th* Z*-ordinal) are exactly the degrees of the sets in $\{(A_\eta, C) \mid C \in \mathcal{C}_\eta \cup \{\varnothing\}\}$, where for each $\eta > 0$, A_η is a subset of λ coding the first η* Z*-ordinals, and $\mathcal{C}_\eta$ is the set of $L_{\alpha_\eta}[\bar{\mu}]$-generic Prikry sequences for $\mathbb{P}_{\bar{\mu}}$, the diagonal (tree) Prikry forcing associated to $\bar{\mu}$.*

Using the property (I.16), one can split a Prikry generic sequence C in (2-b) into two disjoint cofinal subsequences to get two mutually generic sequences, and hence two $\leq_{\mathsf{Z}}$-incomparable subsets of λ. This gives the "only if" direction of Theorem 1.1. The pictures in the $L[\bar{\mu}]$-models beyond ω-many measurable cardinals below the inaccessible limit of measurable cardinals are pretty much the same, except that κ_ω in clause (2) can be replaced by any ω-limit of measurable cardinals and in clause (1), λ is any singular cardinal that is not an ω-limit of measurable cardinals.

But the property (I.16) is a bit of an overkill for proving Theorem 1.1. In $L[\bar{\mu}]$-models beyond the inaccessible limit of measurable cardinals, since only full measures are used in constructing $L[\bar{\mu}]$, the argument for the "if" direction of Theorem 1.1 still works — the constructions in the vicinity of non-ω-limit-of-measurable singular cardinals is essentially L-like, thus the Z-degrees at these cardinals are also wellordered on a cone. Now suppose $\lambda = \sup_n \kappa_n$ is an ω-limit of measurable cardinals, and $\bar{\mu} = \langle \mu_n : n < \omega \rangle$ be the witnessing measure sequence, apply the product of diagonal (tree) Prikry forcing $\mathbb{P}_{\bar{\mu}} \times \mathbb{P}_{\bar{\mu}}$ over the minimal Z-model in question, one obtains a pair of mutually Prikry generic sequence, hence a pair of $\leq_{\mathsf{Z}}$-incomparable subsets of λ. In fact, we now know the width of the partial orders of such Z-degrees — in 2022, Zhang showed that there are anti-chains of λ-degrees of maximum possible size.

THEOREM 1.3 (Zhang [18]). *Assume $V = L[\bar{\mu}]$ and $\lambda = \sup_n \kappa_n$ is an ω-limit of measurable cardinals, and $\bar{\mu} = \langle \mu_n : n < \omega \rangle$ be the witnessing measure sequence. Consider the λ-degrees. Above every degree $\boldsymbol{a} \geq_\lambda \deg(J_\lambda(\bar{\mu}))$, there are λ^ω many pairwise incomparable λ-degrees.*

Zhang stated his theorem for α-degree notions, but his argument works for any notion of definability degrees that are stronger than α-degree (i.e. larger equivalence classes and hence coarser equivalence relations). Thus the same holds for Z-degrees. Therefore, as we mentioned earlier in the footnote for Theorem 1.1, one can remove the second assumption in Theorem 1.1.

The next step of the project is to look into the degree structures in fine structure extender models $L[\mathcal{E}]$ beyond sequences of measurable cardinals.

Question 1. Is it still true in $L[\mathcal{E}]$ that the Z-degrees at singular cardinals is well-ordered on a cone if and only if it is not an ω-limit of measurable cardinals?

[3] In addition to the original statement in Theorem 2.13 of [14], here we include in this clause the case $\mathrm{cf}(\lambda) > \omega$, as Sy Frieman's argument is applicable here and yields the same conclusion.

Unfortunately, the (full) covering lemmas are only available for core models for sequences of measures and for non-overlapping extenders (see Mitchell's handbook article [8]). Once entering the realm of overlapping extenders, only the weak covering (which asserts that under a suitable anti-large cardinal assumption, for every singular cardinal in V, the core model computes its successor correctly) is known for the core models (see [6, 7]). Apart from this, unlike in the construction of $L[\bar{\mu}]$-models only full measures are used, the construction of $L[\mathcal{E}]$ allows partial extenders in $\mathcal{E}$. This adds another layer of complication. Woodin suggested looking into the degrees at $\aleph_\omega$ as a test problem before tackling Question 1.

Question 2 (Woodin). Are the Z-degrees at $\aleph_\omega$ still well-ordered on a cone in $L[\mathcal{E}]$?

In the summer of 2019 at the Berkeley core model conference, Schimmerling announced his joint work with Mitchell on covering limit cardinals in Jensen-Steel's core model (see [6]). Although their main theorem isn't stated quite the same as the (full) covering lemmas for non-overlapping extenders, their machinery for finding the covering set for subsets of limit cardinals enables us to find the missing piece of the puzzle. In this paper, we shall answer both questions.

Throughout the rest of this paper, let Γ be a rich fragment of ZFC that is sufficent to prove the existence of the core model K.

§2. Degrees at $\aleph_\omega$.

Definition 2.1. Let λ be a cardinal. For a set $A \subset \lambda$, a Γ-*model for* A is a model of the form $L_\alpha[\chi_A, A]$, where χ_A is a fixed well ordering of H_λ (the set of sets whose transitive closure have size $< \lambda$) and α is an ordinal $> \lambda$ such that $L_\alpha[\chi_\lambda, A] \models \Gamma$. Such α is the height of that Γ-model, it is called an Γ-*ordinal for* A. The *minimal* Γ-*model for* A, denoted as M_A, is the Γ-model for A with the least α. We shall only be interested in the case λ is a strong limit cardinal, hence the ordertype of χ_λ equals to λ and can be coded by (hence always treated as) a subset of λ.

Given two sets $A, B \subseteq \lambda$, we say A *is* Γ-reducible to B, written as $A \leq_\Gamma B$, if $M_A \cap \mathscr{P}(\lambda) \subseteq M_B \cap \mathscr{P}(\lambda)$. Let $A \equiv^\lambda_\Gamma B$ if and only if $A \leq^\lambda_\Gamma B$ and $B \leq^\lambda_\Gamma A$. $\equiv^\lambda_\Gamma$ is an equivalence relation on $\mathscr{P}(\lambda)$, the equivalence classes are called Γ-degrees, and the induced strict relation $<^\lambda_\Gamma$ is a partial order on the set of Γ-degrees at λ, $\mathscr{D}^\lambda_\Gamma = \mathscr{P}(\lambda)/\equiv^\lambda_\Gamma$. We drop the supscription λ when it is clear from context. *The* Γ-*jump of* A, denoted as $J_\Gamma(A)$, is the theory of M_A. Just like $0^\sharp$ as a subset of ω, we treat $J_\Gamma(A)$ as a subset of λ.

A frequently used fact in our argument is the following theorem of Steel.

Theorem (Steel). *Suppose M is a pure extender mouse and M models "Γ + there is no proper class model with a Woodin". Then "$V = K$" holds in M.*

Steel first proved it for 1-small $L[E]$ in the 90's, and then for the n-Woodin cardinal model M_n, $n \leq \omega$ (see [12] and [17]). Although nowadays this is more or less a folklore among inner model theorists, there is no write-up on this until Schlutzenberg's note [13] on 2019, which generalizes Steel's result to sufficiently

self-iterable meek mice satisfying "There is a proper class of measurables". The version quoted here, which removes the measurable cardinal constraint and is applicable to the modern Jensen-Steel's core model ([5]), is from the correspondence between Steel and the second author.

Let us first answer Woodin's test question.

THEOREM 2.1. *Assume no transitive model of* Γ *has an inner model with a Woodin cardinal. Let* $L[E]$ *be a fully iterable pure extender model. Then*

$$L[E] \models \text{“}(\mathscr{D}_\Gamma^{\aleph_\omega}, <_\Gamma^{\aleph_\omega}) \text{ is a wellorder”}.$$

PROOF. We first verify the following.

CLAIM. Let M be a transitive model of Γ with $H_{\aleph_\omega} = K^M|\aleph_\omega \in M$. Then $M \models \text{“}\mathscr{P}(\aleph_\omega) \subset K\text{”}$.

PROOF. Fix an $X \in \mathscr{P}(\aleph_\omega) \cap M$. Let $e : \aleph_\omega \to K^M|\aleph_\omega$ be bijective and definable over $K^M|\aleph_\omega$. Then $\langle e^{-1}(A \cap \aleph_n) : n < \omega\rangle \in {}^\omega\aleph_\omega \cap M$. By applying Mitchell-Schimmering's covering argument (see [6]) inside M, there is some $Y \in K^M$ such that $\{e^{-1}(X \cap \aleph_n) \mid n < \omega\} \subset Y \subset \aleph_\omega$ and $K^M \models |Y| < \aleph_1$. Let $f : \gamma \to Y$ be the Mostowski's collapse of Y (so $f \in K^M$ and $\gamma < \aleph_\omega$). As $H_{\aleph_\omega} = K^M|\aleph_\omega \in M$, $f^{-1}\text{“}\{e^{-1}(X \cap \aleph_n) \mid n < \omega\} \in H_{\aleph_\omega} \subset K^M$, hence

$$\{e^{-1}(X \cap \aleph_n) \mid n < \omega\} = f\text{“}f^{-1}\text{“}\{e^{-1}(X \cap \aleph_n) \mid n < \omega\} \in K^M.$$

Since $e \in K^M$, we have $\{X \cap \aleph_n \mid n < \omega\} \in K^M$, and hence $X = \bigcup\{X \cap \aleph_n \mid n < \omega\} \in K^M$. ⊣

For $A \subset \aleph_\omega$, write $\beta_A = \aleph_\omega^{+M_A}$. Note that in M_A, by the above argument, $\aleph_\omega^{+K^{M_A}} = \beta_A$. By Lemma 2.1 of [1], the iterability of premouse is absolute between M_A and $L[E]$, thus $K^{M_A}\|\beta_A$ is iterable in $L[E]$. Since $M_A|\aleph_\omega = L[E]|\aleph_\omega$, by the inductive definition of K (see §6 in [16]), $K^{L[E]}\|\aleph_\omega = K^{M_A}\|\aleph_\omega$, and hence $K^{L[E]}\|\aleph_\omega \trianglelefteq K^{M_A}\|\beta_A$. Cofinally many initial segments of $K^{M_A}\|\beta_A$ are ω-sound and project to $\aleph_\omega$, by Lemma 3.5 of [4], $K^{M_A}\|\beta_A \triangleleft K^{L[E]}$, thus $K^{M_A}\|\beta_A = K^{L[E]}\|\beta_A = L[E]\|\beta_A$, the last equality is because "$V = K$" holds in $L[E]$.

In $L[E]$, for $A \in \mathscr{P}(\aleph_\omega)$, since $H_{\aleph_\omega} = L[E]|\aleph_\omega = K^{M_A}|\aleph_\omega \in M_A$, applying the claim to $M = M_A$, one gets $A \in K^{M_A}|\aleph_\omega^{+K^{M_A}} = K^{M_A}|\beta_A$. Now let $A, B \subset \aleph_\omega$. If $\beta_A \leq \beta_B$, then

$$A \in K^{M_A}|\beta_A = L[E]|\beta_A \subset L[E]|\beta_B = K^{M_B}|\beta_B \subset M_B.$$

So $A \leq_\Gamma B$ if and only if $\beta_A \leq \beta_B$. This shows that $<_\Gamma$ is a wellorder. ⊣

Given an set Γ-degree $\boldsymbol{a}$, the set $\mathscr{D}_{\boldsymbol{a}} = \{\boldsymbol{d} \mid \boldsymbol{d} \geq_\Gamma \boldsymbol{a}\}$ is called *the cone above* $\boldsymbol{a}$. We say $\leq_\Gamma^\lambda$ *is a wellorder on a cone* if if there is an $A \subseteq \lambda$ such that $\mathscr{D}_{\boldsymbol{a}}$ is wellordered by $<_\Gamma$, where $\boldsymbol{a} = \deg_\Gamma(A)$ is the Γ-degree of A. The next theorem extends the wellordered (on a cone) degree structure to singular cardinals (of countable cofinality) that are not ω-limit of measurable cardinals.

THEOREM 2.2. *Assume there is no transitive model of* Γ *has an inner model with a Woodin cardinal. Let* $L[E]$ *be a fully iterable pure extender model. Let* λ

be a singular cardinal in $L[E]$. If λ is not an ω-limit of measurable cardinals (in $L[E]$), then

$$L[E] \models \text{“}<^\lambda_\Gamma \text{ is wellordered on a cone”}.$$

Proof. By hypothesis, there are two cases: $\mathrm{cf}^V(\lambda) > \omega$ or

$$\{\mu < \lambda \mid \mu \text{ is measurable in } L[E]\} \text{ is bounded below } \lambda.$$

As mentioned in the introduction, the case $\mathrm{cf}^V(\lambda) > \omega$ follows from Sy Friedman's argument (see Lemma 2 of [2]). Here we look into the other case.

For $A \subset \lambda$, write $\beta_A = (\lambda^+)^{M_A}$. Argue as in the proof of Theorem 2.1,

- $K^{M_A}\|\beta_A$ is iterable in M_A, and hence in $L[E]$ (by Lemma 2.1 of [1]);
- Since $M_A|\lambda = L[E]|\lambda$, $K^{L[E]}\|\lambda = K^{M_A}\|\lambda$, and hence $K^{L[E]}\|\lambda \trianglelefteq K^{M_A}\|\beta_A$.

Cofinally many initial segments of $K^{M_A}\|\beta_A$ are ω-sound and project to λ, by Lemma 3.5 of [4], $K^{M_A}\|\beta_A \lhd K^{L[E]} = L[E]$.

Let $\gamma_0 \in (\lambda, \lambda^{+L[E]})$ be such that $\rho_\omega(L[E]\|\gamma_0) = \lambda$ and

$$L[E]\|\gamma_0 \models \text{“}\lambda \text{ is singular”}.$$

Let $h : \lambda \to L[E]\|\gamma_0$ be a bijection and definable over $L[E]\|\gamma_0$. h induces some $A_0 \subset \lambda$ which codes $L[E]\|\gamma_0$. For all $A \geq^\lambda_\Gamma A_0$, $L[E]\|\gamma_0 \in M_A$, and hence $M_A \models \text{“}\lambda \text{ is singular”}$.

To show that $<^\lambda_\Gamma$ is wellordered for $A \geq^\lambda_\Gamma A_0$, the key is to prove the following analogue of the claim in the proof of Theorem 2.1.

Claim. For $A \geq^\lambda_\Gamma A_0$, $M_A \models \text{“}\mathscr{P}(\lambda) \subset K\text{”}$.

For that, we need the following covering property of M_A,

Subclaim. If $X \in [\lambda]^{\mathrm{cf}^V(\lambda)} \cap M_A$, then there is some $Y \in K^{M_A}$, $|Y| < \lambda$, $Y \supset X$.

Work in M_A. Fix an $X \in [\lambda]^{\mathrm{cf}(\lambda)}$. Let H, π be such that $\pi : H \prec M_A|\theta$, where θ is sufficiently large, H is transitive, ${}^\omega H \subset H$ and $|H| = |X|^{\aleph_0} < \lambda$, $\mathrm{ran}(\pi) \supset X \cup \{\lambda\}$ and $\sup(\lambda \cap \mathrm{ran}(\pi)) = \lambda$. Write $\bar\lambda = \pi^{-1}(\lambda)$ and $\bar K = \pi^{-1}(K^{M_A}\|\lambda)$. Furthermore, we may assume that $\mathrm{ran}(\pi)$ satisfies the two requirements in Mitchell-Schimmerling [9]. In particular, by Requirement 1 in [9], there is an iteration tree $\mathcal{T}$ on K^{M_A} such that $M^{\mathcal{T}}_\infty \rhd \bar K$. By the hypothesis, the set $\{\mu < \lambda \mid \mu \text{ is measurable in } L[E]\}$ is bounded below λ, so the generators of the extenders used on the main branch of $\mathcal{T}$ are bounded in $\bar\lambda$, then the argument from Mitchell-Schimmerling [9] (for Theorem 4) gives us the desired covering set.

With the covering property, it is routine to prove that $M_A \models \text{“}\mathscr{P}(\lambda) \subset K\text{”}$. Now given any two $A, B \geq^\lambda_\Gamma A_0$, as in the proof of Theorem 2.1, $A \leq^\lambda_\Gamma B$ if and only if $K^{M_A}|\beta_A \subseteq K^{M_B}|\beta_B$ if and only if $L[E]|\beta_A \subseteq L[E]|\beta_B$ if and only if $\beta_A \leq \beta_B$. Thus $<^\lambda_\Gamma$ is wellordered on the cone with base the Γ-degree of A_0. $\dashv$

There are two questions left from Theorem 2.2. One is what happens if λ is an ω-limit of measurable cardinals? And the other is what one can say about the structures outside that cone?

§3. Incomparable degrees. In this section, assume that no transitive Γ-model has an inner model with a Woodin cardinal. We deal with the first question — what if λ is an ω-limit of measurables. Below are two basic facts which will be needed in the proof of our next theorem.

Let λ be a singular cardinal with $\mathrm{cf}(\lambda) = \omega$ and that there is $\langle \kappa_n : n < \omega \rangle$ increasing and cofinal in λ such that each κ_n is a measurable cardinal. Let U_n be a measure on κ_n, for $n < \omega$.

PROPOSITION 3.1. *Let λ and $\langle \kappa_n, U_n : n < \omega \rangle$ be as above. Suppose M is an inner model of a sufficently rich fragment of* ZFC *such that $H_\lambda \cup \{\kappa_n : n < \omega\} \subset M$ and $|^\omega H_\lambda \cap M| = \lambda$. Then there exists a sequence $\langle \alpha_n : n < \omega \rangle$ such that for every $\langle X_n : n < \omega \rangle \in M$ with $X_n \in U_n$ for all $n < \omega$, $\alpha_n \in X_n$ for all sufficiently large $n < \omega$.*

PROOF. As $|^\omega H_\lambda \cap M| = \lambda$, there is an enumeration $\langle \langle X^i_n : n < \omega \rangle : i < \lambda \rangle$ of all $\langle X_n : n < \omega \rangle \in M$ such that $X_n \in U_n$ for every $n < \omega$. Let $\kappa_{-1} := \omega$. For $n < \omega$, let $Y_n = \bigcap \{X^i_n \mid i < \kappa_{n-1}\}$. For each $n < \omega$, by the κ_n-completeness of U_n, $Y_n \in U_n$. Pick $\alpha_n \in Y_n$ for all $n < \omega$. We claim that $\langle \alpha_n : n < \omega \rangle$ satisfies the conclusion of the claim.

To see this, let $Z_n = \{\langle X^i_n : n < \omega \rangle : i < \kappa_{n-1}\}$, $n < \omega$. Let $\langle X_n : n < \omega \rangle \in M$ be such that $X_n \in U_n$ for all $n < \omega$. There is some $n_0 < \omega$ such that $\langle X_n : n < \omega \rangle \in Z_m$ for all $m \geq n_0$. Then if $m > n_0$, $Y_m \subset X_m$ and hence $\alpha_m \in X_m$ for all $m > n_0$. ⊣

DEFINITION 3.1. The *diagonal Prikry forcing with the measure sequence* $\langle U_n : n < \omega \rangle$ is partial order $\mathbb{P}_{\langle U_n : n < \omega \rangle}$ defined as follows:

1. Conditions $p \in \mathbb{P}_{\langle U_n : n < \omega \rangle}$ are ω-sequence of the form $\langle p_n : n < \omega \rangle$ with some $k_p < \omega$ such that for $n < k_p$, $p_n < \kappa_n$ and for $n \geq k_p$, $p_n \in U_n$.
2. For two conditions $p, q \in \mathbb{P}_{\langle U_n : n < \omega \rangle}$, $p \leq_{\mathbb{P}_{\langle U_n : n < \omega \rangle}} q$ if and only if $k_p \geq k_q$, $p{\restriction}k_q = q{\restriction}k_q$, for $n \in [k_q, k_p)$, $p_n \in q_n$, and for $n \geq k_p$, $p_n \subset q_n$.

Next is Mathias' characterization of generics for diagonal Prikry forcing.

PROPOSITION 3.2. *Let U_n, $n < \omega$, be as above. Suppose M is an inner model for a sufficiently rich fragment of* ZFC *and such that $\langle U_n \mid n \in \omega \rangle \in M$. An ω-sequence of ordinals $\langle \alpha_n : n < \omega \rangle$ is generic for the diagnoal Prikry forcing $\mathbb{P}_{\langle U_n : n < \omega \rangle}$ if and only if for every ω-sequence $\langle X_n : n < \omega \rangle$ such that $X_n \in U_n$ for every $n < \omega$, $\alpha_n \in X_n$ for sufficiently large $n < \omega$.*

The proof is a routine exercise, we leave it to the reader. Now we prove the main theorem for the case of ω-limit of measurables.

THEOREM 3.1. *Assume no transitive model of Γ has an inner model with a Woodin cardinal. Let $L[E]$ be a fully iterable pure extender model. If λ is an ω-limit of measurable cardinals in $L[E]$, then*

$$L[E] \models \text{“}\forall A \exists B, C \geq^\lambda_\Gamma A\,(B, C \textit{ are } \Gamma\textit{-incomparable})\text{”}.$$

In particular, there are $\leq^\lambda_\Gamma$-incomparable degrees.

PROOF. Set $V = L[E]$. Fix an $A \geq^\lambda_\Gamma A_0$, and noticing that $L[E]\|\beta_A = K^{M_A}\|\beta_A$, where A_0 and β_A are as in the proof of Theorem 2.2, we may assume that $K^{M_A} \models$ “λ is singular”.

Take an $X \in [\lambda]^{\mathrm{cf}(\lambda)} \cap M_A$. We analyze the situation that we may fail to cover X with a set in K^{M_A}. Work in M_A. Let $\pi, H, \bar{K}$ and $\mathcal{T}$ be as in the proof Theorem 2.2. By Requirement 1 in [9], there is an iteration tree $\mathcal{T}$ on K^{M_A} such that $M_\infty^{\mathcal{T}} \rhd \bar{K}$. If the generators of the extenders used on the main branch of $\mathcal{T}$ are bounded in $\bar{\lambda} = \pi^{-1}(\lambda)$, then we get the desired covering set. So let us suppose otherwise.

Let $\alpha \in [0,\infty)_{\mathcal{T}}$ be such that $\bar{\lambda} \in \mathrm{ran}(\pi_{\alpha\infty}^{\mathcal{T}})$ (and no drop). Write $\lambda' = {\pi_{\alpha\infty}^{\mathcal{T}}}^{-1}(\bar{\lambda})$. If $\beta \in [0,\infty)_{\mathcal{T}} \setminus \alpha$, then $\mathrm{crit}(\pi_{\beta\infty}^{\mathcal{T}}) \leq \pi_{\alpha\beta}^{\mathcal{T}}(\lambda')$. There are two cases.

Case I. If we always have $\mathrm{crit}(\pi_{\beta\infty}^{\mathcal{T}}) < \pi_{\alpha\beta}^{\mathcal{T}}(\lambda')$, then for each such β there must be some $\gamma > \beta$ on the main branch with $\mathrm{crit}(\pi_{\gamma\infty}^{\mathcal{T}}) > \pi_{\beta\gamma}^{\mathcal{T}}(\mathrm{crit}(\pi_{\alpha\beta}))$; but then $\bar{\lambda}$ is a limit of measurables in $\bar{K}$, hence (by elementarity) so is λ a limit of measurables in $L[E]|\lambda$. Let $\{\beta_n : n < \omega\}$ on the main branch be such that

$$\mathrm{crit}(\pi_{\beta_{n+1}\infty}) > \pi_{\beta_n\beta_{n+1}}(\mathrm{crit}(\pi_{\beta_n\infty})), \text{ for } n < \omega.$$

Write $\bar{\kappa}_n = \mathrm{crit}(\pi_{\beta_n\infty})$. If $\sup_n \beta_n < \infty$, then $\sup_n \bar{\kappa}_n < \bar{\lambda}$, so $\langle \bar{\kappa}_n : n < \omega\rangle \in \bar{K} = \pi^{-1}(L[E])\|\bar{\lambda}$ and is diagonally Prikry generic over $M_\infty^{\mathcal{T}}$, hence over $\bar{K}$. This is ridiculous! So in this case the β_n's must be cofinal in the length of $[0,\infty)_{\mathcal{T}}$ and hence the $\bar{\kappa}_n$'s are cofinal in $\bar{\lambda}$. By the elementarity of π, in this case λ is an ω-limit of measurable cardinals $\langle \kappa_n : n < \omega\rangle$.

Let U_n be a measure on κ_n, $n < \omega$, in M_A. By Proposition 3.1 and 3.2, there is a sequence $\langle \alpha_n : n < \omega\rangle$ that is $\mathbb{P}_{\langle U_n : n<\omega\rangle}$-generic over M_A. Via the splitting function $\langle p_n : n < \omega\rangle \mapsto (\langle p_{2n} : n < \omega\rangle, \langle p_{2n+1} : n < \omega\rangle)$, $\mathbb{P}_{\langle U_n : n<\omega\rangle}$ is densely embedding into $\mathbb{P}_{\langle U_{2n} : n<\omega\rangle} \times \mathbb{P}_{\langle U_{2n+1} : n<\omega\rangle}$, $(\langle \alpha_{2n} : n < \omega\rangle, \langle \alpha_{2n+1} : n < \omega\rangle)$ is $\mathbb{P}_{\langle U_{2n} : n<\omega\rangle} \times \mathbb{P}_{\langle U_{2n+1} : n<\omega\rangle}$-generic over M_A, so $\langle \alpha_{2n} : n < \omega\rangle$, $\langle \alpha_{2n+1} : n < \omega\rangle$ are mutually generic (via this product forcing) over M_A. By the minimality of $M_{A,*}$ models, $M_A[\langle \alpha_{2n} : n < \omega\rangle] = M_{A,\langle \alpha_{2n} : n<\omega\rangle}$, $M_A[\langle \alpha_{2n+1} : n < \omega\rangle] = M_{A,\langle \alpha_{2n+1} : n<\omega\rangle}$, thus $\langle \alpha_{2n+1} : n < \omega\rangle \notin M_{A,\langle \alpha_{2n} : n<\omega\rangle}$ and $\langle \alpha_{2n} : n < \omega\rangle \notin M_{A,\langle \alpha_{2n+1} : n<\omega\rangle}$. Therefore $A \oplus \langle \alpha_{2n} : n < \omega\rangle$ and $A \oplus \langle \alpha_{2n+1} : n < \omega\rangle$ are $\leq_\Gamma^\lambda$-incomparable.

Case II. Suppose that $\mathrm{crit}(\pi_{\beta\infty}^{\mathcal{T}}) = \pi_{\alpha\beta}^{\mathcal{T}}(\lambda')$ from some point on. We show that this case is excluded by the assumption that λ is singluar in K^{M_A}.

Let $\langle \beta_n : n < \omega\rangle$ enumerate the first ω such β, and again write $\bar{\kappa}_n = \mathrm{crit}(\pi_{\beta_n\infty}^{\mathcal{T}})$. Again $\langle \bar{\kappa}_n : n < \omega\rangle$ would be generic over $M_\infty^{\mathcal{T}}$, hence over $\bar{K}$, for a Prikry forcing $P_{\bar{\mu}}$. Let $\kappa_n = \pi(\bar{\kappa}_n)$, $n < \omega$. If $\langle \kappa_n : n < \omega\rangle$ is bounded below λ, then it would be in $K^{M_A}|\lambda$ and is generic over $K^{M_A}\|\lambda$ for $\mathbb{P}_{\pi(\bar{\mu})}$ (using Mathias' characterization, Theorem 1.12 in [3]).[4] This is ridiculous! So it must be that $\langle \kappa_n : n < \omega\rangle$ is cofinal in λ.

We may assume that on a tail and use the measure of Mitchell order 0 on κ_n to form $\pi_{\beta_n\,\mathcal{T}\text{-}\mathrm{succ}(\beta_n)}^{\mathcal{T}}$, as otherwise λ gets a limit of measurables again of cofinality ω. But then $\langle \kappa_n : n < \omega\rangle$ is in fact Prikry generic for the order 0 measure on $\bar{\lambda}$ in

[4]Strictly speaking, Theorem 1.12 in [3] is stated for ZFC model M, it may not be applicable to $K^{M_A}\|\lambda$ directly. Instead, one could take an $N \subset K^{M_A}\|\lambda$ such that $N \supset K^{M_A}|\sup_n \kappa_n$ and models a sufficiently rich fragment of ZFC as assumed in Proposition 3.2. By the elmentarity of π, every $\pi(\bar{\mu})$-measure one subset of $\sup_n \kappa_n$ in $K^{M_A}|\lambda$ contains a tail of $\langle \kappa_n : n < \omega\rangle$. This is also true over N, hence $\langle \kappa_n : n < \omega\rangle$ is Prikry generic over N.

$M_\infty^{\mathcal{T}}$ which gives as in Mitchell-Schimmerling [9] that λ is measurable in K^{M_A}. But we assumed that $K^{M_A} \models$ "λ is singular".

By the above analysis, the case that λ is an ω-limit of measurables only happens in CASE I, and hence $\leq_\Gamma^\lambda$-incomparable degrees exist in every cone of subsets of λ. $\dashv$

Combining Theorem 2.2 and Theorem 3.1, we have the following characterization of singular cardinals in $L[E]$ that are ω-limit of measurable cardinals in terms of the existence of a cone over which the Γ-degree is wellordered.

THEOREM 3.2. *Assume no transitive model of Γ has an inner model with a Woodin cardinal. Let $L[E]$ be a fully iterable pure extender model. Let λ be a singular cardinal of $L[E]$.*

1. *If λ is not an ω-limit of measurable cardinals (in $L[E]$), then*
$$L[E] \models \text{"}\leq_\Gamma^\lambda \textit{ is wellordered on a cone}\text{"}.$$
2. *If λ is an ω-limit of measurable cardinals (in $L[E]$), then*
$$L[E] \models \text{"}\textit{There are } \leq_\Gamma^\lambda\textit{-incomparable degrees in every cone}\text{"}.$$

§4. An example outside the cone. In this section, assume no transitive Γ-model has an inner model with a Woodin cardinal, we construct an example of an $L[E]$ which has a singular cardinal λ (not necessarily ω-limit of measurable cardinals) and $A, B \subset \lambda$ with $A \not\leq_\Gamma B$ and $B \not\leq_\Gamma A$.

Let us fix $M = L[E]$, fully iterable, and some λ such that

(a) λ is inaccessible in M,

and there are $\nu < \mu$ such that

(b) E_ν^M has critical point λ and λ as its only generator,
(c) E_ν^M is total on $M|\mu$, and
(d) E_μ^M has critical point $> \nu$ and $\mathrm{crit}(E_\mu^M)$ as its only generator, and $\rho_1(M\|\mu) > \lambda$.

For example, work with some $L[E]$ which has two measurables, $\lambda < \mathrm{crit}(E_\mu^M)$.

Let $\sigma : Q \cong X = \mathrm{Hull}_{\Sigma_1}^{M\|\mu}(\lambda \cup \{\lambda, \nu\}) \prec_{\Sigma_1} M\|\mu$, where Q is transitive. Then $Q \in M|\mu$ by (d), $\rho_1(Q) = \lambda$, $\lambda^{+Q} < \lambda^{+M\|\mu}$, and in fact $Q = M\|\bar\mu$, where $\bar\mu$ is least such that $\lambda^{+Q} < \bar\mu < \lambda^{+M\|\mu}$ and $\rho_1(M\|\bar\mu) \leq \lambda$.

Let $\bar\nu = \sigma^{-1}(\nu)$ and
$$\overline{U} = \{X \in Q \cap \mathscr{P}(\lambda) \mid \{\{\xi\} : \xi \in X\} \in (E_{\bar\nu}^M)_{\{\lambda\}}\}$$
be the normal measure component of $E_{\bar\nu}^M$, and
$$U = \{X \in M|\mu \cap \mathscr{P}(\lambda) \mid \{\{\xi\} : \xi \in X\} \in (E_\nu^M)_{\{\lambda\}}\}$$
be the normal measure component of E_ν^M. By $\sigma : Q \prec_{\Sigma_1} M\|\mu$, $\sigma(E_{\bar\nu}^M) = E_\nu^M$, we have that $\overline{U} \subset U$; also $M\|\mu \models$ "$|\bar U| = \lambda$", further $E_{\bar\nu}^M$ is total on $M|\bar\mu$.

Now let $Y \prec_{\Sigma_\omega} M\|\mu$ be countable, $Y \in M$, and $\{\bar\mu, \bar\nu, \nu\} \subset Y$. Let $Y^+ = \mathrm{Hull}_{\Sigma_\omega}^{M\|\mu}(Y \cup \gamma)$, where $\gamma = \sup(Y \cap \lambda)$. By (a), if $\bar\gamma \in Y \cap \lambda$, and $s \in [Y]^{<\omega}$, then for each Skolem term τ,
$$\sup\{\tau^{M\|\mu}(\xi, \bar\gamma, s) : \xi < \bar\gamma\} \in Y \cap \lambda;$$

This implies that

$$\gamma \leq \sup(Y^+ \cap \lambda) = \sup(Y \cap \lambda) = \gamma,$$

so that $\gamma = Y^+ \cap \lambda$.

Let $\sigma^* : P \cong Y^+ \prec_{\Sigma_\omega} M\|\mu$. By (a), there is a bijection $h : \lambda \to H_\lambda$ definable in M. Let $\bar{h} = \sigma^{*-1}(h)$. As $\mathrm{crit}(\sigma^*) = \gamma$, $\sigma^*(\gamma) = \lambda$, $\bar{h}$ is an initial segment of h. Thus $H^P_\gamma = H^M_\gamma$. Therefore γ is a (limit of) cardinal(s) in P as well as in $M\|\mu$. In fact, by the definition of P, $P = M|\mu_0$, where $\mu_0 = P \cap \mathsf{Ord}$. Let $(\bar{\nu}_0, \bar{\mu}_0) = \sigma^{*-1}((\bar{\nu}, \mu_0))$. If $\overline{U}_0$ (U_0) is the normal measure component of $E^M_{\bar{\nu}_0}$ ($E^M_{\nu_0}$, resp.), then $\overline{U}_0 \subset U_0$; moreover, $M\|\mu_0 \models$ "$|\overline{U}_0| = \gamma$" and $\overline{U}_0$ is total on $M|\bar{\mu}_0$.

Let $\langle X_i : i < \gamma\rangle \in M|\mu_0$ be such that $\overline{U}_0 = \{X_i \mid i < \gamma\}$, and let $X = \Delta_{i<\gamma} X_i$. By $\overline{U}_0 \subset U_0$ and the normality of $\overline{U}_0$, $X \in U_0$ and $X \setminus (i+1) \subset X_i$ for all $i < \gamma$. As $\gamma = \sup(Y \cap \lambda)$, $|Y| = \aleph_0$, γ has cofinality ω (in M). Let $\langle \alpha_n : n < \omega\rangle$ be strictly increasing and cofinal in γ. Let $\kappa_n = \sup(X \setminus \alpha_n)$, $n < \omega$. If $\alpha_n > i$ then $\kappa_n \in X_i$. In other word, for every $Y \in \overline{U}_0$, a tail end of $\langle \kappa_n : n < \omega\rangle$ is contained in Y.

Now as $\overline{U}_0$ is total on $M|\bar{\mu}_0$, and $E^M_{\bar{\mu}_0}$ has critical point $> \bar{\nu}_0$, $\overline{U}_0$ is a total measure on $L[M|\bar{\nu}_0]$. But then $\langle \kappa_n : n < \omega\rangle$ is Prikry generic over $L[M|\bar{\nu}_0]$ with respect to the measure $\overline{U}_0 \approx (E^M_{\bar{\nu}_0})_{\{\gamma\}}$.

Recall $C = \langle \kappa_n : n < \omega\rangle \in M$. Let $C_0 = \langle \kappa_{2n} : n < \omega\rangle$ and $C_1 = \langle \kappa_{2n+1} : n < \omega\rangle$. Let α be least such that $\alpha > \bar{\nu}_0$ and $L_\alpha[M|\bar{\nu}_0] \models \Gamma$. Then $L_\alpha[M|\bar{\nu}_0, C_0, C_1] \models \Gamma$, so that $M_{C_0} \subset L_\alpha[M|\bar{\nu}_0, C_0]$ and $M_{C_1} \subset L_\alpha[M|\bar{\nu}_0, C_1]$. But $C_0 \notin L_\alpha[M|\bar{\nu}_0, C_1]$ and $C_1 \notin L_\alpha[M|\bar{\nu}_0, C_0]$, therefore $C_0 \not\leq_\Gamma C_1$ and $C_1 \not\leq_\Gamma C_0$. This completes the construction of our example.

Remark. The pair of $\leq^\lambda_\Gamma$-incomparable degrees constructed in this example is at a singular cardinal which is clearly not a limit of measurable cardinals in $L[E]$, which seemly contradicts to the characterization in Theorem 3.2. The subtle point is that the pair (C_0, C_1) in this example is constructed outside any cone of degrees that can compute an ω-sequence that is cofinal in γ (in $L[E]$), while the two $\leq^\lambda_\Gamma$-incomparable Γ-degrees obtained in the proof Theorem 3.1 — the degrees of $A \oplus \langle \alpha_{2n} : n < \omega\rangle$ and $A \oplus \langle \alpha_{2n+1} : n < \omega\rangle$ — are above the degree of A which codes an ω-sequence of measurables that is cofinal in λ (in $L[\mathcal{E}]$).

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INSTITUT FÜR MATHEMATISCHE LOGIK UND GRUNDLAGENFORSCHUNG UNIVERSITÄT MÜNSTER EINSTEINSTR 62, 48149 MÜNSTER, GERMANY

E-mail: rds@uni-muenster.de

SCHOOL OF MATHEMATICAL SCIENCES, BEIJING NORMAL UNIVERSITY, KEY LABORATORY OF MATHEMATICS AND COMPLEX SYSTEMS, MINISTRY OF EDUCATION, BEIJING 100875, P.R. CHINA

E-mail: shi.bnu@gmail.com