A weak (?) consequence of determinacy

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Abstract

It is shown that if every real has a sharp and every subset of ω_1 is constructible from a real, then there is an inner model with a Woodin cardinal.

The following theorem was produced at the AIM meeting "Descriptive Inner Model Theory," June 02–06, 2014,[1] in a working group whose participants were the authors listed above. The authors would like to thank AIM for their generous hospitality.

Theorem 0.1 *Assume that every real has a sharp and every subset of ω_1 is constructible from a real. Then there is an inner model with a Woodin cardinal.*

Let $\mathcal{C}$ denote the club filter on ω_1, i.e., $\mathcal{C} = \{X \subset \omega_1 : \exists C \subset \omega_1 \text{ club } C \subset X\}$.

Lemma 0.2 (Folklore) *Assume that every real has a sharp and every subset of ω_1 is constructible from a real. Then $\mathcal{C}$ is an ultrafilter.*

[1] Cf. `http://aimath.org/pastworkshops/innermodel.html`

Proof. Let $X \subset \omega_1$. Say $X \in L[x]$, where $a \in \mathbb{R}$. As $x^\#$ exists, we may write $X = \tau^{L[x]}(\eta_1, \ldots, \eta_k, \omega_1^V)$, where $\eta_1, \ldots, \eta_k$ are x–indiscernibles below ω_1^V. Let $C \subset \omega_1^V$ denote the club of countable x–indiscernibles η with $\max\{\eta_1, \ldots, \eta_k\} < \eta$. If $\eta, \eta' \in C$, then $\eta \in X$ iff $\eta \in \tau^{L[x]}(\eta_1, \ldots, \eta_k, \omega_1^V)$ iff $\eta' \in \tau^{L[x]}(\eta_1, \ldots, \eta_k, \omega_1^V)$ iff $\eta' \in X$, i.e., $C \subset X$ or $C \cap X = \emptyset$. □

Lemma 0.3 (Folklore) *Assume that every real has a sharp and every subset of ω_1 is constructible from a real. Then $\boldsymbol{\delta}^1_2 = \omega_2$.*

Proof. Let $\alpha < \omega_2$, and let $X \subset \omega_1$ code some surjection $f\colon \omega_1 \to \alpha$. If $X \in L[x]$, where $x \in \mathbb{R}$, then $f \in L[x]$ and hence $\alpha < (\omega_1^V)^{+L[x]}$. We have shown that $\boldsymbol{\delta}^1_2 = \sup\{(\omega_1^V)^{+L[x]} \colon x \in \mathbb{R}\} = \omega_2$. □

If there is no inner model with a Woodin cardinal, then by K we denote the core model as constructed in HOD, cf. [3].

Lemma 0.4 *Suppose that there is no inner model with a Woodin cardinal. If $\mathcal{C}$ is an ultrafilter, then $\omega_2 = (\omega_1^V)^{+K}$.*

Proof. Write $\alpha = (\omega_1^V)^{+K}$. Assume that $\alpha < \omega_2$, and let $f\colon \omega_1 \to \alpha$ be onto, $f \in V$. Pick $A \subset \mathrm{OR}$ such that $\mathsf{HOD} = L[A]$. Let $W = L[A, f, \mathcal{C}]$, the inner model of ZFC constructed from the predicates A, f, and $\mathcal{C}$. We have that $f \in W$, $\mathcal{C} \cap W \in W$, and $A \cap \xi \in W$ for all ordinals ξ. It is thus easy to see that $W = \mathsf{HOD}[f, \mathcal{C} \cap W]$, which is hence a generic extension of HOD via the Vopěnka algebra. In particular, $K = K^W$ by the forcing absoluteness of K. Also, $\mathcal{C} \cap W$ witnesses that ω_1^V is a measurable cardinal in W. Therefore by [6],

$$(\omega_1^V)^{+K} = (\omega_1^V)^{+K^W} = (\omega_1^V)^W > \alpha.$$

Contradiction! □

Proof of Theorem 0.1. Suppose not. By Lemmas 0.2 and 0.4, $\omega_2 = (\omega_1^V)^{+K}$.
Claim 1 in the proof of [1, Theorem 0.4] then gives that $K|\omega_1^V$ is universal with respect to countable mice, and on the other hand, by Lemma 0.3 and Claim 2 in the proof of [1, Theorem 0.4], $K|\omega_1^V$ is *not* universal with respect to countable mice. For the reader's convenience, let us reproduce these arguments from [1].

Claim 1. If $(\omega_1^V)^{+K} = \omega_2$, then $K|\omega_1^V$ is universal with respect to countable mice with no definable Woodin cardinals.

Proof. Let $\mathcal{M}$ be a countable mouse. Let us assume that $\mathcal{M}$ does not have a definable Woodin cardinal. As $K|\omega_2^V$ is universal with respect to countable mice (cf. [4]), there must in fact be some $\delta < \omega_2^V$ such that $K||\delta$ wins the comparison against $\mathcal{M}$. Say $\rho_1(K||\delta) = \omega_1^V$. Let $\mathcal{T}$ and $\mathcal{U}$ denote the normal iteration trees on $\mathcal{M}$ and $K||\delta$, respectively, arising from the comparison of $\mathcal{M}$ with $K||\delta$. Notice that both $\mathcal{M}$ and $K||\delta$ have unique iteration strategies.
Let $f\colon \omega_1^V \to K||\delta$ be bijective, where $f \in K$. Let us pick

$$\pi\colon H \to H_\theta$$

such that H is countable and transitive, θ is large enough, and

$$\{\mathcal{M}, K||\delta, \mathcal{T}, \mathcal{U}, f\} \subset \text{ran}(\pi).$$

Set $\bar{K} = \pi^{-1}(K||\delta)$, $\bar{\mathcal{T}} = \pi^{-1}(\mathcal{T})$, and $\bar{\mathcal{U}} = \pi^{-1}(\mathcal{U})$. By our hypotheses, the iteration trees $\bar{\mathcal{T}}$ and $\bar{\mathcal{U}}$ are according to the unique iteration strategies for $\mathcal{M}$ and $\bar{K}$, respectively, and they witness that $\bar{K}$ wins the comparison against $\mathcal{M}$.

But $\bar{K}$ is the transitive collapse of ran$(f \restriction \text{crit}(\pi))$, and therefore $\bar{K} \in K$ and has size $< \omega_1^V$ in K. Inside K, $K|\omega_1^V$ is certainly universal with respect to mice of size $< \omega_1^V$, and therefore the fact that $\bar{K} \in K$ wins the comparison against $\mathcal{M}$ implies that $K|\omega_1$ wins the comparison against $\mathcal{M}$, too. □ (Claim 1)

Claim 2. Suppose that $x^\#$ exists for every $x \in \mathbb{R}$, and $\boldsymbol{\delta}^1_2 = \aleph_2$. Then $K|\omega_1^V$ is not universal with respect to countable mice with no definable Woodin cardinals, and in fact the mouse order on the set of all such countable mice has length ω_2.

Proof. Jensen has shown that the hypothesis of this Claim implies that $x^\dagger$ exists for every real x (cf. [2]).

Let us fix $x \in \mathbb{R}$ for a while, and let $\kappa = \kappa_x < \Omega = \Omega_x$ denote the two measurable cardinals of $x^\dagger$. Let K_x denote the (lightface) core model of $x^\dagger$ of height Ω. By absoluteness, K_x is a mouse in V. Let

$$(\mathcal{N}^x_i, \pi^x_{ij} \colon i \leq j \leq \omega_1)$$

denote the linear iteration of $\mathcal{N}^x_0 = x^\dagger$ obtained by iterating the unique measure on κ and its images ω_1 times. By [5], $\pi^x_{ii+1} \restriction \pi^x_{0i}(K_x)$ is an iteration of $\pi^x_{0i}(K_x)$, and there is hence a (not necessarily normal) iteration tree $\mathcal{T}$ on K_x of length $\omega_1 + 1$ such that

$$\mathcal{M}^{\mathcal{T}}_{\omega_1} = \pi^x_{0\omega_1}(K_x).$$

By [6],

$$\kappa^{+x^\dagger} = \kappa^{+K_x},$$

so that

$$\omega_1^{+\mathcal{N}^x_{\omega_1}} = \omega_1^{+\pi^x_{0\omega_1}(K_x)}.$$

Now by $\boldsymbol{\delta}^1_2 = \aleph_2$,

$$\sup(\{\omega_1^{\mathcal{N}^x_{\omega_1}} \colon x \in \mathbb{R}\}) = \aleph_2,$$

and therefore the supremum of all $\mathcal{P} \cap \text{OR}$ such that there is some countable mouse $\mathcal{M}$ (with no definable Woodin cardinal) and some iteration tree $\mathcal{T}$ on $\mathcal{M}$ of length $\omega_1 + 1$ such that $\mathcal{P} = \mathcal{M}^{\mathcal{T}}_{\omega_1}$ is equal to $\aleph_2$. On the other hand, a boundedness argument shows that for a fixed countable mouse $\mathcal{M}$, the supremum of all $\mathcal{P} \cap \text{OR}$ such that there is some iteration tree $\mathcal{T}$ on $\mathcal{M}$ of length $\omega_1 + 1$ such that $\mathcal{P} = \mathcal{M}^{\mathcal{T}}_{\omega_1}$ is smaller that $\omega_1^{+L[\mathcal{M}]}$ (cf. [7, p. 56f.]).

This shows that the mouse order on the set of all countable mice has length ω_2. This readily implies that $K|\omega_1$ cannot be universal with respect to countable mice (with no definable Woodin cardinals), as otherwise $\{K|\delta \colon \delta < \omega_1\}$ would be cofinal in the mouse order on the set of all countable mice. □ (Claim 2)

We have reached a contradiction! □

Question 1. Is the hypothesis of Theorem 0.1 stronger than one Woodin cardinal?

Question 2. Assume that every real has a sharp and $\boldsymbol{\delta}^1_2 = \omega_2$. Must there be an inner model with a Woodin cardinal?

References

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