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Tall Cardinals in Extender Models

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Abstract We obtain a carachterization of λ-tall cardinals in terms of the function $o(\alpha)$ in extender models $L[E]$ which have no inner model with a Woodin caridnal and $L[E] \models$ " I am iterable". This implies in the equivalence between tall cardinals and strong cardinals in such extender models.

1 Introduction

Tall cardinals apperead in varying extents as hypothesis in the work of Woodin and Gitik but they were only named as a distinc type of large cardinals by Hamkins in [9] where Hamkins does a systematic study of this large cardinal concept. Also Apter in [3], [4], [5], [2], Apter and Gitik in [1] and Apter and Cummings in [6] investigated tall cardinals.

Assuming that there is no inner model with a Woodin cardinal by [10] we can isolate $\mathscr{K}$, the core model, and then make sense of the following:

Definition 1.1 We say $o(\kappa) = \alpha$ if and only if $\alpha = \text{otp}(\{\beta \mid crit(E^{\mathscr{K}}_{\beta}) = \kappa\})$ and $O(\kappa) = \alpha$ if and only if $\alpha = sup\{\beta \mid crit(E^{\mathscr{K}}_{\beta}) = \kappa\}$.

Remark 1.2 In the cases we are interested in $o(\kappa)$ we will have $o(\kappa) = O(\kappa)$. See proposition 2.5.

A special case of the results obtained by Gitik in [7] is the following:

Theorem 1.3 *(Gitik) Suppose $\neg 0^{\P}$ and that κ is a measurable cardinal such that*

$$2^{\kappa} > \mu > \kappa^{+}$$

where μ is regular, then

$$o(\kappa) \geq \mu$$

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Our main result is theorem 1.5 which is an attempt of generalizing theorem 1.3 to larger core models. Instead of the notion of a measurable cardinal κ with $2^\kappa > \mu$ we work with a more general notion of μ-tall cardinal.

Definition 1.4 κ is a α-*tall cardinal* if and only if there is an elementary embedding $j : V \to M$ where $crit(j) = \kappa$, $M^\kappa \subseteq M$ and $j(\kappa) > \alpha$. We say that κ is a *tall cardinal* if and only if κ is α-tall for every ordinal α.

Note that in theorem 1.3 κ is μ-tall, so a natural question to ask is whether changing $\neg 0^\P$ for "There is no inner model with a Woodin cardinal" and "κ measurable with $2^\kappa > \mu > \kappa^{+}$" for "$\kappa$ is μ-tall for some regular $\mu > \kappa^{+}$" we still obtain $o(\kappa) \geq \mu$. It happens that this is false, already by corollary 2.7 and theorem 2.10 in [9] (see proposition 2.13 below) if κ is measurable and $sup\{\beta < \kappa \mid o(\beta) > \mu\} = \kappa$ then κ is μ-tall. We can ask whether this is the only exception, and if κ is μ-tall for some regular cardinal $\mu > \kappa^+$ implies that

$$\mathscr{K} \models o(\kappa) \geq \mu \ \vee \Big(sup\{\nu < \kappa \mid o(\nu) \geq \mu\} = \kappa \ \& \ \kappa \text{ is a measurable cardinal} \Big) \tag{1}$$

We do not know how to answer this question in this generality, we then shift to a very restricted version of this question:

We keep the hypothesis that there is no inner model with a Woodin cardinal and we assume that V, the universe, is a $L[E]$ model such $L[E] \models$ "I am iterable", and in this settings we ask whether κ is μ-tall for some regular cardinal $\mu > \kappa^+$ implies (1).

We answer this last question positively in section 2 for many cases:

Theorem 1.5 *Suppose that $V = L[E]$, there is no inner model with a Woodin cardinal, $L[E] \models$ "I am iterable" and let κ, μ be regular cardinals in $L[E]$. Suppose further that if $\mu = \lambda^+$ for some singular cardinal λ and $cf(\lambda) \geq \kappa$ then $cf(\lambda)$ is not the critical point of a total measure on $L[E]$ indexed on E.* *Then*

$$L[E] \models \ \kappa \textit{ is } \mu\textit{-tall}$$

if and only if

$$L[E] \models o(\kappa) > \mu \ \vee \Big(sup\{\nu < \kappa \mid o(\nu) > \mu\} = \kappa \ \& \ \kappa \textit{ is a measurable cardinal} \Big)$$

Remark 1.6 The direction ($\Leftarrow$) is due to Hamkins, see corollary 2.7 and theorem 2.10 in [9].

Remark 1.7 If κ is a regular cardinals such that $o(\kappa) \in (\lambda, \lambda^+)$ where

$$L[E] \models cf(\lambda) \text{ is a measurable cardinal}$$

and $L[E] \models cf(\lambda) \geq \kappa$, then using arguments of section 2 in [9] and the fact that if U is a measure on $cf(\lambda)$ then $\pi_U(\lambda) > \lambda^+$, we can conclude that κ is λ^+-tall. So we can not drop the hypothesis on $cf(\lambda)$ in theorem 1.5.

Corollary 1.8 *Suppose that $V = L[E]$, there is no inner model with a Woodin cardinal, $L[E] \models$ "I am iterable". Then*

$$L[E] \models \kappa \textit{ is a tall cardinal},$$

if and only if

$$L[E] \models \ \kappa \textit{ is a strong cardinal or a measurable limit of strong cardinals.}$$

Indexing: We use λ-indexing, also called Jensen indexing, so if $\mathscr{M}$ is a potential premouse, $E_\beta^{\mathscr{M}} \neq \emptyset$ and $\mathscr{N} = Ult_0(\mathscr{M}^*, E_\beta^{\mathscr{M}})$, then $\beta = \pi_{E_\beta^{\mathscr{M}}}(crit(E_\beta^{\mathscr{M}}))^{+\mathscr{N}}$.

Iteration trees: If $\mathscr{T}$ is an iteration tree on a premouse $\mathscr{M}$, then we will write $\nu_\xi^{\mathscr{T}} = \nu(E_\xi^{\mathscr{T}})$ where $\nu(E_\xi^{\mathscr{T}})$ is the ordinal where $E_\xi^{\mathscr{T}}$ is indexed in the sequence $E^{\mathscr{M}_\xi^{\mathscr{T}}}$ of $\mathscr{M}_\xi^{\mathscr{T}}$. We may some times omit the superscript $\mathscr{T}$.

2 Equivalence

We will need the following results:

Lemma 2.1 (lemma 1.1 in [11]) *Let $\mathscr{M} = \langle J_\alpha^E, \in, E, F\rangle$ be a 0-iterable premouse, where $F \neq \emptyset$. Suppose that for no $\mu \leq \mathscr{M} \cap OR$ do we have*

$$\mathscr{J}_\mu^{\mathscr{M}} \models ZFC + \text{ there is a Woodin cardinal}.$$

Set $\kappa = crit(F)$ and let $\xi \in (\kappa, \rho_1(\mathscr{M}))$. Then there is some $\tilde{\nu} \in (\xi, \xi^{+\mathscr{M}})$ and $crit(E_{\tilde{\nu}}) = crit(F)$. Moreover if $\rho_1(\mathscr{M}) > crit(F)^{+\mathscr{M}}$, then $(F|\kappa^+)^*$ is indexed on $\mathscr{M}$.

Theorem 2.2 (lemma 3.5 in [8]) *If there is no inner model with a Woodin cardinal, $\mathscr{K}$ is the core model and $\kappa \geq \aleph_2^V$ is a $\mathscr{K}$-cardinal, then for all sound iterable mouse $\mathscr{M}$ such that $\mathscr{M}|\kappa = \mathscr{K}|\kappa$ and $\rho_\omega(\mathscr{M}) \leq \kappa$ it holds that*

$$\mathscr{M} \lhd \mathscr{K}$$

Theorem 2.3 (theorem 2.1 in [12]) *If there is no inner model with a Woodin cardinal and $j : V \to M$ is an elementary embedding, $\mathscr{M}^\omega \subseteq \mathscr{M}$, $\mathscr{P}(\mathbb{R}) \subseteq \mathscr{M}$ [1], then there is an iteration tree $\mathscr{T}$ on $\mathscr{K}^V$ which does not drop along the main branch such that $\mathscr{M}_\infty^{\mathscr{T}} = \mathscr{K}^{\mathscr{M}}$ and $j|\mathscr{K} = i_{0,\infty}^{\mathscr{T}}$.*

Definition 2.4 Suppose htere is no inner model with a Woodin cardinal. Given $j : V \longrightarrow M$ an elementary embedding. Let $\mathscr{T}$ and $\mathscr{U}$ be the iteration trees obtained by comparing $\mathscr{K}^V$ and $\mathscr{K}^M$ respectvely. Then we say that $\mathscr{T}$ is the *iteration tree induced by j*.

Proposition 2.5 *Suppose there is no inner model with a Woodin cardinal and that $\kappa < \mu$ are cardinals in $\mathscr{K}$, the core model. Then*

$$O(\kappa) > \mu \longleftrightarrow o(\kappa) > \mu,$$

moreover

$$O(\kappa) = \mu \longleftrightarrow o(\kappa) = \mu$$

Proof It is easy to see that $o(\kappa) > \mu$ implies $O(\kappa) > \mu$. For the other direction we have two cases. Suppose first that μ is a limit cardinal and $O(\kappa) > \mu$ we will verify that $o(\kappa) > \mu$. Consider $\langle \nu_\eta \mid \eta < cf(\mu)\rangle$ be a increasing continuous sequence of cardinals which is cofinal in μ and such that each $\nu_{\eta+1}$ is a regular cardinal and $\nu_0 > \kappa$.

For each $\eta + 1 < cf(\mu)$ we will verify:

$$\text{otp}(\{\beta < \nu_{\eta+1} \mid crit(E_\beta^{\mathscr{K}}) = \kappa\}) = \nu_{\eta+1}$$

for that it is enough to verify that

$$sup(\{\beta < \nu_{\eta+1} \mid crit(E_\beta^{\mathscr{K}}) = \kappa\}) = \nu_{\eta+1} \tag{2}$$

Let $\alpha_{\eta+1} > \nu_{\eta+1}$ be such that $crit(E^{\mathscr{K}}_{\alpha_{\eta+1}}) = \kappa$. Let $\mathscr{M} = \mathscr{J}^{\mathscr{K}}_{\alpha_{\eta+1}}$, then $\rho_1(\mathscr{M}) \geq \nu_{\eta+1}$. Fix $\xi < \nu_{\eta+1}$, by lemma 2.1 there is $\tilde{\xi} \in (\xi, \xi^{+\mathscr{M}})$ such that $crit(E^{\mathscr{M}}_{\tilde{\xi}}) = \kappa$. Since ξ was arbitrary the equality in (2) follows. Since (2) holds for every $\nu_{\eta+1}$ with $\eta < cf(\mu)$ we have

$$|\{\beta < \mu \mid crit(E^{\mathscr{K}}_{\beta}) = \kappa\}| = \mu$$

which implies

$$o(\kappa) \geq \mu$$

Since $O(\kappa) > \mu$ it follows that $o(\kappa) > \mu$.

For the case where $\mu = \theta^+$, we suppose $O(\kappa) > \mu$, we fix α such that $\alpha > \mu$ and consider $\mathscr{M} = \mathscr{J}^{\mathscr{K}}_{\alpha}$, then $\rho_1(\mathscr{M}) \geq \mu$ and give $\xi \in (\kappa, \mu)$ by lemma 2.1 there is $\tilde{\xi} \in (\xi, \xi^{+\mathscr{M}})$ such that $crit(E^{\mathscr{M}}_{\tilde{\xi}})) = \kappa$. Hence

$$sup(\{\beta < \mu \mid crit(E^{\mathscr{K}}_{\beta}) = \kappa\}) = \mu$$

which implies together with $O(\kappa) > \mu$ that $o(\kappa) > \mu$.

For the moreover part, suppose $o(\kappa) = \mu$, then $O(\kappa) \leq \mu$, otherwise by the first part we would have $o(\kappa) > \mu$. Hence $O(\kappa) = \mu$.

Suppose $O(\kappa) = \mu$. If μ is a limit cardinal and consider $\langle \nu_\eta \mid \eta < cf(\mu)\rangle$ as above. Then by the first part $o(\kappa) > \nu_\eta$ for each $\eta < cf(\mu)$. Hence $o(\kappa) \geq \mu$ and thus $o(\kappa) = \mu$. If $\mu = \theta^+$ for some cardinal θ, then $O(\kappa) = \mu$ implies trivially $o(\kappa) = \mu$ because μ is a regular cardinal.

Before we start proving lemmas 2.6, 2.9, 2.11 and 2.15, which later will be used to prove theorem 1.5 we try to give a brief description of the proof of theorem 1.5 and how these lemmas will be used.

The proof of theorem 1.5 goes by contradiction. We suppose that theorem 1.5 is false for a cardinal κ which is λ-tall. So there is an ordinal $\mu < \kappa$ such that for all $\xi \in (\mu, \kappa]$ we have $o(\xi) \leq \lambda$. Lemma 2.11 will give that there is a $\Theta \in (\kappa, \lambda]$ which is a cut point. We fix j an elementary embedding which witness that κ is λ-tall and apply lemma 2.6 and theorem 2.3 to find an iteration tree $\mathscr{T}$ on $\mathscr{K}$ such that $j = \pi^{\mathscr{T}}_{0,lh(\mathscr{T})-1}$. The contradiction comes from showing that $\lambda < j(\kappa) = \pi^{\mathscr{T}}_{0,lh(\mathscr{T})-1}(\kappa) \leq \lambda$. For that we need to control where κ gets mapped along the iteration tree $\mathscr{T}$. Lemma 2.9 will tell us that $\mathscr{T}$ must be finite, and by induction we keep track of how the cut point Θ provided by lemma 2.11 is moved along the iteration. This will allow us to bound $\pi^{\mathscr{T}}_{0,lh(\mathscr{T})-1}(\kappa) \leq \lambda$.

Lemma 2.6 (Steel) *Suppose there is no inner model with a Woodin cardinal, $L[E] = V$ and $L[E] \models$ "I am iterable", then $L[E] \models V = \mathscr{K}$.*

Proof

Claim 2.7 $\mathscr{K}|\aleph_2 = L[E]|\aleph_2$

Proof By our anti-large cardinal hypothesis we can build $\mathscr{K}^{L[E]}$ which we will denote by $\mathscr{K}$. Because of acceptability and soundness there are cofinally many

$\alpha < \omega_1$ such that $\rho_\omega(L[E]|\alpha) = \omega$. Fix such $\alpha < \omega_1$, since $L[E] \models$ "I am iterable" we can compare $\mathscr{K}$ and $L[E]|\alpha$. By a well known argument [2] $L[E]|\alpha \lhd \mathscr{K}$. Thus

$$\mathscr{K}|\aleph_1 = L[E]|\aleph_1$$

Again by acceptability and soundness there are unboundedly many $\alpha < \aleph_2$ such that $\rho_\omega(L[E]|\alpha) = \omega_1$. We fix some such α and compare $\mathscr{K}$ and $L[E]|\alpha$. Note that there is no extender E_β indexed above ω_1 such that $crit(E_\beta) < \omega_1$, for $crit(E_\beta) < \omega_1$ implies that

$$L[E]|\beta \models crit(E_\beta) \text{ is a cardinal} \tag{3}$$

while there is $\xi < \omega_1$ such that

$$L[E]|\xi \models crit(E_\beta) \text{ is countable}$$

so $\beta < \xi < \omega_1$. Thus if there is an extender indexed above ω_1 in $L[E]|\alpha$, its critical point must be $\geq \omega_1$. Hence $L[E]|\alpha$ does not move in the comparison. We can conclude like in the previous case that $L[E]|\alpha \lhd \mathscr{K}$.

Suppose κ is a singular cardinal and for every cardinal $\mu < \kappa$ we have $L[E]|\mu = \mathscr{K}|\mu$, then $L[E]|\kappa = \mathscr{K}|\kappa$. Now suppose $\kappa \geq \aleph_2$ is a successor cardinal, say $\kappa = \mu^+$ and $\mathscr{K}|\mu = L[E]|\mu$. Then by theorem 2.2 for every $\alpha \in (\mu, \kappa)$ such that $\rho_\omega(L[E]|\alpha) \leq \mu$ we have $L[E]|\alpha \lhd \mathscr{K}$. Thus $\mathscr{K}|\kappa = L[E]|\kappa$.

Then $L[E] = \mathscr{K}$.

Definition 2.8 We define the following hypothesis:

$$(\Delta) \longleftrightarrow \Big(\big(\text{there is no inner model with a Woodin cardinal}\big) \;\&\; \big(V = L[E]\big) \;\&\; \big(L[E] \models \text{ "I am iterable"}\big)\Big)$$

Lemma 2.9 *Assume (Δ). If $j : L[E] \longrightarrow M$ is a witness to the α-tallness of κ, and $\mathscr{T}$ is the iteration tree induced by j, then $\mathscr{T}$ is finite*

Proof Note that it makes sense to say that $\mathscr{T}$ is the tree induce by j because we are in the hypothesis of theorem 2.3. We also have $\pi^{\mathscr{T}}_{0,\infty} = j$ since, by lemma 2.6, $V = \mathscr{K}$. There is a first order formula that express $x \in \mathscr{K}$, then $L[E] \models \forall x(x \in \mathscr{K})$ implies that $M \models \forall x(x \in \mathscr{K})$ so $M = \mathscr{K}^M = \mathscr{M}^{\mathscr{T}}_\infty$.

Suppose that $lh(\mathscr{T}) \geq \omega + 1$ and that b is the cofinal branch on ω such that $\mathscr{M}^{\mathscr{T}}_\omega = dirlim_{n \in (b \setminus n_0)} \mathscr{M}^{\mathscr{T}}_n$ for some large enough $n_0 \in \omega$. Consider $\langle \kappa_n \mid n \in \omega \cap b \rangle$ where

$$\forall n \in (b \setminus n_0) \left(\kappa_n = crit(\pi^{\mathscr{T}}_{n,\omega})\right)$$

where n_0 is large enough such that $\pi^{\mathscr{T}}_{n,\omega}$ is defined and $\kappa_n = \emptyset$ for $n \in n_0 \cap b$.

Let us verify that $\langle \kappa_n \mid n \in \omega \cap b \rangle \notin \mathscr{M}^{\mathscr{T}}_\omega$. For a contradiction suppose $\langle \kappa_n \mid n \in \omega \cap b \rangle \in \mathscr{M}^{\mathscr{T}}_\omega$, then there is $m \in \omega \cap b$ and $\bar{x} \in \mathscr{M}^{\mathscr{T}}_m$ such that

$$\pi^{\mathscr{T}}_{m,\omega}(\bar{x}) = \langle \kappa_n \mid n \in \omega \cap b \rangle$$

and then

$$crit(\pi^{\mathscr{T}}_{m,\omega}) = \pi^{\mathscr{T}}_{m,\omega}(\bar{x})(m) \in ran(\pi^{\mathscr{T}}_{m,\omega})$$

This is a contradiction so $\langle \kappa_n \mid n \in \omega \cap b \rangle \notin \mathscr{M}^{\mathscr{T}}_{\omega}$.

Below we will get a contradiction to our assumption that $lh(\mathscr{T}) \geq \omega + 1$ by verifying that $\langle \kappa_n \mid n \in \omega \cap b \rangle \in \mathscr{M}^{\mathscr{T}}_{\omega}$.

If $lh(\mathscr{T}) = \omega + 1$, the sequence $\langle \kappa_n \mid n \in b \cap \omega \rangle \in M = \mathscr{M}^{\mathscr{T}}_{\infty}$ because M is κ-closed and we are done in this case.

Subclaim 2.10 *Suppose $\mathscr{T}$ is not finite and has length $> \omega + 1$. Then $\langle \kappa_n \mid n \in \omega \cap b \rangle \in \mathscr{M}^{\mathscr{T}}_{\omega}$*

Proof The normality of $\mathscr{T}$ implies the following,

$$sup_{n\in\omega}\kappa_n \leq \sup\{\nu^{\mathscr{T}}_m \mid m+1 \in (\omega \cap b)\} \overset{(*)}{\leq} \nu^{\mathscr{T}}_{\omega} \tag{4}$$

By the agreement of the models along a normal iteration tree and acceptability it will follow that $\nu^{\mathscr{T}}_{\omega}$ is a successor cardinal in $\mathscr{M}^{\mathscr{T}}_{\omega+1}$ and $\sup\{\nu^{\mathscr{T}}_m \mid m+1 \in (\omega \cap b)\}$ is a limit cardinal in $\mathscr{M}^{\mathscr{T}}_{\omega+1}$, then the strict inequality holds in (*) (4). Write $\nu^* := \sup\{\nu^{\mathscr{T}}_m \mid m+1 \in (\omega \cap b)\}$, then

$$\{\kappa_n \mid n \in \omega\} \subseteq sup_{n\in\omega}\kappa_n < (\nu^*)^{+\mathscr{M}^{\mathscr{T}}_{\omega+1}} \leq \nu^{\mathscr{T}}_{\omega} \tag{5}$$

Then

$$\mathscr{M}^{\mathscr{T}}_{\omega+1}|\nu^{\mathscr{T}}_{\omega+1} \models sup_{n\in\omega}\kappa_n < (\nu^*)^{+} \leq \nu^{\mathscr{T}}_{\omega} \tag{6}$$

Since

$$\mathscr{M}^{\mathscr{T}}_{\omega+1}|\nu^{\mathscr{T}}_{\omega+1} = \mathscr{M}_{\infty}|\nu^{\mathscr{T}}_{\omega+1} = \mathscr{M}^{\mathscr{T}}_{\omega}|\nu^{\mathscr{T}}_{\omega},$$

by (6) and $\langle \kappa_n \mid n \in b \cap \omega \rangle \in \mathscr{M}^{\mathscr{T}}_{\infty}$ we have $\langle \kappa_n \mid n \in b \cap \omega \rangle \in \mathscr{M}^{\mathscr{T}}_{\infty}|\nu^{\mathscr{T}}_{\omega}$. We also have

$$\mathscr{M}^{\mathscr{T}}_{\omega+1}|\nu^{\mathscr{T}}_{\omega} = \mathscr{M}_{\infty}|\nu^{\mathscr{T}}_{\omega} = \mathscr{M}^{\mathscr{T}}_{\omega}|\nu^{\mathscr{T}}_{\omega}$$

thus

$$\langle \kappa_n \mid n \in \omega \cap b \rangle \in \mathscr{M}^{\mathscr{T}}_{\omega} \tag{7}$$

Lemma 2.11 *Assume (Δ). Suppose that $o(\kappa) \leq \lambda$ for some cardinal λ and $\{\beta < \kappa \mid o(\beta) > \lambda\}$ is bounded in κ. Let*

$$\mu = sup\{\ \vartheta < \kappa \mid O(\vartheta) > \lambda\ \}. \tag{8}$$

and

$$\Theta = sup\{\ O(\alpha) + 1 \mid \mu < \alpha \leq \kappa\ \} \leq \lambda \tag{9}$$

Then there is no $\eta \in (\kappa, \Theta]$ such that $O(\eta) > \Theta$.

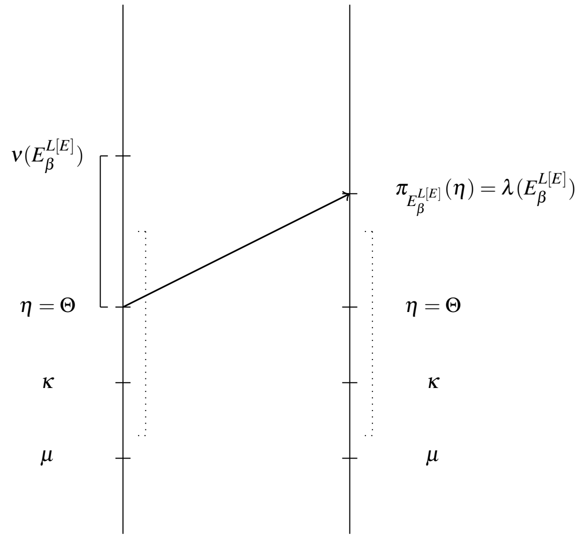

Figure 1 Case 1

Proof Suppose otherwise. Let $E_\beta^{L[E]}$ be such that $\beta > \Theta$ and $\eta := crit(E_\beta^{L[E]}) \in (\mu, \Theta]$ and denote by $\mathscr{M}^*$ the largest initial segment of $L[E]$ where we can apply $E_\beta^{L[E]}$.

Case 1: $\eta = \Theta$. In this case $\eta = \Theta$ is a limit ordinal. Then by Σ_0-elementarity it follows that $\pi(\Theta)$ is a limit ordinal and a limit of indexes of extenders with critical points in the interval $(\mu, \kappa]$, more precisely:

$$L[E] \models \underbrace{\forall\gamma < \Theta(\exists\gamma' < \Theta(\gamma < \gamma' \;\& E_{\gamma'}^{L[E]} \neq \emptyset \;\& crit(E_\gamma^{L[E]}) \in (\mu, \kappa]))}_{\varphi(\Theta,\mu,\kappa)},$$

and

$$\mathscr{M}^* \models \underbrace{\forall\gamma < \Theta(\exists\gamma' < \Theta(\gamma < \gamma' \;\& E_{\gamma'}^{L[E]} \neq \emptyset \;\& crit(E_\gamma^{L[E]}) \in (\mu, \kappa]))}_{\varphi(\Theta,\mu,\kappa)},$$

then

$$Ult_0(\mathscr{M}^*, E_\beta^{L[E]}) \models \varphi(\pi_{E_\beta^{L[E]}}(\Theta), \underbrace{\mu,}_{=\pi_{E_\beta^{L[E]}}(\mu)}, \underbrace{\kappa}_{=\pi_{E_\beta^{L[E]}}(\kappa)}).$$

Using coherence of $E_\beta^{L[E]}$ we find extenders that contradict the definition of Θ.

Case 2 i) $\eta < \Theta \;\&\; \lambda(E_\beta^{L[E]}) \leq \Theta$. Then

$$\pi_{E_\beta^{L[E]}}(\Theta) \geq \pi_{E_\beta^{L[E]}}(\lambda(E_\beta^{L[E]})) > \nu(E_\beta^{L[E]})$$

and by Σ_0-elementarity there is a $\gamma \in (\nu, \pi_{E_\beta^{L[E]}}(\Theta))$ such that $E_\gamma^{Ult_0(\mathscr{M}^*, E_\beta^{L[E]})} \neq \emptyset$ and

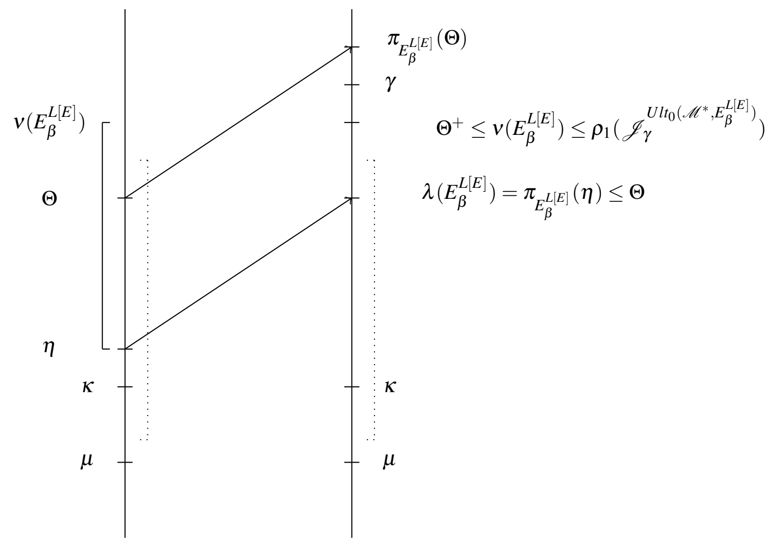

Figure 2 Case 2 i)

$$crit(E_\gamma^{Ult_0(\mathscr{M}^*, E_\beta^{L[E]})}) \in (\mu, \kappa]$$

Since

$$Ult_0(\mathscr{M}^*, E_\beta^{L[E]}) \models \nu(E_\beta^{L[E]}) \text{ is a cardinal} \tag{10}$$

it follows that

$$\rho_1(\mathscr{J}_\gamma^{Ult_0(\mathscr{M}^*, E_\beta^{L[E]})}) \geq \nu(E_\beta^{L[E]}) \tag{11}$$

for any $\gamma \in (\nu(E_\beta^{L[E]}), \pi_{E_\beta^{L[E]}}(\Theta))$. Since

$$\rho_1(\mathscr{J}_\gamma^{Ult_0(\mathscr{M}^*, E_\beta^{L[E]})}) \geq \nu(E_\beta^{L[E]}) \geq \Theta^{+Ult_0(\mathscr{M}^*, E_\beta^{L[E]})} > \Theta$$

it follows by lemma 2.1 that there is $\gamma' \in (\Theta, \Theta^{+Ult_0(\mathscr{M}^*, E_\beta^{L[E]})})$ such that

$$E_{\gamma'}^{Ult_0(\mathscr{M}^*, E_\beta^{L[E]})} \neq \emptyset$$

and

$$crit(E_{\gamma'}^{Ult_0(\mathscr{M}^*, E_\beta^{L[E]})}) = crit(E_\gamma^{Ult_0(\mathscr{M}^*, E_\beta^{L[E]})}) \tag{12}$$

As $\Theta^{+Ult_0(\mathscr{M}^*, E_\beta^{L[E]})} \leq \nu(E_\beta^{L[E]})$ this extender is in the $L[E]$ sequence by coherence, which contradicts the definition of Θ.

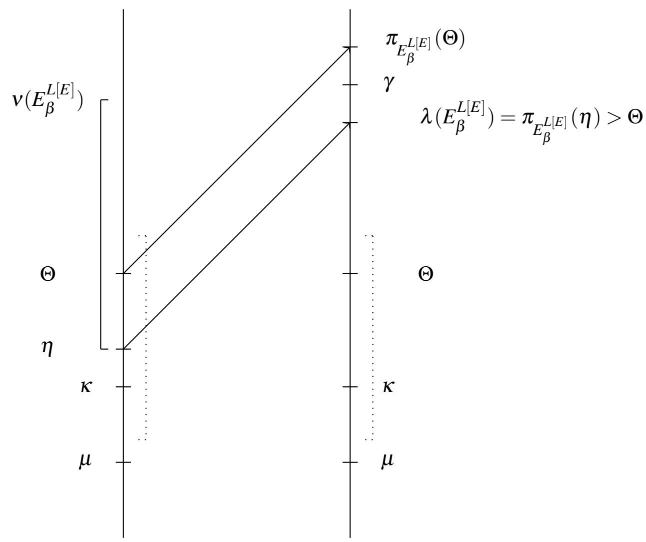

Figure 3 Case 2 ii)

Case 2 ii) $\lambda(E_\beta^{L[E]}) > \Theta$: In this case $\pi_{E_\beta^{L[E]}}(\Theta) > \lambda(E_\beta^{L[E]}) > \Theta$. Then for all γ such that

$$\gamma \in (\lambda(E_\beta^{L[E]}), \pi_{E_\beta^{L[E]}}(\Theta))$$

we have

$$\rho_1(\mathscr{J}_\gamma^{Ult_0(\mathscr{M}^*, E_\beta^{L[E]})}) \geq \lambda(E_\beta^{L[E]}) > \Theta \tag{13}$$

Then as in case 2i) we can find an extender in the $L[E]$ sequence indexed between $(\Theta, \lambda(E_\beta^{L[E]}))$ contradicting the definition of Θ.

Definition 2.12 Let μ be an ordinal and κ be a cardinal, κ is a μ-strong cardinal if and only if there is an elementary embedding $j : V \longrightarrow M$ such that $crit(j) = \kappa$ and $V_\mu \subseteq M$.

The next proposition will be used later but it also serves as a warm up for theorem 1.5.

Proposition 2.13 *Assume* (Δ). *If* μ *is a cardinal and*

$$L[E] \models \kappa \textit{ is } \mu\textit{-strong}$$

then

$$L[E] = \mathscr{K} \models o(\kappa) > \mu$$

Proof Let $j : L[E] \to M$ witness that κ is μ-strong. Because $cf(\mu) > \kappa$ we can assume that $M^\kappa \subseteq M$ and we can apply theorem 2.3 and consider $\mathscr{T}$ the tree induced by j. Let us verify that $\nu_0^{\mathscr{T}} > \mu$. Suppose not then $\nu_0^{\mathscr{T}} \leq \mu$ and since we do not index extenders on cardinals it follows that $\nu_0 < \mu$. We have

$$(L[E]|\mu) \models cf(\nu_0) \leq (crit(E_0^{\mathscr{T}})^{+L[E]|\nu_0}) < \nu_0 < \mu$$

while

$$\mathscr{M}_{lh(\mathscr{T})-1}^{\mathscr{T}} = M \models \nu_0^{\mathscr{T}} \text{ is regular cardinal.}$$

which contradicts

$$M|\mu = \mathscr{M}_{lh(\mathscr{T})-1}^{\mathscr{T}}|\mu \supseteq V_\mu^{L[E]} \supseteq L[E]|\mu.$$

Then $\nu_0^{\mathscr{T}} > \mu$. Let $E_\alpha^{\mathscr{T}}$ be the first extender applied in $\mathscr{M}_0^{\mathscr{T}}$ that leads to the second model on the main branch. If $\alpha = 0$, then $crit(E_0^{\mathscr{T}}) = \kappa$ which implies $o(\kappa) > \mu$. Suppose $\alpha > 0$, then we have $\nu_\alpha^{\mathscr{T}} > \nu_0^{\mathscr{T}}$ and $crit(E_\alpha^{\mathscr{T}}) = \kappa$. We have that

$$\mathscr{M}_\alpha^{\mathscr{T}} \models \nu_0^{\mathscr{T}} \text{ is as successor cardinal}$$

thus for all $\gamma \in (\nu_0^{\mathscr{T}}, \alpha)$ we have

$$\rho_1(\mathscr{M}_\alpha^{\mathscr{T}}|\gamma) \geq \nu_0^{\mathscr{T}}$$

Hence by lemma 2.1 it follows that there is $\widetilde{\gamma} \in (\mu, \nu_0^{\mathscr{T}})$ such that $E_{\widetilde{\gamma}}^{\mathscr{M}_\alpha^{\mathscr{T}}} \neq \emptyset$ and $crit(E_{\widetilde{\gamma}}^{\mathscr{M}_\alpha^{\mathscr{T}}}) = \kappa$. Since

$$\mathscr{M}_\alpha^{\mathscr{T}}|\nu_0^{\mathscr{T}} = \mathscr{M}_0^{\mathscr{T}}|\nu_0^{\mathscr{T}}$$

it follows that $E_{\widetilde{\gamma}}^{\mathscr{M}_\alpha^{\mathscr{T}}} = E_{\widetilde{\gamma}}^{\mathscr{M}_0^{\mathscr{T}}}$ which implies that $o(\kappa) > \mu$.

For the moreover part, notice that if μ

Definition 2.14 Let $\mathscr{T}$ be iteration tree on a mouse $\mathscr{M}$ of length $< \mu$ and Θ an ordinal such that $\Theta \leq \mu$. We say that θ *tracks* Θ *along* $\mathscr{T}$ *if and only if* θ *is a partial function* $\theta : \mathscr{T} \to OR$ *such that* $\theta_0 = \Theta$*, and if* $\alpha \in \mathscr{T}$ *is a successor ordinal and* θ_β *is defined for all* $\beta \in [0, \eta]_T$ *where* $\eta := pred_T(\alpha)$ *and* $\theta_\eta \in (\mathscr{M}_\alpha^{\mathscr{T}})^*$ *we define* $\theta_\alpha = (\pi_\alpha^{\mathscr{T}})_\alpha^*(\theta_\eta)$ *where* $(\pi_\alpha^{\mathscr{T}})^* : (\mathscr{M}_\alpha^{\mathscr{T}})^* \to Ult_{n_\alpha}((\mathscr{M}_\alpha^{\mathscr{T}})^*, E_\alpha^{\mathscr{T}})$*, otherwise we left it undefined for the successor case. If* α *is a limit ordinal and* θ_β *is defined for all* $\beta \in [0, \alpha]_T$ *then we define* $\theta_\alpha = \pi_{\beta,\alpha}^{\mathscr{T}}(\theta_\beta)$ *where* β *is the least ordinal such that* $\pi_{\beta,\alpha}^{\mathscr{T}}$ *is defined, otherwise we let* θ_α *undefined.*

Lemma 2.15 *Let* μ *be a regular cardinal and* $\kappa < \mu$ *an ordinal. Suppose that if* $\mu = \lambda^+$ *for some singular cardinal* λ *then* κ, μ *and* λ *satisfy:*

(*) If $cf(\lambda) \geq \kappa$ *then* $cf(\lambda)$ *is not the critical point of a total measure on* $L[E]$ *indexed on* E*.*

Then for any $\Theta \leq \mu$ *and* $\theta : \mathscr{T} \to OR$ *that tracks* Θ *along* $\mathscr{T}$ *where* $\mathscr{T}$ *is an iteration tree on* $L[E]|\mu$ *such that*

1. $lh(\mathscr{T}) < \mu$,

2. the critical point of all extenders used on $\mathscr{T}$ are $\geq \kappa$

the following hold:

1. (Proposition 4.8 in [10]) $\mathscr{M}_\alpha^{\mathscr{T}} \cap OR \leq \mu$ for all $\alpha < lh(\mathscr{T})$ and

2. for all $\beta \in T$ such that θ_β is defined, it follows that $\theta_\beta \leq \mu$.

Proof The condition (a) is proposition 4.8 in [10], we only verify condition (b). If $\theta_0 = \Theta < \mu$ then (b) follows from (a), so we only need to verify it for the case $\theta_0 = \mu$, which is equivalent to verify that $\theta_\beta = \mu$ whenever θ_β is defined. Using (a) all we need to verify is that for every $\beta \in \mathscr{T}$ if θ_β is defined then $(\pi_\beta^{\mathscr{T}})^*$ is continuous at $\mu = \theta_\eta$ where $\eta = pred_T(\beta)$. We will verify it by induction.

Let $\beta = 0$, $\theta_0 = \mu$ is defined, suppose that $\theta_1 = (\pi_1^{\mathscr{T}})^*(\theta_0)$ is also defined. Since μ is a cardinal in $L[E]$, $\mathscr{T}$ lives on $L[E]|\mu$ and $\mu \in \mathscr{M}_1^*$, it follows that $E_0^{\mathscr{T}}$ is a total extender on $L[E]$. Let

$$[a,f]_{E_0^{\mathscr{T}}} \in \pi_{0,1}^{\mathscr{T}}(\mu)$$

since $crit(E_0^{\mathscr{T}}) < \mu$ and μ is a regular cardinal, we have that $ran(f)$ is bounded in μ and thus there is a $\xi < \mu$ such that

$$[a,f]_{E_0^{\mathscr{T}}} \in \pi_{0,1}^{\mathscr{T}}(\xi)$$

Hence

$$\pi_{0,1}^{\mathscr{T}}(\mu) = sup_{\xi\in\mu}\pi_{0,1}^{\mathscr{T}}(\xi)$$

By lemma 2.15

$$sup_{\xi\in\mu}\pi_{0,1}^{\mathscr{T}}(\xi) \leq \mu$$

which gives

$$\pi_{0,1}^{\mathscr{T}}(\mu) = \mu.$$

The general successor step $\beta+1$ is similar to 0 case, we carry the induction hypothesis that whenever θ_β is defined then $D^{\mathscr{T}} \cap [0,\beta]_T = \emptyset$. If $\theta_{\beta+1}$ is defined, then θ_{β^*} is defined and $\mathscr{M}_{\beta^*}^{\mathscr{T}}$ is a class sized model by the induction hypothesis. Since we are assuming we are in the hypothesis of lemma 2.15 we have $\nu_\beta^{\mathscr{T}} \in \mu$ and then $crit(E_\beta^{\mathscr{T}}) < \mu$. Since $\mu = \theta_{\beta^*} \in (\mathscr{M}_{\beta+1}^{\mathscr{T}})^*$, $\mathscr{M}_{\beta^*}^{\mathscr{T}}$ is class sized and μ is a cardinal, it follows that $\beta+1 \notin D^{\mathscr{T}}$. Now the computations for the case $\beta+1$ can be done exactly as in case 0.

If $\beta \in \mathscr{T}$ is a limit ordinal and θ_β is defined, then given

$$x \in \theta_\beta$$

there are $\overline{\beta} \in [0,\beta)_T$ and $\overline{x} \in \mathscr{M}_{\overline{\beta}}^{\mathscr{T}}$ such that

$$\pi_{\overline{\beta},\beta}^{\mathscr{T}}(\overline{x}) = x \in \theta_\beta = \pi_{\overline{\beta},\beta}^{\mathscr{T}}(\theta_{\overline{\beta}})$$

let $\xi \in (\overline{x}, \theta_{\overline{\beta}}) = (\overline{x}, \mu)$ then

$$\pi_{\overline{\beta},\beta}^{\mathscr{T}}(\overline{x}) < \pi_{\overline{\beta},\beta}^{\mathscr{T}}(\xi) < \pi_{\overline{\beta},\beta}^{\mathscr{T}}(\theta_{\overline{\beta}}) = \theta_\beta$$

and by lemma 2.15

$$sup_{\xi\in\mu}\pi_{\overline{\beta},\beta}^{\mathscr{T}}(\xi) \leq \mu.$$

Hence

$$\theta_\beta = \mu$$

□

Remark 2.16 For example $L[E]|\lambda^{++}$ always satisfies the hypothesis of lemma 2.15.

Lemma 2.17 (lemma 2.3 in [9]) *If $V = L[E]$ and there is $j : L[E] \longrightarrow M$ where ${}^{\kappa}M \subseteq M$ and $j(\kappa) \geq \theta$, then*

$$L[E] \models \kappa \text{ is } \theta\text{-tall.}$$

Proof sketch By elementarity of j it follows that $j(\kappa)$ is measurable in M, let $U \in M$ be a total measure on M with $crit(U) = j(\kappa)$ and i the ultrapower embedding from U, then $i \circ j$ witness that κ is θ-tall. □

Proposition 2.18 (theorem 2.10 in [9]) *If μ is a cardinal, $cf(\mu) > \kappa$ and $L[E] \models o(\kappa) > \mu$ then $L[E] \models \kappa$ is μ-tall.*

Proposition 2.19 (corollary 2.7 in [9]) *If $\mu > \kappa$ is a cardinal, $cf(\mu) > \kappa$ and*

$$L[E] \models \kappa \text{ is a measurable cardinal } \& \ sup\{\alpha < \kappa \mid o(\alpha) > \mu\} = \kappa,$$

then $L[E] \models \kappa$ is μ-tall. Moreover

$$L[E] \models \kappa \text{ is a measurable cardinal } \& \ sup\{\alpha < \kappa \mid \alpha \text{ is a strong cardinal }\} = \kappa,$$

implies

$$L[E] \models \kappa \text{ is a tall cardinal}$$

Remark 2.20 The fact that the following lemma holds under much weaker hypothesis is a theorem due Schlutzenberg obtained in his Ph.d thesis [13]. Here we are working with the hypothesis that (Δ) holds which makes it easy to verify the lemma, so we provide a proof for this case.

Lemma 2.21 *Assume (Δ). Suppose $L[E] \models \kappa$ is a measurable cardinal, then there is an ordinal β such that E_β is a total measure on $L[E]$ with critical point κ.*

Proof Let $j : L[E] \longrightarrow Ult(L[E], U)$ where U is a total measure on κ. Then ${}^{\kappa}Ult(L[E], U) \subseteq Ult(L[E], U)$ and theorem 2.3 and lemma 2.6 and we obtain that $j = i^{\mathscr{T}}$ for some iteration tree on $L[E]$ such that there is on drop along the main branch b of $\mathscr{T}$. Let us verify the following:

$$\forall \alpha \in lh(\mathscr{T})(crit(E_\alpha^{\mathscr{T}}) \geq \kappa) \tag{14}$$

Suppose for a contradiction that $crit(E_\alpha^{\mathscr{T}}) < \kappa$: We have from the continuity of $\pi_{E_\alpha^{\mathscr{T}}}$ at $crit(E_\alpha^{\mathscr{T}})^{+\mathscr{J}_{\nu_\alpha^{\mathscr{T}}}^{\mathscr{M}_\alpha^{\mathscr{T}}}}$ that

$$(\kappa \geq crit(E_\alpha^{\mathscr{T}})^{+\mathscr{J}_{\nu_\alpha^{\mathscr{T}})}^{\mathscr{M}_\alpha^{\mathscr{T}}}} \geq cf(\nu_\alpha^{\mathscr{T}})^{\mathscr{M}_\alpha^{\mathscr{T}}})^{L[E]}$$

and

$$\forall \beta > \alpha(\mathscr{M}_\beta^{\mathscr{T}} \models \nu_\beta^{\mathscr{T}} \text{ is a regular cardinal}).$$

in particular

$$M = \mathscr{M}^{\mathscr{T}}_{lh(\mathscr{T})-1} \models \nu^{\mathscr{T}}_{\beta} \text{ is a regular cardinal.}$$

Then

$$(\mathscr{M}^{\mathscr{T}}_{\infty})^{\kappa} \not\subseteq \mathscr{M}^{\mathscr{T}}_{\infty} = M,$$

which is a contradiction. Thus (14) holds.

Since $crit(i^{\mathscr{T}}) = \kappa$ and $\mathscr{T}$ is normal it follows that $E^{\mathscr{T}}_{\alpha}$ the first extender used on the main branch has critical point κ. If $\alpha = 0$ by lemma 2.1 $(E^{\mathscr{T}}_0|\kappa^{+L[E]})^*$ is indexed on $L[E]$ and we are done, so suppose we are in the case $\alpha > 0$. Since there is no drop along b and

$$\mathscr{M}^{\mathscr{T}}_{\alpha} \models \kappa^+ < \nu^{\mathscr{T}}_0$$

it follows that $\nu^{\mathscr{T}}_0 > \kappa^{+L[E]}$, as otherwise

$$\kappa^{+\mathscr{M}^{\mathscr{T}}_{\alpha}} < \kappa^{+L[E]}$$

which would imply that $\alpha \in D^{\mathscr{T}}$.

Let $\beta \in \mathscr{T}$ be the least extender used on the branch $[0,\alpha]_{\mathscr{T}}$. Since $\nu^{\mathscr{T}}_{\beta} \geq \nu_0(E^{\mathscr{T}}_0) > \kappa^+$, it follows that $\rho_1((\mathscr{M}^{\mathscr{T}}_{\beta+1})^*) \geq \kappa^+$. Using (14) we have by an easy induction that $\rho_1(\mathscr{M}^{\mathscr{T}}_{\alpha}) > \kappa^{+L[E]}$. Then by lemma 2.1 we can find $E^{\mathscr{M}^{\mathscr{T}}_{\alpha}}_{\gamma}$ with $crit(E^{\mathscr{M}^{\mathscr{T}}_{\alpha}}_{\gamma}) = \kappa$ and $\gamma \in (\kappa^{+L[E]}, \rho_1(\mathscr{M}^{\mathscr{T}}_{\alpha}))$. Since $\mathscr{T}$ is normal, $E^{\mathscr{M}^{\mathscr{T}}_{\alpha}}_{\gamma} = E^{L[E]}_{\gamma}$ and $(E_{\gamma}|\kappa^{+L[E]})^*$ is indexed in $L[E]$. □

Theorem 2.22 *1.1.4 Assume* (Δ). *Suppose that* μ *is a regular cardinal and if* $\mu = \lambda^+$ *for some singular cardinal* λ *and* $cf(\lambda) \geq \kappa$ *then* $cf(\lambda)$ *is not the critical point of a total measure on* $L[E]$ *indexed on* E. *Then* $L[E] \models \kappa$ *is* μ*-tall if and only if*

$$L[E] \models (o(\kappa) > \mu) \vee (\kappa \text{ is a measurable and } \kappa = sup\{\alpha < \kappa \mid o(\alpha) > \mu\}).$$

Proof ($\Leftarrow$) It follows from 2.18 and 2.19.
($\Rightarrow$) We have that κ is measurable and by 2.21 there is a total measure indexed on E with index κ.

Suppose that $o(\kappa) \leq \mu$ and $\kappa > sup(\{\alpha < \kappa \mid o(\alpha) > \mu\})$. Then if $\beta^* := sup(\{\beta < \kappa \mid o(\beta) > \mu\}) < \kappa$ by 2.5

$$\beta^* := sup(\{\beta < \kappa \mid O(\beta) > \mu\})$$

and

$$sup(\{O(\beta) \mid \beta^* < \beta < \kappa\}) =: \Theta \leq \mu$$

Let $j : V \to M$ witness the μ-tallness of κ and by 2.3 we can consider $\mathscr{T}$ the iteration induced by j on $\mathscr{K}$ and by 2.6 $\mathscr{K} = L[E] = M^{\mathscr{T}}_0$. Notice that $\mathscr{M}^{\mathscr{T}}_{\infty} = \mathscr{K}^M = M$ and $\pi^{\mathscr{T}}_{0,lh(\mathscr{T})-1} = j$. Let us verify the following:

$$\forall n \in lh(\mathscr{T})(crit(E^{\mathscr{T}}_n) \geq \kappa) \tag{15}$$

Suppose for a contradiction that $crit(E_n^{\mathscr{T}}) < \kappa$: We have from the continuity of $\pi_{E_n^{\mathscr{T}}}$ at $crit(E_n^{\mathscr{T}})^{+\mathscr{J}_{\nu_n^{\mathscr{T}}}^{\mathscr{M}_n^{\mathscr{T}}}}$ that

$$(\kappa \geq crit(E_n^{\mathscr{T}})^{+\mathscr{J}_{\nu_n^{\mathscr{T}}}^{\mathscr{M}_n^{\mathscr{T}}}} \geq cf(\nu_n^{\mathscr{T}})^{\mathscr{M}_n^{\mathscr{T}}})^{L[E]}$$

and

$$\forall m > n(\mathscr{M}_m^{\mathscr{T}} \models \nu_n^{\mathscr{T}} \text{ is a regular cardinal}).$$

in particular

$$M = \mathscr{M}_{lh(\mathscr{T})-1}^{\mathscr{T}} \models \nu_n^{\mathscr{T}} \text{ is a regular cardinal}.$$

Then

$$(\mathscr{M}_\infty^{\mathscr{T}})^\kappa \not\subseteq \mathscr{M}_\infty^{\mathscr{T}} = M,$$

which is a contradiction. Thus (15) holds.

Consider a partial function $\theta : \mathscr{T} \to OR$ defined as in lemma 2.15 with $\theta_0 = \Theta$, by 2.9 $\mathscr{T}$ is finite, and by (15) all extenders have critical point $\geq \kappa$ so if we manage to prove that $\mathscr{T}$ lives in $L[E]|\mu$ this will imply by lemma 2.15 that

$$j(\kappa) = \pi_{0,lh(\mathscr{T})-1}^{\mathscr{T}}(\kappa) \leq \theta_{lh(\mathscr{T})-1} \leq \mu$$

and we will reach a contradiction.

Let b be the main branch of $\mathscr{T}$. We know by theorem 2.3 that there is no drop along b so we can define:

$$t_0 = min\Big(\{m \in b \mid \pi_{m,lh(\mathscr{T})-1}^{\mathscr{T}}(\pi_{0,m}^{\mathscr{T}}(\kappa)) = \pi_{0,m}(\kappa)\} \cup \{lh(\mathscr{T})-1\}\Big)$$

Note t_0 is the least ordinal in b where $\pi_{0,t_0}(\kappa) = \pi_{0,m}$ for all $m \geq t_0$. Now we start verifying that $\mathscr{T}|(t_0+1)$ lives in $L[E]|\mu$.

Claim 2.23 *For all $n \in \mathscr{T}|(t_0+1)$, $\nu_n^{\mathscr{T}|(t_0+1)} \leq \theta_n$ and $\theta_n \leq \mu$ or θ_n is not defined (so $\mathscr{T}|(t_0+1)$ lives in $L[E]|\mu$)*

Proof We prove this by induction on $n \leq t_0$, and to simplify notation we assume $t_0 = lh(\mathscr{T})-1$ and we write $\mathscr{T}$ instead of $\mathscr{T}|(t_0+1)$.

For **n=0** we will verify that it is not the case that $\nu_0^{\mathscr{T}} > \theta$ & $crit(E_0^{\mathscr{T}}) > \theta_0$. We will check by induction that

$$\forall k > 0 \ \mathscr{M}_k^{\mathscr{T}} \models \nexists \gamma > \theta_0 \Big(E_\gamma^{\mathscr{M}_k^{\mathscr{T}}} \neq \emptyset \ \& \ crit(E_\gamma^{\mathscr{M}_k^{\mathscr{T}}}) \in (\mu, \theta_0)\Big) \tag{16}$$

Before starting the induction, note that from (16) plus (15) and normality we will get that $crit(E_k^{\mathscr{T}}) > \theta_0$ for all $k \in lh(\mathscr{T})$ so θ_0 and κ will be fixed by the iteration maps. This will be a contradiction, because this implies

$$\pi_{0,\infty}^{\mathscr{T}}(\kappa) = \kappa \leq \mu.$$

Suppose it holds for k and let us verify it for $k+1$. By the normality of $\mathscr{T}$ we have $\nu_k^{\mathscr{T}} > \nu_0^{\mathscr{T}} > \theta_0$. By induction the following holds for k :

$$\mathscr{M}_k^{\mathscr{T}} \models \nexists \gamma > \theta_0 (E_\gamma^{\mathscr{M}_k^{\mathscr{T}}} \neq \emptyset \ \& \ crit(E_\gamma^{\mathscr{M}_k^{\mathscr{T}}}) \in (\mu, \theta_0)).$$

Then $crit(E_k^{\mathscr{T}}) > \theta_0$ or $crit(E_k^{\mathscr{T}}) \leq \mu$, but the last case is ruled out by (15) so $crit(E_k^{\mathscr{T}}) > \theta_0$. We have $\eta_{k+1}^{\mathscr{T}} > crit(E_k^{\mathscr{T}})$ and by induction hypothesis

$$\mathscr{M}_{(k+1)^*}^{\mathscr{T}} \models \nexists \gamma > \theta_0 \left(E_\gamma^{\mathscr{M}_{(k+1)^*}^{\mathscr{T}}} \neq \emptyset \ \&\ crit(E_\gamma^{\mathscr{M}_{(k+1)^*}^{\mathscr{T}}}) \in (\mu, \theta_0) \right)$$

so the following holds:

$$(\mathscr{M}_{k+1}^{\mathscr{T}})^* \models \nexists \gamma > \theta_0 \left(E_\gamma^{(\mathscr{M}_{k+1}^{\mathscr{T}})^*} \neq \emptyset \ \&\ crit(E_\gamma^{(\mathscr{M}_{k+1}^{\mathscr{T}})^*}) \in (\mu, \theta_0) \right)$$

Thus by Σ_1-elementarity we have

$$\mathscr{M}_{k+1}^{\mathscr{T}} \models \nexists \gamma > \theta_0 \left(E_\gamma^{\mathscr{M}_{k+1}^{\mathscr{T}}} \neq \emptyset \ \&\ crit(E_\gamma^{\mathscr{M}_{k+1}^{\mathscr{T}}}) \in (\mu, \theta_0) \right)$$

and this verifies.

For $n = 0$ claim lemma 2.11 excludes the case $\nu_0^{\mathscr{T}} > \theta$ and $\kappa_0 \in [\mu, \theta]$. Then, we must have

$$\nu_0^{\mathscr{T}} \leq \Theta = \theta_0 \leq \mu$$

So $\mathscr{T}|1$ lives in $L[E]|\mu$. If $\pi_{0,1}(\theta_0)$ is defined, we have by lemma 2.15 that

$$\theta_1 = \pi_{0,1}^{\mathscr{T}}(\theta_0) \leq \mu$$

This verifies case $n = 0$.

Case $\mathbf{n = k+1}$, a): θ_{k+1} is not defined. Then $\mathscr{T}|(k+2)$ lives in $L[E]|\mu$ since by induction hypothesis $\mathscr{T}|(k+1)$ and $\mathscr{T}|((k+2)^*+1)$ live in $L[E]|\mu$ and $\theta_{(k+1)^*}$ is not defined or $\eta_{k+1}^{\mathscr{T}} < \theta_{(k+1)^*}$.

Case $\mathbf{n = k+1}$, b): θ_{k+1} is defined. Suppose $\nu_{k+1}^{\mathscr{T}} > \theta_{k+1}$. Using claim *lemma*2.11 we can verify by induction that [3]

$$\mathscr{M}_{k+1}^{\mathscr{T}} \models \nexists \nu > \theta_{k+1} \ (E_\nu^{\mathscr{M}_{k+1}^{\mathscr{T}}} \neq \emptyset \ \&\ crit(E_\nu^{\mathscr{M}_k^{\mathscr{T}}}) \in (\mu, \theta_{k+1})). \tag{17}$$

We have by induction that $\mathscr{T}|(k+1)$ lives in $L[E]|\mu$ then by lemma 2.15

$$(\pi_{(k+1)}^{\mathscr{T}})^*(\theta_{(k+1)^*}) = \theta_{k+1} \leq \mu \tag{18}$$

We will verify that

$$\nu_k^{\mathscr{T}} < \theta_{k+1}^{\mathscr{T}}. \tag{19}$$

We have by induction that

$$\nu_{(k+1)^*}^{\mathscr{T}} \leq \theta_{(k+1)^*}$$

and

$$crit(E_k^{\mathscr{T}}) < crit(E_k^{\mathscr{T}})^{+\mathscr{M}_{(k+1)^*}^{\mathscr{T}}} < \nu_{(k+1)^*}^{\mathscr{T}} < \theta_{(k+1)^*}$$

which implies the following desired inequality:

$$\begin{gathered}\theta_{k+1} = \pi_{(k+1)^*,k+1}^{\mathscr{T}}(\theta_{(k+1)^*}) \\ \geq \\ \pi_{(k+1)^*,k+1}^{\mathscr{T}}(\nu_{(k+1)^*}^{\mathscr{T}}) > \pi_{(k+1)^*,k+1}^{\mathscr{T}}(crit(E_k^{\mathscr{T}})^{+\mathscr{M}_{k+1}^{\mathscr{T}}}) = \nu_k^{\mathscr{T}}.\end{gathered}$$

We will verify by induction, similar to what we did in case n=0, that $k+1<\gamma$ implies :

$$k+1 <_{\mathscr{T}} \gamma \\ \& \\ \mathscr{M}_\gamma^{\mathscr{T}} \models \nexists \gamma > \theta_{k+1}\ (E_\gamma^{\mathscr{M}_\gamma^{\mathscr{T}}} \neq \emptyset \ \&\ crit(E_\gamma) \in (\mu, \theta_{k+1})) \tag{20}$$

Suppose this holds for all l such that $k+1<l\leq m$, let us verify it for $m+1$:

By normality $\nu_m^{\mathscr{T}} > \nu_0^{\mathscr{T}}$. By induction hypothesis or 17 we have only two cases: $crit(E_m^{\mathscr{T}}) \leq \mu$ or $crit(E_m^{\mathscr{T}}) > \theta_{k+1}$. The first case is excluded by (15). Let us deal with the case $crit(E_m^{\mathscr{T}}) > \theta_{k+1}$. By (19)

$$crit(E_m^{\mathscr{T}}) > \theta_{k+1} \ \longrightarrow\ crit(E_m^{\mathscr{T}}) > \nu_k^{\mathscr{T}}$$

which implies that

$$k+1 \leq i = (m+1)^*.$$

Then by induction hypothesis or (17) we have that

$$\mathscr{M}_i^{\mathscr{T}} \models \nexists \gamma > \theta_{k+1}\ (E_\gamma^{\mathscr{M}_i^{\mathscr{T}}} \neq \emptyset \ \&\ crit(E_\gamma^{\mathscr{M}_i^{\mathscr{T}}}) \in (\mu, \theta_{k+1})\ )$$

and consequently

$$(\mathscr{M}_{(m+1)^*})^{\mathscr{T}} \models \nexists \gamma > \theta_{k+1}\ (E_\gamma^{(\mathscr{M}_{(m+1)^*})^{\mathscr{T}}} \neq \emptyset \ \&\ crit(E_\gamma^{(\mathscr{M}_{(m+1)^*})^{\mathscr{T}}}) \in (\mu, \theta_{k+1})\ )$$

and by Σ_1-elementarity we have

$$\mathscr{M}_{m+1}^{\mathscr{T}} \models \nexists \gamma > \theta_{k+1}\ (E_\gamma^{\mathscr{M}_{m+1}^{\mathscr{T}}} \neq \emptyset \ \&\ crit(E_\gamma^{\mathscr{M}_{m+1}^{\mathscr{T}}}) \in (\mu, \theta_{k+1})\ )$$

and since $k+1 \leq_{\mathscr{T}} i$ it follows that

$$k+1 <_{\mathscr{T}} m+1.$$

This concludes the induction and verifies (20).

It follows that $k+1 \in [0, lh(\mathscr{T})-1]_T$, so $\pi_{0,k}^{\mathscr{T}}$ is defined and

$$\kappa < \theta_0 = \Theta \longrightarrow \pi_{0,k+1}^{\mathscr{T}}(\kappa) < \pi_{0,k+1}^{\mathscr{T}}(\theta_0) = \theta_{k+1} \tag{21}$$

By (20)

$$\pi_{k+1,lh(\mathscr{T})-1}^{\mathscr{T}} | \theta_{k+1} = id | \theta_{k+1} \tag{22}$$

Hence (21), (22) and (18) give

$$\pi_{0,lh(\mathscr{T})-1}^{\mathscr{T}}(\kappa) = \pi_{k+1,lh(\mathscr{T})-1}^{\mathscr{T}} \circ \pi_{0,k+1}^{\mathscr{T}}(\kappa) = id \circ \pi_{0,k+1}^{\mathscr{T}}(\kappa) \leq \mu$$

which is a contradiction.

Then it must hold that $\nu_{k+1}^{\mathscr{T}} \in (\mu, \theta_{k+1}]$. So $\mathscr{T}|k+2$ lives in $L[E]|\mu$.

This verifies the case $n=k+1$.

□

From 2.23 it follows that $\pi_{0,\infty}^{\mathscr{T}}(\kappa) \leq \mu$ which is a contradiction. □

Definition 2.24 (Hamkins) A cardinal κ is $<\alpha$-tall if and only if for all $\beta<\alpha$ κ is β-tall.

Corollary 2.25 *Assume* (Δ). *Suppose*

$$L[E]\models \alpha \text{ is a limit cardinal and } cf(\alpha)>\kappa$$

Then

$$L[E]\models \kappa \text{ is } <\alpha\text{-tall}$$

if and only if

$$L[E]\models \big(o(\kappa)\geq\alpha\big)\vee$$
$$\big(\kappa \text{ is a measurable cardinal and } \kappa=sup\{\beta<\kappa \mid o(\beta)\geq\alpha\}\big)$$

Proof ($\Leftarrow$) It follows from 2.18 and 2.19. ($\Rightarrow$) Let $\langle\mu_\xi \mid \xi<cf(\alpha)\rangle$ be a cofinal sequence in α. Note that for $\mu=\mu_\xi^{++}$ we are in hypothesis of theorem 1.5. If for each $\mu_\xi^{++}>\kappa$ we always obtain $o(\kappa)>\mu_\xi^{++}$ then $o(\kappa)\geq\alpha$ and we are done. So suppose this is not the case and $o(\kappa)<\alpha$, then we need to find $B\subseteq\kappa$ such that B is cofinal in κ and for all $\beta\in B$ we have $o(\beta)\geq\alpha$. Fix $\xi<cf(\alpha)$ such that $\mu_\xi^{++}>o(\kappa)$ and apply theorem 1.5 for μ_ξ^{++}. This gives $B\subseteq\kappa$ cofinal such that for all $\beta\in B$ we have $o(\beta)>\mu_\xi^{++}$.

For each $\xi<cf(\alpha)$, let

$$B_\xi=\{\beta<\kappa \mid o(\beta)>\mu_\xi^{++}\}$$

Then $\langle B_\xi \mid \xi<cf(\alpha)\rangle$ is a sequence of cofinal subsets of κ such that

$$\forall\xi\Big(\xi<\zeta<cf(\alpha)\longrightarrow B_\zeta\subsetneq B_\xi\Big)$$

This is a contradiction since $cf(\alpha)>\kappa$.

Corollary 2.26 *1.1.5 Assume* (Δ).

$$L[E]\models \kappa \text{ is tall}$$

if and only if

$$L[E]\models \kappa \text{ is a strong cardinal or a measurable limit of strong cardinals.}$$

Question 2.27 *If* $V\models$ *"κ is λ-tall" and there is no inner model with a Woodin cardinal, does it imply that*

$$\mathscr{K}\models\Big(o(\kappa)>\lambda \vee(\kappa \text{ is measurable and } \kappa=sup\{\mu<\kappa \mid o(\mu)>\lambda\})\Big)$$

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Notes

1. The hypothesis that $\mathscr{P}(\mathbb{R}) \subseteq M$ is not necessary when we assume that there is no inner model with a Woodin cardinal, we could omit it.

2. $\mathscr{K}$ is universal so the $L[E]|\alpha$ side does not drop, but $\rho_\omega(L[E]|\alpha) = \omega$ implies that $L[E]|\alpha$ can not move without dropping, so $L[E]|\alpha$ is an initial segment of the final model on the $\mathscr{K}$ side. It is actually a proper initial segment because otherwise the final model on the $\mathscr{K}$ side is not sound and is equal to $L[E]|\alpha$ which is sound. If some extender is used on the $\mathscr{K}$ side its index is a cardinal in the last model but it is collapsed by $L[E]|\alpha = Hull_\omega^{L[E]|\alpha}(\omega)$, so $\mathscr{K}$ also does not move.

3. We use induction like in case $n = 0$, there may be drops in model along $[0, k+1]_\mathscr{T}$ but by hypothesis θ_m is defined for all $m \in [0, k+1]_\mathscr{T}$

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