A weak (?) consequence of determinacy

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Abstract

It is shown that if every real has a sharp and every subset of $\omega_1$ is constructible from a real, then there is an inner model with a Woodin cardinal.

The following theorem was produced at the AIM meeting “Descriptive Inner Model Theory,” June 02–06, 2014,\(^1\) in a working group whose participants were the authors listed above. The authors would like to thank AIM for their generous hospitality.

**Theorem 0.1** Assume that every real has a sharp and every subset of $\omega_1$ is constructible from a real. Then there is an inner model with a Woodin cardinal.

Let $\mathcal{C}$ denote the club filter on $\omega_1$, i.e., $\mathcal{C} = \{X \subset \omega_1 : \exists C \subset \omega_1 \text{ club } C \subset X\}$.

**Lemma 0.2 (Folklore)** Assume that every real has a sharp and every subset of $\omega_1$ is constructible from a real. Then $\mathcal{C}$ is an ultrafilter.

\(^1\)Cf. http://aimath.org/pastworkshops/innermodel.html
Lemma 0.3 (Folklore) Assume that every real has a sharp and every subset of \( \omega_1 \) is constructible from a real. Then \( \delta_2^1 = \omega_2 \).

Proof. Let \( X \subseteq \omega_1 \). Say \( X \in L[x] \), where \( a \in \mathbb{R} \). As \( x^# \) exists, we may write \( X = \tau^{L[x]}(\eta_1, \ldots, \eta_k, \omega_1^V) \), where \( \eta_1, \ldots, \eta_k \) are \( x \)-indiscernibles below \( \omega_1^V \). Let \( C \subseteq \omega_1^V \) denote the club of countable \( x \)-indiscernibles \( \eta \) with \( \max\{\eta_1, \ldots, \eta_k\} < \eta \). If \( \eta, \eta' \in C \), then \( \eta \in X \) iff \( \eta \in \tau^{L[x]}(\eta_1, \ldots, \eta_k, \omega_1^V) \) iff \( \eta' \in \tau^{L[x]}(\eta_1, \ldots, \eta_k, \omega_1^V) \). Hence \( \delta_2^1 = \omega_2 \).

Lemma 0.4 Suppose that there is no inner model with a Woodin cardinal. If \( C \) is an ultrafilter, then \( \omega_2 = (\omega_1^V)^+K \).

Proof. Write \( \alpha = (\omega_1^V)^+K \). Assume that \( \alpha < \omega_2 \), and let \( f: \omega_1 \to \alpha \) be onto, \( f \in V \). Pick \( A \in OR \) such that \( \text{HOD} = L[A] \). Let \( W = L[A, f, C] \), the inner model of \( \text{ZFC} \) constructed from the predicates \( A, f, \) and \( C \). We have that \( f \in \text{W}, C \cap \text{W} \in \text{W} \), and \( A \cap \xi \in \text{W} \) for all ordinals \( \xi \). It is thus easy to see that \( W = \text{HOD}[f, C \cap \text{W}] \), which is hence a generic extension of \( \text{HOD} \) via the Vopěnka algebra. In particular, \( K = K^W \) by the forcing absoluteness of \( K \). Also, \( C \cap \text{W} \) witnesses that \( \omega_1^V \) is a measurable cardinal in \( W \). Therefore by [6],

\[
(\omega_1^V)^+K = (\omega_1^V)^+K^W = (\omega_1^V)^W > \alpha.
\]

Contradiction!

Proof of Theorem 0.1. Suppose not. By Lemmas 0.2 and 0.4, \( \omega_2 = (\omega_1^V)^+K \).

Claim 1 in the proof of [1, Theorem 0.4] then gives that \( K[\omega_1^V] \) is universal with respect to countable mice, and on the other hand, by Lemma 0.3 and Claim 2 in the proof of [1, Theorem 0.4], \( K[\omega_1^V] \) is not universal with respect to countable mice. For the reader’s convenience, let us reproduce these arguments from [1].

Claim 1. If \( (\omega_1^V)^+K = \omega_2 \), then \( K[\omega_1^V] \) is universal with respect to countable mice with no definable Woodin cardinals.

Proof. Let \( \mathcal{M} \) be a countable mouse. Let us assume that \( \mathcal{M} \) does not have a definable Woodin cardinal. As \( K[\omega_2^V] \) is universal with respect to countable mice (cf. [4]), there must in fact be some \( \delta < \omega_2^V \) such that \( K[\delta] \) wins the comparison against \( \mathcal{M} \). Say \( \rho_1(\mathcal{K}[\delta]) = \omega_1^V \). Let \( \mathcal{T} \) and \( \mathcal{U} \) denote the normal iteration trees on \( \mathcal{M} \) and \( 
\mathcal{K}[\delta] \), respectively, arising from the comparison of \( \mathcal{M} \) with \( K[\delta] \). Notice that both \( \mathcal{M} \) and \( K[\delta] \) have unique iteration strategies.

Let \( f: \omega_1^V \to \mathcal{K}[\delta] \) be bijective, where \( f \in \mathcal{K} \). Let us pick

\[
\pi: H \to H_\delta
\]
such that $H$ is countable and transitive, $\theta$ is large enough, and
\[ \{\mathcal{M}, K||\delta, \mathcal{T}, \mathcal{U}, f\} \subset \text{ran}(\pi). \]

Set $\bar{K} = \pi^{-1}(K||\delta)$, $\bar{T} = \pi^{-1}(\mathcal{T})$, and $\bar{U} = \pi^{-1}(\mathcal{U})$. By our hypotheses, the iteration trees $\mathcal{T}$ and $\mathcal{U}$ are according to the unique iteration strategies for $\mathcal{M}$ and $K$, respectively, and they witness that $K$ wins the comparison against $\mathcal{M}$.

But $\bar{K}$ is the transitive collapse of $\text{ran}(f \upharpoonright \text{crit}(\pi))$, and therefore $\bar{K} \in K$ and has size $< \omega_1^V$ in $K$. Inside $K$, $K|\omega_1^V$ is certainly universal with respect to mice of size $< \omega_1^V$, and therefore the fact that $\bar{K} \in K$ wins the comparison against $\mathcal{M}$ implies that $K|\omega_1$ wins the comparison against $\mathcal{M}$, too. \(\Box\) (Claim 1)

Claim 2. Suppose that $x^\#$ exists for every $x \in \mathbb{R}$, and $\delta_2 = \aleph_2$. Then $K|\omega_1^V$ is not universal with respect to countable mice with no definable Woodin cardinals, and in fact the mouse order on the set of all such countable mice has length $\omega_2$.

**Proof.** Jensen has shown that the hypothesis of this Claim implies that $x^\dagger$ exists for every real $x$ (cf. [2]).

Let us fix $x \in \mathbb{R}$ for a while, and let $\kappa = \kappa_x < \Omega = \Omega_x$ denote the two measurable cardinals of $x^\dagger$. Let $K_x$ denote the (lightface) core model of $x^\dagger$ of height $\Omega$. By absoluteness, $K_x$ is a mouse in $V$. Let
\[ (N_i^x, \pi_i^x : i \leq j \leq \omega_1) \]
denote the linear iteration of $N_0^x = x^\dagger$ obtained by iterating the unique measure on $\kappa$ and its images $\omega_1$ times. By [5], $\pi_{i+1}^x | \pi_i^x(K_x)$ is an iteration of $\pi_i^x(K_x)$, and there is hence a (not necessarily normal) iteration tree $\mathcal{T}$ on $K_x$ of length $\omega_1 + 1$ such that
\[ \mathcal{M}_{\omega_1} = \pi_{\omega_1}^x(K_x). \]

By [6],
\[ \kappa^+ x^\dagger = \kappa^+ K_x, \]
so that
\[ \omega_1^+ N_x = \omega_1^+ \pi_{\omega_1}^x(K_x). \]

Now by $\delta_2 = \aleph_2$,
\[ \sup(\{\omega_1^{N_x} : x \in \mathbb{R}\}) = \aleph_2, \]
and therefore the supremum of all $\mathcal{P} \cap \text{OR}$ such that there is some countable mouse $\mathcal{M}$ (with no definable Woodin cardinal) and some iteration tree $\mathcal{T}$ on $\mathcal{M}$ of length $\omega_1 + 1$ such that $\mathcal{P} = \mathcal{M}_{\omega_1}^\mathcal{T}$ is equal to $\aleph_2$. On the other hand, a boundedness argument shows that for a fixed countable mouse $\mathcal{M}$, the supremum of all $\mathcal{P} \cap \text{OR}$ such that there is some iteration tree $\mathcal{T}$ on $\mathcal{M}$ of length $\omega_1 + 1$ such that $\mathcal{P} = \mathcal{M}_{\omega_1}^\mathcal{T}$ is smaller that $\omega_1^\text{L(M)}$ (cf. [7, p. 56f.]).

This shows that the mouse order on the set of all countable mice has length $\omega_2$. This readily implies that $K|\omega_1$ cannot be universal with respect to countable mice (with no definable Woodin cardinals), as otherwise $\{K|\delta : \delta < \omega_1\}$ would be cofinal in the mouse order on the set of all countable mice. \(\Box\) (Claim 2)
We have reached a contradiction!

**Question 1.** Is the hypothesis of Theorem 0.1 stronger than one Woodin cardinal?

**Question 2.** Assume that every real has a sharp and $\delta_2^1 = \omega_2$. Must there be an inner model with a Woodin cardinal?

**References**


