Long games and Woodin cardinals

joint work with Juan Aguilera

We consider "long games". Let $A \subseteq \omega \times (\omega)^2$ for some fixed $\alpha < \omega_1$.

\[
\begin{array}{c|ccccccc}
\hline
& n_0 & h_2 & \cdots & n_w & h_{w+2} & \cdots \\
\hline
I & h_1 & h_3 & \cdots & n_{w+1} & h_{w+3} & \cdots \\
\hline
\end{array}
\]

Player I wins iff $(n_0, n_1, \ldots) \in A$, o/w Player II wins.

Remark: Determinacy for games of length $\omega \cdot (n+1)$ with $\text{II}_n$ payoff implies determinacy for games of length $\omega$ with $\text{II}_{n+1}$ payoff.

Idea: We can simulate projections by $\omega$ moves (one round) in a game, where we only consider the moves of one of the two players.

Example: Let $A = pB$, $B = (\omega)^2$, $B \in \text{II}_1$.

$G^*$

\[
\begin{array}{c|cccc}
\hline
& x_0 & x_2 & \cdots & y_1 \ y_1 \ y_2 \\
\hline
I & x_1 & x_3 & \cdots \\
\hline
\end{array}
\]

Player I wins iff $(x, y) \in B$.

$G$

\[
\begin{array}{c|cccc}
\hline
& x_0 & x_2 & \cdots \\
\hline
I & x_1 & x_3 & \cdots \\
\hline
\end{array}
\]

Player I wins iff $x \in A$.

Let $G$ be a w.s. for I in $G^*$. Then $\exists \text{Nw}$ (restricted to the first $\omega$ moves of the game) is a w.s. for I in $G$.

Analogous for Player II.
Thm (Neeman): Let \( \alpha > 1 \) be a ctble ordinal and suppose that there are \(-1+\alpha\) Woodin cardinals with a \# above them all. Then all games of length \( w \cdot \alpha \) with \( \Sigma_1 \) payoff (in fact even \( < \omega^2 \)-\( \Sigma_1 \) payoff) are determined.

Note: Woodin showed this earlier for ordinals \( \alpha \) of the form \( \alpha = \omega \cdot \beta \), where \( \beta \) is additively closed.

Question: Is this result optimal, i.e. can we prove the converse?

Theorem: Suppose games of length \( w \cdot (w+1) \) with \( \Sigma_1 \) payoff are determined. Then there is a premouse with \( w+1 \) Woodin cardinals.

Remark: In fact, we will consider a game of length \( w \cdot w \) with \( \Sigma_2 \) payoff.

Note: The same proof works for games of length \( w \cdot (w+n) \) with \( \Sigma_1 \) payoff and \( w+n \) Woodin cardinals for every new.

Idea of the proof: From now on suppose games of length \( w \cdot (w+1) \) with \( \Sigma_1 \) payoff are determined.

- We first show that there is a model of the form \( M_1(A) \) for some \( A \in P_{\omega_1}(\mathbb{R}) \) s.t. \( M_1(A) \cap \mathbb{R} = A \) and \( M_1(A) \models \text{AD} \) (and in fact \( M_1(A) \models \text{AD}^+ \)).

- Use \( \text{AD} \) to "generate" \( w \) Woodin cardinals in a generic extension of \( M_1(A) \) (with a Prikry-type forcing).

- Use a \( \mathbb{P} \)-construction to add the remaining Woodin cardinal on top.
Lemma 1: There is a club $C_1 \subseteq \mathcal{P}_\omega^\omega$ (TR) such that $TR \cap M_\omega(A) = A$ for all $A \in C_1$.

We thank John Steel for pointing out to us that a variant of our proof of Lemma 2 below shows Lemma 1.

Lemma 2: There is a club $C_2 \subseteq \mathcal{P}_\omega^\omega$ (TR) such that $M_\omega(A) = ZF + AD$ for all $A \in C_2$.

Proof: Spn not, i.e., there is a stationary set of $B \in \mathcal{P}_\omega^\omega$ (TR) with $M_\omega(B) \models \Delta$. With card in $M_\omega(B)$

Now we play the following game $G$.

\[
\begin{array}{ccccc}
I & z & a & v_1, x_1 & v_3, x_3 & \ldots \\
II & b & x_2 & x_4 & \ldots & \text{where} \\
\end{array}
\]

\cdot $z, x_1, x_2, \ldots \epsilon w$ (played as sequences of natural numbers)
\cdot $a, b \epsilon w$ are obtained by alternating moves of $I$ and $II$
\cdot $v_i \epsilon \{0, 1\}$ are interpreted as truth values of formulae $\phi_i$

where \{$\phi_i : i \epsilon w$\} is a fixed enumeration of all $L_{\omega_1}(\{x_i : i \epsilon w\})$ - formulae $\forall i \phi_i$ if $j \leq i$.

$\rightarrow$ Every play determines a complete theory $T$ in the language

$L_{\omega_1}(\{x_i : i \epsilon w\})$.

Player I wins $G$ iff

1. $x_1 \models T a \& b$
2. For each $i \epsilon w$, $T$ contains the sentence $x_i \epsilon w$ and for each $j, m \epsilon w$, $T$ contains the sentence $x_i(m) = j$ iff $x_i(m) = j$, where we let $x_0 = z$.
(3) Let $m$ and $n$ be fixed maps, mapping each $x_i (i \leq \omega)$, formula $\phi$ to an even natural number $m\phi$ (or $n\phi$) such that $m$ and $n$ are recursive, have disjoint ranges, and $m\phi$ and $n\phi$ are larger than $\max\{i : x_i \text{ occurs in } \phi\}$.

For every formula $\phi(x)$ with one free variable, $T$ contains the sentences

$$\exists x \phi(x) \rightarrow \exists x \exists \alpha \in \text{Ord} (\phi(x) \land \theta(\alpha, x_{m\phi}, x))$$

and

$$\exists x (\phi(x) \land x \in X) \rightarrow \phi(x_{n\phi}).$$

Here $\theta(\cdot, \cdot, \cdot)$ is a formula defining a well-order relative to a countable set $X$ for an $X$-premouse, and $X$ is the symbol for $X$ in $\text{L}_{m}$.  

(4) $T$ is a complete, consistent theory s.t. for every model $M$ of $T$ and every model $N^*$ which is the definable closure of $\{x_i : i \leq \omega\}$ in $M \cap \text{L}_{m}$, $N^*$ is well-founded and if $N$ denotes the transitive collapse of $N^*$,

(a) $N$ is a 1-smallest $X$-premouse, where $X = \{x_i : i \leq \omega\} = N^* \cap \text{N},$

(b) $N \models \exists \text{ZF} + \text{ "there are no Woodin cardinals"},$

(c) $N \not\models \text{AD},$

(d) if $P \not\models N$ and $P$ satisfies (4b), then $P \models \text{AD},$

(i.e. $N$ is minimal with (b) and (c)),

(e) $N$ is $\text{T}_{2}^{\dagger}$-definable,

(4f) there is a non-determined set of reals in $N$ definable from $\phi$ and if $Z(x,N)$ is the least such set (in the well-order relative to $x$ defined by $\theta(\cdot, x, \cdot)$),

then $a \oplus b \in Z(x, N)$.
Remark: Rule (4) can be followed by Player I by playing an appropriate move as then the model $N$ is uniquely determined since all possible $N^*$'s are isomorphic.

By our hypothesis this game is determined.

Case 1: Player I has a winning strategy $\sigma$ in $G$.

Let

$$W \subseteq \gamma < \gamma$$

and consider $M_1(\mathcal{TR}_W)$.

Since there is a club of $\mathcal{TR}_w \in \mathcal{P}_1(R)$ for such $W$ we can assume

$$M_1(\mathcal{TR}_W) \cap R = \mathcal{TR}_W$$

(by Lemma 1) and

$$M_1(\mathcal{TR}_W) \cap S_{\mathcal{TR}} \neq \emptyset$$

(by hypothesis).

Note that the game $G$ is definable in $W$ and let $\bar{\gamma} \in W$

be s.t. $\Pi(\bar{\gamma}) = \bar{\gamma}$ (i.e. $\bar{\gamma} = 5^\gamma W$). Then $\bar{\gamma}$ is a winning strategy for I in $G$ inside $W$.

Let $\Pi: w \to \mathcal{TR}_W$ be an enumeration of $\mathcal{TR}_W$, $\Pi \in V$.

Consider the following play $p$ of $G$ in $V$:

- I plays according to his winning strategy $\sigma$.
- II plays some real $b = \omega^z$ and then $\Pi$.

Since every $\Pi^\gamma \in W$ for new $\omega$, every initial segment of this play is in the domain of $5^\gamma (\Pi^\gamma W)$.

Hence I's moves are in $W$ as well and $2, x_1, x_2, \ldots$ enumerate $\mathcal{TR}_W$.

Let $\mathcal{N}_p$ be the $T_{\Pi}^{\mathcal{TR}_W}$-premouse obtained from the fact that $\sigma$ is a w.s. for I.
By comparing \( N_p \) with \( M_1(T^{\omega}_w) \), we actually get that \( N_p \) is \( w_1 \)-iterable using the minimality property (4d) of \( N_p \).

Let \( Z = Z(\omega, N_p) \) for \( \omega = \bar{\omega}(\emptyset) = \bar{\omega}(\emptyset) \) be the least non-determined set definable from \( \omega \) in \( N_p \).

Consider the Gale-Stewart game \( G(\omega) \) and the following strategy \( \tau \) for \( I \):

\[ a = \tau(b) \quad \text{iff} \quad (\omega, a) = \bar{\omega}(b). \]

Note: \( \tau \in W \) and \( \tau \) can be coded by a real, so \( \tau \in N_p \).

Then we can show:

**Claim:** \( \tau \) is a winning strategy for \( I \) in \( G(\omega) \) in \( N_p \).

This contradicts the fact that \( \omega \) is not determined in \( N_p \).

**Case 2:** Player II has a winning strategy in \( G \).

Analogous, asking \( I \) to play the theory of the shortest initial segment of \( M_1(T^{\omega}_w) \) satisfying \( ZF + \text{AD} \) (for \( W \) as in Case 1), together with some real \( a_0 \in \omega^\omega \) and an enumeration \( \lambda \) of \( TR^w \), while II plays according to his winning strategy.

**Lemma 3:** Sps. \( A \in P_{w_1}(\mathbb{R}) \) with \( M_1(A) \vdash ZF + \text{AD} \) and \( M_1(A) \cap TR = A \). Then \( M_1(A) \vdash DC \) and moreover we can sps. \( M_1(A) \vdash \text{AD}^+ + \Theta = \Theta_0 \).

Let's fix an \( A \in P_{w_1}(\mathbb{R}) \) as in Lemma 3.
"Picture for the rest of the proof."

This part of the argument follows ideas from John Steel's unpublished notes "An optimal consistency strength lower bound for $\text{HOD}^\omega$."

This is a premouse with $\omega + 1$ Woodin cardinals.

A detailed preprint containing a full proof of this result will be on my webpage soon.

Check https://muller-sandra.github.io/publications.