Theorem 1. It is consistent, relative to a \((\Sigma_2^-)\) reflecting cardinal, that "BPFA + NS_{\omega_1} is \(\Pi_1\) definable in the parameter \(\omega_1\)" holds true.

Proof: We force over \(L\), assuming that
\(L \models \"\kappa\) is a reflecting cardinal.\) Let \(\mathbb{T}\) be the countable support iteration of proper forcings of length \(\kappa\) as in the proof of Theorem 2.11 of [GoSh], p. 65. Let \(g\) be \(\mathbb{T}\)-generic over \(L\). By Theorem 2.11 of [GoSh],
\[L[g] \models \text{BPFA}.
\]

We now aim to see that there is a \(\Sigma_1\) formula \(\varphi\) such that
\[L[g] \models \forall S (S \text{ is a stationary subset of } \omega_1 \longleftrightarrow \varphi(S, \omega_1)). \quad (*)\]
For every $\alpha < \kappa$, we may write

$$P \equiv P_\alpha * Q_\alpha,$$

where $P_\alpha$ denotes the initial segment of $P$ of length $\alpha$ and if $g^\alpha$ is the $P_\alpha$-generic filter over $L$ induced by $g$, then $(Q_\alpha)^{g^\alpha} \equiv P / g^\alpha$ is the "tail end" of the iteration.

Fact 1. For each $\alpha < \kappa$, forcing with $(Q_\alpha)^{g^\alpha}$ over $L[g^{\geq \alpha}]$ preserves stationary subsets of $\omega_1$.

Proof of Fact 1: Theorem 3.1 of [Mi], p.58.

This immediately implies, as $P$ has the $\kappa$-ccc:

Fact 2. Let $S \in \mathcal{P}(\omega_1 \cap L[g])$. Then $S$ is stationary in $L[g]$ iff for all/some $\alpha < \kappa$, if $S \subseteq L[g^{\geq \alpha}]$ (equivalently, $S \subseteq H^{L[g^{\geq \alpha}]}$), then $L[g^{\geq \alpha}] \models "S \text{ is stationary}"$ (equivalently, $H^{L[g^{\geq \alpha}]} \models "S \text{ is stationary}"$).
Let us now fix an inaccessible cardinal \( \kappa < \nu \) for a while, where
\[
\textrm{L}[g_{\kappa}] = \text{"} \kappa = \omega_2 \text{"}.
\]
Let \((C_\alpha : \alpha < \kappa^+)\) be the canonical \( D_\gamma \)-sequence of \( L \). Let \( D \in \textrm{L}[g] \) be club in \( \kappa^+ \) of order type \( \omega_1 \). We may define a tree \( T \) on \( D \) by setting \( \alpha < T \beta \) iff \( \alpha \) is a limit point of \( C_\beta \), for \( \alpha, \beta \in D \). As \( \text{MA}_{\omega_1} \) holds true in \( \textrm{L}[g] \), the tree \((D; \leq T)\) is special in \( \textrm{L}[g] \), i.e., there is a function \( f : D \to \omega \) in \( \textrm{L}[g] \) such that if \( \alpha, \beta \in D \), \( \alpha < T \beta \), then \( f(\alpha) \neq f(\beta) \). See [To], in particular the proof of Lemma 4 of [To], p. 136f. See also [To'A].

By [CaVe], \( H_{\omega_2} \) satisfies the theory \( T \) from the proof of Theorem 2 of [CaVe], p. 397, where we take the canonical ladder system of \( L \) as the relevant one. We have thus shown \((1) \implies (2)\) of the following.
Claim 1. The following are equivalent.

(1) \( S \in \mathcal{P}(\omega_1) \cap L[\mathcal{G}] \) is stationary in \( L[\mathcal{G}] \).

(2) Inside \( L[\mathcal{G}] \), there is a model \( L_\mathcal{G}[\mathcal{G}] \) such that
(a) \( \omega_1 = \omega_1^{L_\mathcal{G}[\mathcal{G}]} \),
(b) \( L_\mathcal{G}[\mathcal{G}] \models \text{"there is a largest cardinal, call it } \delta, \text{ such that } \delta \text{ is inaccessible in } L, \"} \)
(c) \( G \subseteq \delta \), and \( L_\delta[\mathcal{G}] = H_{\omega_2}^{L_\mathcal{G}[\mathcal{G}]} \),
(d) there is some \( D \) club in \( \mathcal{G} \), \( \text{otp}(D) = \omega_1 \), and the tree \( (D, <_T) \) defined as above is special,
(e) \( L_\delta[\mathcal{G}] \models \text{"} T, \text{"} \) and
(f) \( S \in \mathcal{P}(\omega_1) \cap L_\delta[\mathcal{G}] \) and \( L_\delta[\mathcal{G}] \models \text{"} S \text{ is stationary. } \"

Let us verify (2) \( \Rightarrow \) (1) of Claim 1. By (d) (and (b)), \( \mathcal{G} = \delta^+ \). See [To]. Therefore, \( \mathcal{G} \) is an inaccessible cardinal of \( L \). By (a) (and) (e), \( H_{\omega_2}^{L_\mathcal{G}[\mathcal{G}]} = L_\delta[\mathcal{G} \upharpoonright \mathcal{G}] \), as any transitive model \( M \) containing the canonical ladder system from \( L \), \( M \models T \), is uniquely determined.
by its ordinal height, see [CaVe], p. 397.

Hence $L_\delta[g] = L_\delta[g \upharpoonright \delta]$. By (f) and Fact 2, $S$ is then stationary in $L[g]$.

It is now easy to see that (2) of Claim 1 may be written in a $\Sigma_1$ fashion with parameter $\omega_1$, so that for this $\Sigma_1$ formula, (6) on p. 1 holds true.

\[ \text{Recall the weak proper forcing axiom WPFA from [BaGiSch], see Definition 6.2 of [BaGiSch].} \]

The above proof may easily be amalgamated with the proof of Theorem 6.3 of [BaGiSch] to produce the following.

**Theorem 2.** It is consistent, relative to a remarkable cardinal, that "WPFA + $\forall \omega \exists \delta$ $\exists \omega_1$ is $\Pi_1$ definable in the parameter $\omega_1"$ holds true.
References.


