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Long games and Woodin cardinals

joint work with Juan Aguilera

We consider "long games". Let $A \subseteq {}^{\omega \cdot \alpha}\omega \approx ({}^{\omega}\omega)^{\alpha}$ for some fixed $\alpha < \omega_1^V$.

I	n_0		n_2	...	n_ω		$n_{\omega+2}$ ...
II		n_1	n_3	...		$n_{\omega+1}$	$n_{\omega+3}$ ...

Player I wins iff $(n_0, n_1, \dots) \in A$, o/w Player II wins.

Rmk: Determinacy for games of length $\omega \cdot (n+1)$ with Π_1^1 payoff implies determinacy for games of length ω with Π_{n+1}^1 payoff.

Idea: We can simulate projections by ω moves (one round) in a game, where we only consider the moves of one of the two players.

Example: Let $A = pB$, $B \subseteq (\omega^\omega)^2$, $B \in \underset{\sim}{\Pi}_1^1$.

$$G^* \quad \begin{array}{c|cccccccc} \mathrm{I} & x_0 & & x_2 & & \dots & y_0 \; y_1 \; y_2 \; \dots \\ \hline \mathrm{II} & & x_1 & & x_3 & \dots & \end{array}$$

II plays something but we don't care what

Player I wins iff $(x,y) \in B$

$$G \quad \begin{array}{c|ccccc} \mathrm{I} & x_0 & & x_2 & & \dots \\ \hline \mathrm{II} & & x_1 & & x_3 & \dots \end{array}$$

Player I wins iff $x \in A$.

Let σ be a w.s. for I in G^*. Then $\sigma \restriction \omega$ (restricted to the first ω moves of the game) is a w.s. for I in G.

Analogous for Player II.

<u>Thm (Neeman)</u>: Let $\alpha > 1$ be a ctble ordinal and suppose that there are $-1+\alpha$ Woodin cardinals with a # above them all. Then all games of length $\omega \cdot \alpha$ with Π^1_1 payoff (in fact even $<\omega^2$-Π^1_1 payoff) are determined.

<u>Note</u>: Woodin showed this earlier for ordinals α of the form $\alpha = \omega \cdot \beta$, where β is additively closed.

<u>Question</u>: Is this result optimal, i.e. can we prove the converse?

<u>Theorem</u>: Suppose games of length $\omega \cdot (\omega+1)$ with Π^1_1 payoff are determined. Then there is a premouse with $\omega+1$ Woodin cardinals

<u>Rmk</u>: In fact, we will consider a game of length $\omega \cdot \omega$ with Π^1_2 payoff.

<u>Note</u>: The same proof works for games of length $\omega \cdot (\omega+n)$ with Π^1_1 payoff and $\omega+n$ Woodin cardinals for every $n \in \omega$.

<u>Idea of the proof</u>: From now on suppose games of length $\omega \cdot (\omega+1)$ with Π^1_1 payoff are determined.

- We first show that there is a model of the form $M_1(A)$ for some $A \in \mathcal{P}_{\omega_1}(\mathbb{R})$ s.t. $M_1(A) \cap \mathbb{R} = A$ and $M_1(A) \models AD$ (and in fact $M_1(A) \models AD^+$).
- Use AD to "generate" ω Woodin cardinals in a generic extension of $M_1(A)$ (with a Prikry-type forcing)
- Use a $\mathcal{P}$-construction to add the remaining Woodin cardinal on top.

<u>Lemma 1</u>: There is a club $\mathcal{C}_1 \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ such that $\mathbb{R} \cap M_1(A) = A$ for all $A \in \mathcal{C}_1$.

We thank John Steel for pointing out to us that a variant of our proof of lemma 2 below shows Lemma 1.

<u>Lemma 2</u>: There is a club $\mathcal{C}_2 \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ such that $M_1(A) \models ZF + AD$ for all $A \in \mathcal{C}_2$.

<u>Proof</u>: Sps. not, i.e. there is a stationary set of $B \in \mathcal{P}_{\omega_1}(\mathbb{R})$ with $M_1(B) \mid \delta_B \not\models AD$.

δ_B: Wdn. card in $M_1(B)$

Now we play the following game G.

I	z a v_0, x_1		v_1, x_3		...	
II		b		x_2		x_4 ...

where

- $z, x_1, x_2, \ldots \in {}^\omega\omega$ (played as sequences of natural numbers)
- $a, b \in {}^\omega\omega$ are obtained by alternating moves of I and II
- $v_i \in \{0,1\}$ are interpreted as truth values of formulae ϕ_i, where $\{\phi_i : i < \omega\}$ is a fixed enumeration of all $\mathcal{L}_{pm}(\{\dot{x}_i : i < \omega\})$ - formulae (s.t. $\dot{x}_i$ does not appear in ϕ_j if $j \leq i$).

$\rightarrow$ Every play determines a complete theory T in the language $\mathcal{L}_{pm}(\{\dot{x}_i : i < \omega\})$.

Player I wins G iff

(1) $x_1 \geq_T a \oplus b$

(2) For each $i \in \omega$, T contains the sentence $\dot{x}_i \in {}^\omega\omega$ and for each $j, m \in \omega$, T contains the sentence $\dot{x}_i(m) = j$ iff $x_i(m) = j$, where we let $x_0 = z$.

(3) Let m and n be fixed maps, mapping each $\mathcal{L}_{pm}(\{\dot{x}_i \;\, i<\omega\})$-formula ϕ to an even natural number m_ϕ (or n_ϕ) such that m and n are recursive, have disjoint ranges, and m_ϕ and n_ϕ are larger than $\max\{i \mid \dot{x}_i \text{ occurs in } \phi\}$.

For every formula $\phi(x)$ with one free variable, T contains the sentences

$$\exists x\, \phi(x) \longrightarrow \exists x\, \exists \alpha \in \mathrm{Ord}\, (\phi(x) \wedge \theta(\alpha, \dot{x}_{m_\phi}, x)) \quad \text{and}$$

$$\exists x\, (\phi(x) \wedge x \in \dot{X}) \rightarrow \phi(\dot{x}_{n_\phi}).$$

Here $\theta(\cdot,\cdot,\cdot)$ is a formula defining a well-order relative to a ctble set X for an X-premouse and $\dot{X}$ is the symbol for X in $\mathcal{L}_{pm}$.

(4) T is a complete, consistent theory s.t. for every model $\mathcal{M}$ of T and every model $\mathcal{N}^*$ which is the defble closure of $\{x_i : i<\omega\}$ in $\mathcal{M} \upharpoonright \mathcal{L}_{pm}$, $\mathcal{N}^*$ is well-founded and if $\mathcal{N}$ denotes the transitive collapse of $\mathcal{N}^*$,

(a) $\mathcal{N}$ is a 1-small X-premouse, where $X = \{x_i : i<\omega\} = \mathcal{N} \cap \mathbb{R}$,

(b) $\mathcal{N} \models ZF$ + "there are no Woodin cardinals",

(c) $\mathcal{N} \not\models AD$,

(d) if $\mathcal{P} \ntrianglelefteq \mathcal{N}$ and $\mathcal{P}$ satisfies (4b), then $\mathcal{P} \models AD$,
(i.e. $\mathcal{N}$ is minimal with (b) and (c)),

(e) $\mathcal{N}$ is Π^1_2-iterable,

(f) there is a non-determined set of reals in $\mathcal{N}$ defble from z and if $Z(z,\mathcal{N})$ is the least such set (in the well-order relative to z defined by $\theta(\cdot, z, \cdot)$), then $a \oplus b \in Z(z,\mathcal{N})$:

<u>Rmk</u>: Rule (4) can be followed by Player I by playing an appropriate theory as then the model $\mathcal{N}$ is uniquely determined since all possible $\mathcal{N}^*$'s are isomorphic.

By our hypothesis this game is determined.

<u>Case 1</u>: Player I has a winning strategy σ in G.

Let $\underset{\substack{\uparrow \\ \text{trans. collapse}}}{W} \overset{\pi}{\cong} \underset{\substack{\Psi \\ \sigma}}{Y} \overset{\text{ctble}}{\prec} \underset{\substack{\uparrow \\ \text{large enough}}}{V_\eta}$ and consider $M_1(\mathbb{R}^W)$

Since there is a club of $\mathbb{R}^W \in \mathcal{P}_{\omega_1}(\mathbb{R})$ for such W we can assume $M_1(\mathbb{R}^W) \cap \mathbb{R} = \mathbb{R}^W$ (by Lemma 1) and

$M_1(\mathbb{R}^W) \mid \delta_{\mathbb{R}^W} \nvDash AD$ (by hypothesis).

Note that the game G is definable in W and let $\bar{\sigma} \in W$ be s.t. $\pi(\bar{\sigma}) = \sigma$ (i.e. $\bar{\sigma} = \sigma \cap W$). Then $\bar{\sigma}$ is a winning strategy for I in G inside W.

Let $h : \omega \to \mathbb{R}^W$ be an enumeration of $\mathbb{R}^W$, $h \in V$.

Consider the following play p of G in V:

- I plays according to his winning strategy σ.
- II plays some real $b_0 \in {}^\omega\omega$ and then h.

Since every $h \restriction n \in W$ for $n \in \omega$, every initial segment of this play is in the domain of $\bar{\sigma}$ $(= \sigma \cap W)$.

Hence I's moves are in W as well and $z, x_1, x_2, \ldots$ enumerate $\mathbb{R}^W$.

Let $\mathcal{N}_p$ be the Π^1_2-ctble $\mathbb{R}^W$-premouse obtained from the fact that σ is a w.s for I.

By comparing N_p with $M_1(\mathbb{R}^W) \mid \delta_{\mathbb{R}^W}$ we actually get that N_p is ω_1-iterable using the minimality property (4d) of N_p.

Let $Z = Z(z, N_p)$ for $z = \sigma(\emptyset) = \bar{\sigma}(\emptyset)$ be the least non-determined set defble from z in N_p.

Consider the Gale-Stewart game $\mathcal{G}(Z)$ and the following strategy τ for I:

$$a = \tau(b) \quad \text{iff} \quad (z, a) = \bar{\sigma}(b).$$

Note: $\tau \in W$ and τ can be coded by a real, so $\tau \in N_p$.

Then we can show:

<u>Claim</u>: τ is a winning strategy for I in $\mathcal{G}(Z)$ in N_p.

This contradicts the fact that Z is not determined in N_p.

<u>Case 2</u>: Player II has a winning strategy σ in $\mathcal{G}$.

Analogous, asking I to play the theory of the shortest initial segment of $M_1(\mathbb{R}^W) \mid \delta_{\mathbb{R}^W}$ satisfying $ZF + \neg AD$ (for W as in Case 1), together with some real $a_0 \in {}^\omega\omega$ and an enumeration h of $\mathbb{R}^W$, while II plays according to his winning strategy.

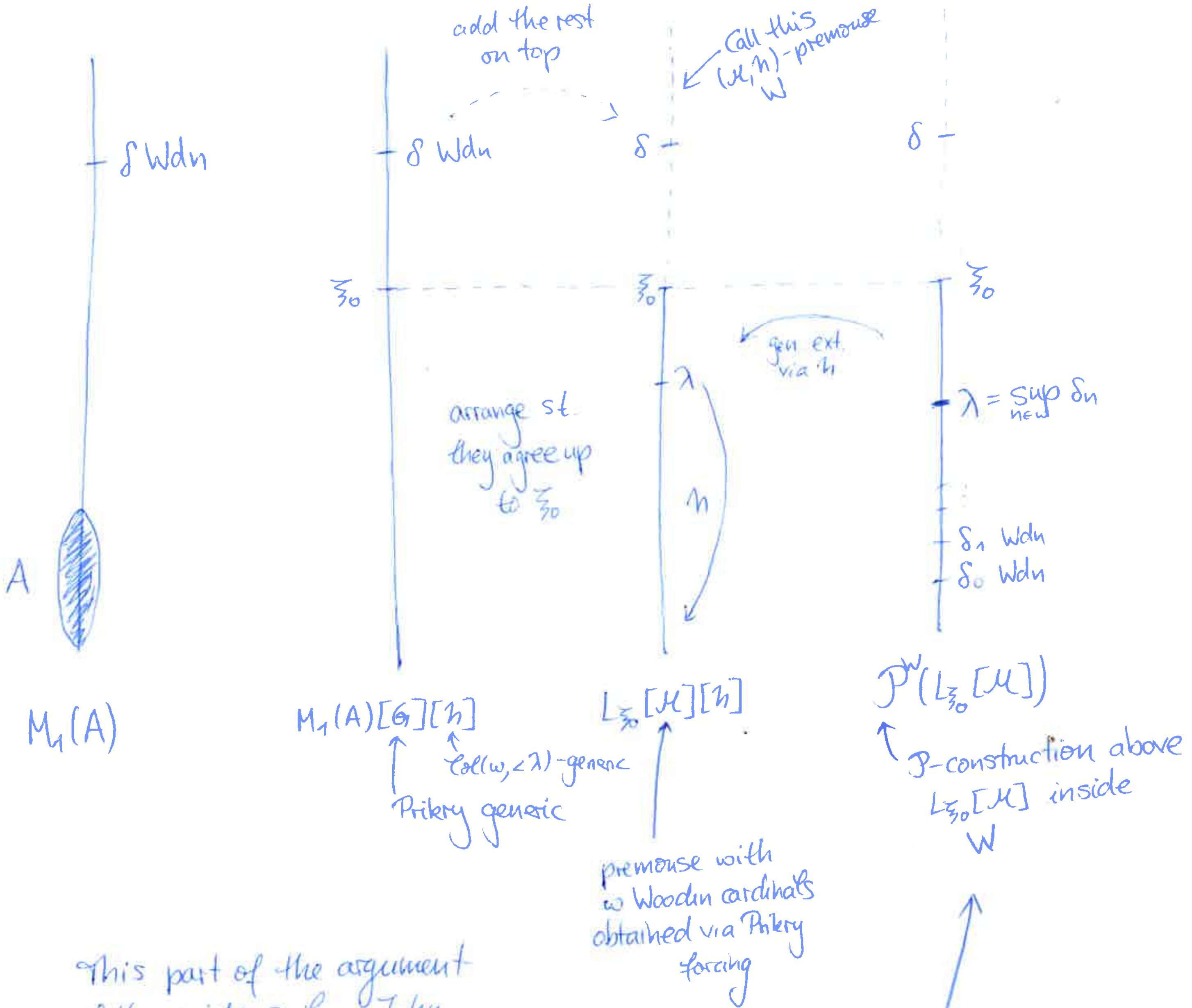
$\square$

<u>Lemma 3</u>: Sps $A \in \mathcal{P}_{\omega_1}(\mathbb{R})$ with $M_1(A) \models ZF + AD$ and $M_1(A) \cap \mathbb{R} = A$. Then $M_1(A) \models DC$ and moreover we can

sps. $M_1(A) \models AD^+ + \Theta = \Theta_0$.

Let's fix an $A \in \mathcal{P}_{\omega_1}(\mathbb{R})$ as in Lemma 3.

Picture for the rest of the proof.



This part of the argument follows ideas from John Steel's unpublished notes "An optimal consistency strength lower bound for $AD_{\mathbb{R}}$".

This is a premouse with $\omega+1$ Woodin cardinals.

□

A detailed preprint containing a full proof of this result will be on my webpage soon.
Check https://muellersandra.github.io/publications.