F. Schlutzenberg observed that the methods from [We] could be used to strengthen said theorem. Let $M^#$ denote the sharp for an inner model with a proper class of measurable cardinals. Let $D < w_1$ be a club such that each $\alpha \in D$ is an $M^#_\alpha$-admissible. Chan-Jackson-Trang had basically shown that $L_{w_1}[D] \models 2FC + GCH$.

We here show that this is an iterate, $M$, of $M^#$ together with a Prikry generic, $C$, over $M$ such that $L_{w_1}[D] = M/\omega_1[C]$.

Let $(\alpha_i : i < w_1)$ be the monotone enumeration of $D$. Let $M'$ denote the iterate of $M^#$ obtained by hitting the top measure of $M^#$ and its images $w_1$ times, so that $w_1$ is the critical point of the top measure of $M'$. 

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A remark on a theorem of Chan-Jackson-Trang
Let $E = w_1$ be the set of critical points used in the iteration from $M^\#$ to $M'$. So $E$ is club, and as every $\alpha \in D$ is $M^\#$-admissible, $E \cap \alpha$ is club in $\alpha$ for every $\alpha \in D$.

We may then iterate $M'$ below $w_1$ and normally in such a fashion that if $M$ denotes the iterate, then $\kappa_{w_i + w}$ is the $i$th measurable cardinal of $M$, $i < w_1$. If $\mathcal{I}$ denotes the (linear) iteration from $M'$ to $M$, then for all $i < w_1$ and all $n < w$, $n > 0$,

$$\pi^\mathcal{I}_{w_i + n, w_i + w}(\kappa_{w_i + n}) = \kappa_{w_i + w}$$

and $\kappa_{w_i + n} = \text{crit}(\pi^\mathcal{I}_{w_i + n, w_i + w})$, so that

$$\kappa_{w_i + n} : 0 < n < w$$

is Prikry-generic over $M$ w.r.t. the (unique) measure of $M$ on $\kappa_{w_i + w}$. Moreover,
\[ C = \{ w_i : i < w, 0 < n < w \} \]

is Parry generic over \( M \), cf. [We].

\( C \) consists exactly of the successor points of \( D \), so that \( L_{w_i}[C] = L_{w_i}[D] \). \( C \) is class generic over \( M_{1w_i} \). We claim that

\[ (*) \quad L_{w_i}[D] = M_{1w_i}[C]. \]

"C" is trivial, so let us verify that "D" holds true. By "C" and \( M^\# \not\in M_{1w_i}[C], \)
\( M^\# \not\in L_{w_i}[C] \), so that \( K^{L_{w_i}[D]} \) exists and is "\( M^\# \)-small," i.e., \( M^\# \not\in K^{L_{w_i}[D]} \). Moreover,
\[ K^{M_{1w_i}[C]} = K^{M_{1w_i}} \] (tail ends of \( C \) don't add new bounded sets, \( K \) doesn't change by set forcing, and \( K \) is locally definable), so that
by \( K^{M_{1w_i}} = M_{1w_i} \), \( M_{1w_i} \) is the core model of \( M_{1w_i}[C] \).

By "C", \( K^{L_{w_i}[D]} \leq^* M_{1w_i} \) then in the
mouse order.

Suppose that $K^{L_{w_1}[C^D]} \prec M\hat{w}_1$, so that a proper initial segment of $M\hat{w}_1$ iterates past $K^{L_{w_1}[C^D]}$. The witnessing station, $U$, on the $M\hat{w}_1$-side must then use a single measure and its images $U^\beta$ times, which implies that the collection of total measures in $K^{L_{w_1}[C^D]}$ is bounded below $w_1$. Let $U$ be the station on $K^{L_{w_1}[C^D]}$ arising in this comparison.

By $K^{L_{w_1}[C^D]} \prec M\hat{w}_1 \prec M^*$ and the fact that $D$ only consists of $M^*$-admissibles, every $\alpha \in D$ is a fixed point of $\Pi_0^U\beta$, hence of $\Pi_0^{U_0\beta}$ for all sufficiently big $\alpha \in D$. Also, the set \[ \{ (\pi_{\beta^0}^\beta)^{-1}(w^\beta_1) : w^\beta_1 \in \text{ran} (\pi_{\beta^0}^\beta) \land (\pi_{\beta^0}^\beta)^{-1}(w^\beta_1) \in \text{crit} (\pi_{\beta^0}^\beta) \} \], which is the set of all critical points of the single measure and
its image which is used on a tail end of $T$, covers a tail end of $D$.

We may then pick $i < w_1$ such that
\[
\pi_{\omega_1} (\kappa_{\omega_1 + w}) = \kappa_{\omega_1 + w} \quad \text{and} \quad \kappa_{\omega_1 + w} = \text{crit} (\pi_{\omega_1 + w}, \omega_1) = (\pi_{\omega_1 + w}, \omega_1)^{-1} (\omega_1).
\]
\(\kappa_{\omega_1 + w}\) is then inaccessible in \(M_{\omega_1}\), hence in \(K^{L_{\nu_1}[D]}\), but \((\kappa_{\omega_1 + n} : n < \omega) \in L_{\nu_1}[D]\)

witnesses that \(\kappa_{\omega_1 + w}\) is of cardinality \(\nu_1\) in \(L_{\nu_1}[D]\). By the Dodd-Jensen covering lemma, then, \(\kappa_{\omega_1 + w}\) must be measurable in \(K^{L_{\nu_1}[D]}\).

But we could have chosen \(\kappa_{\omega_1 + w}\) above the \(\text{sup of the measurable of } K^{L_{\nu_1}[D]}\). Contradiction!

Hence \(K^{L_{\nu_1}[D]} = M_{\nu_1}\) in the mouse order and \(K^{L_{\nu_1}[D]}\) has unboundedly many measurable cardinals in \(\nu_1\), in fact by a theorem of Jensen's there is an elementary embedding
\[ \pi : M^{L_{\omega_1}} \rightarrow K^{L_{\omega_1}[\mathcal{D}]} \] resulting from a
normal situation, call it \( \mathcal{I} \), on \( M^{L_{\omega_1}} \)
\((K^{L_{\omega_1}[\mathcal{D}]}\) was just shown to be a universal
measurable in \( M^{L_{\omega_1}[\mathcal{C}]} \). )

Let us now prove that \( M^{L_{\omega_1}} = K^{L_{\omega_1}[\mathcal{D}]} \),
which will finish the proof of (\( \ast \)), as then
\( M^{L_{\omega_1}} \subseteq L_{\omega_1}[\mathcal{D}] \).

It obviously suffices to verify that the \( \kappa_{\omega_i + w} \),
i.e. the measurable cardinals of \( K^{L_{\omega_1}[\mathcal{D}]} \).

Fix \( \kappa_{\omega_j} \) ad suppose that that's true for all
\( \kappa_{\omega_j} \), \( j < i \). Then \( \kappa_{\omega_i + w} \) is the next measurable
of \( M^{L_{\omega_1}} \), so the next measurable of \( K^{L_{\omega_1}[\mathcal{D}]} \)
must be \( \geq \kappa_{\omega_i + w} \). We have that \( P(\kappa_{\omega_i + w}) \cap K^{L_{\omega_1}[\mathcal{D}]} \)
\( = P(\kappa_{\omega_i + w}) \cap M^{L_{\omega_1}} \) and \( (\kappa_{\omega_i + n} : 0 < n < w) \in K^{L_{\omega_1}[\mathcal{D}]} \)
generates the (unique) measure in
\( M \) on \( \kappa_{\omega_i + w} \). But then this measure is
in $L_{\omega_1}^{\mathbb{D}_J}$ as well and in fact it will be on the $L_{\omega_1}^{\mathbb{D}_J}$-sequence.

References.

[We] Philip Welch, paper on $L[\text{Card}]$.

William Chan, talk at UC Berkeley, July 2019.