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BPFA and the Π_1 definability of NS_{ω_1}

<u>Theorem 1.</u> It is consistent, relative to a (Σ_2-) reflecting cardinal, that "BPFA + NS_{ω_1} is Π_1 definable in the parameter ω_1" holds true.

Proof: We force over L, assuming that $L \models$ "κ is a reflecting cardinal." Let $\mathbb{P}$ be the countable support iteration of proper forcings of length κ as in the proof of Theorem 2.11 of [GoSh], p. 65. Let g be $\mathbb{P}$-generic over L. By Theorem 2.11 of [GoSh],

$L[g] \models$ BPFA.

We now aim to see that there is a Σ_1 formula φ such that

$$L[g] \models \forall S \, (S \text{ is a stationary subset of } \omega_1 \longleftrightarrow \varphi(S, \omega_1)). \qquad (*)$$

For every $\alpha < \kappa$, we may write

$$\mathbb{P} \cong \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha ,$$

where $\mathbb{P}_\alpha$ denotes the initial segment of $\mathbb{P}$ of length α and if $g \restriction \alpha$ is the $\mathbb{P}_\alpha$-generic filter over L induced by g, then $(\dot{\mathbb{Q}}_\alpha)^{g \restriction \alpha} \cong \mathbb{P}/g \restriction \alpha$ is the "tail end" of the iteration.

<u>Fact 1.</u> For each $\alpha < \kappa$, forcing with $(\dot{\mathbb{Q}}_\alpha)^{g \restriction \alpha}$ over $L[g \restriction \alpha]$ preserves stationary subsets of ω_1.

Proof of Fact 1: Theorem 3.1 of [Mi], p. 58. $\dashv$

This immediately implies, as $\mathbb{P}$ has the κ-c.c:

<u>Fact 2.</u> Let $S \in \mathcal{P}(\omega_1) \cap L[g]$. Then S is stationary in $L[g]$ iff for all/some $\alpha < \kappa$, if $S \in L[g \restriction \alpha]$ (equivalently, $S \in H_{\omega_2}^{L[g \restriction \alpha]}$), then $L[g \restriction \alpha] \models$ "S is stationary" (equivalently, $H_{\omega_2}^{L[g \restriction \alpha]} \models$ "S is stationary.")

Let us now fix an inaccessible cardinal $\lambda < \kappa$ for a while, where

$L[g \restriction \lambda] \models$ " $\lambda = \omega_2$."

Let $(C_\alpha : \alpha < \lambda^{+L})$ be the canonical $\square_\lambda$-sequence of L. Let $D \in L[g]$ be club in λ^{+L} and of order type ω_1. We may define a tree T on D by setting $\alpha <_T \beta$ iff α is a limit point of C_β, for $\alpha, \beta \in D$. As MA_{ω_1} holds true in $L[g]$, the tree $(D; <_T)$ is special in $L[g]$, i.e., there is a function $f : D \longrightarrow \omega$ in $L[g]$ such that if $\alpha, \beta \in D$, $\alpha <_T \beta$, then $f(\alpha) \neq f(\beta)$. See [To], in particular the proof of Lemma 4 of [To], p. 136f. See also [To′].

By [CaVe], $H_{\omega_2}^{L[g \restriction \lambda]}$ satisfies the theory T from the proof of Theorem 2 of [CaVe], p. 397, where we take the canonical ladder system of L as the relevant one.

We have thus shown (1) $\Longrightarrow$ (2) of the following.

<u>Claim 1.</u> The following are equivalent.

(1) $S \in \mathcal{P}(\omega_1) \cap L[g]$ is stationary in $L[g]$.

(2) Inside $L[g]$, there is a model $L_\gamma[G]$ such that

(a) $\omega_1 = \omega_1^{L_\gamma[G]}$,

(b) $L_\gamma[G] \models$ "there is a largest cardinal, call it δ, such that δ is inaccessible in L,"

(c) $G \subset \delta$, and $L_\delta[G] = H_{\omega_2}^{L_\gamma[G]}$,

(d) there is some D club in γ, $\mathrm{otp}(D) = \omega_1$, and the tree $(D, <_T)$ defined as above is special,

(e) $L_\delta[G] \models$ "T," and

(f) $S \in \mathcal{P}(\omega_1) \cap L_\delta[G]$ and $L_\delta[G] \models$ "S is stationary."

Let us verify (2) $\Rightarrow$ (1) of Claim 1. By (d) (and (b)), $\gamma = \delta^{+L}$. See [To]. Therefore, γ is an inaccessible cardinal of L. By ((a) and) (e), $H_{\omega_2}^{L_\gamma[G]} = L_\delta[g \restriction \delta]$, as any transitive model M containing the canonical ladder system from L, $M \models T$, is uniquely determined

by its ordinal height, see [CaVe], p. 397.

Hence $L_\delta[G] = L_\delta[g \restriction \delta]$. By (f) and Fact 2, S is then stationary in $L[g]$.

It is now easy to see that (2) of Claim 1 may be written in a Σ_1 fashion with parameter ω_1, so that for this Σ_1 formula, (*) on p. 1 holds true. $\dashv$ (Theorem 1)

Recall the weak proper forcing axiom wPFA from [BaGiSch], see Definition 6.2 of [BaGiSch]. The above proof may easily be amalgamated with the proof of Theorem 6.3 of [BaGiSch] to produce the following.

<u>Theorem 2.</u> It is consistent, relative to a remarkable cardinal, that "wPFA + NS_{ω_1} is Π_1 definable in the parameter ω_1" holds true.

References.

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