HARRINGTON'S PRINCIPLE IN HIGHER ORDER ARITHMETIC

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ABSTRACT. Let Z_2, Z_3, and Z_4 denote 2^{nd}, 3^{rd}, and 4^{th} order arithmetic, respectively. We let Harrington's Principle, HP, denote the statement that there is a real x such that every x–admissible ordinal is a cardinal in L. The known proofs of Harrington's theorem "$Det(\Sigma^1_1)$ implies $0^\sharp$ exists" are done in two steps: first show that $Det(\Sigma^1_1)$ implies HP, and then show that HP implies $0^\sharp$ exists. The first step is provable in Z_2. In this paper we show that Z_2 + HP is equiconsistent with ZFC and that Z_3 + HP is equiconsistent with ZFC + there exists a remarkable cardinal. As a corollary, Z_3 + HP does not imply $0^\sharp$ exists, whereas Z_4 + HP does. We also study strengthenings of Harrington's Principle over 2^{nd} and 3^{rd} order arithmetic.

1. Introduction

Over the last four decades, much work has been done on the relationship between large cardinal and determinacy hypothesis, especially the large cardinal-determinacy correspondence. The first result in this line was proved by Martin and Harrington.

Theorem 1.1. (Martin–Harrington, [6]**)** *In* ZF*,* $Det(\Sigma^1_1)$ *if and only if* $0^\sharp$ *exists.*

Definition 1.2. We let *Harrington's Principle*, HP for short, denote the following statement:

$$\exists x \in 2^\omega \forall \alpha (\alpha \text{ is } x\text{-admissible} \longrightarrow \alpha \text{ is an } L\text{-cardinal}).$$

Theorem 1.3. (Silver, [6]**)** *In* ZF*,* HP *implies* $0^\sharp$ *exists.*

Definition 1.4. (i) $Z_2 = ZFC^- +$ Every set is countable.[1]
(ii) $Z_3 = ZFC^- + \mathcal{P}(\omega)$ exists + Every set is of cardinality $\leq \beth_1$.
(iii) $Z_4 = ZFC^- + \mathcal{P}(\mathcal{P}(\omega))$ exists + Every set is of cardinality $\leq \beth_2$.

Z_2, Z_3, and Z_4 are the corresponding axiomatic systems for second order arithmetic (SOA), third order arithmetic, and fourth order arithmetic, respectively. Note that $Z_3 \vdash H_{\omega_1} \models Z_2$ and $Z_4 \vdash H_{\beth_1^+} \models Z_3$.

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[1] ZFC^- denotes ZFC with the Power Set Axiom deleted and Collection instead of Replacement.

The known proofs of Harrington's theorem "$Det(\Sigma^1_1)$ implies $0^\sharp$ exists" are done in two steps: first show that $Det(\Sigma^1_1)$ implies HP, and then show that HP implies $0^\sharp$ exists. The first step is provable in Z_2. In this paper we prove that Z_2 + HP is equiconsistent with ZFC and Z_3 + HP is equiconsistent with $\mathsf{ZFC}+$ there exists a remarkable cardinal. As a corollary, we have Z_3 + HP does not imply $0^\sharp$ exists. In contrast, Z_4 + HP implies $0^\sharp$ exists.

We also investigate strengthenings of Harrington's Principle, $\mathsf{HP}(\varphi)$, over higher order arithmetic.

Definition 1.5. Let $\varphi(-)$ be a Σ_2–formula in the language of set theory such that, provably in ZFC: for all α, if $\varphi(\alpha)$, then α is an inaccessible cardinal and $L \models \varphi(\alpha)$. Let $\mathsf{HP}(\varphi)$ denote the statement:

$$\exists x \in 2^\omega \forall \alpha (\alpha \text{ is } x\text{-admissible} \longrightarrow L \models \varphi(\alpha)).$$

We show that Z_2 + $HP(\varphi)$ is equiconsistent with ZFC + $\{\alpha|\varphi(\alpha)\}$ is stationary and that Z_3 + $HP(\varphi)$ is equiconsistent with

$$ZFC + \text{ there exists a remarkable cardinal } \kappa \text{ with } \varphi(\kappa) + $$
$$\{\alpha|\varphi(\alpha) \wedge \{\beta < \alpha|\varphi(\beta)\} \text{ is stationary in } \alpha\} \text{ is stationary.}$$

As a corollary, Z_4 is the minimal system of higher order arithmetic to show that HP, $\mathsf{HP}(\varphi)$, and $0^\sharp$ exists are pairwise equivalent with each other.

2. Definitions and preliminaries

Our definitions and notations are standard. We refer to the textbooks [8], [12], [13], or [21] for the definitions and notations we use. For the definition of admissible sets, admissible ordinals, and x-admissible ordinals for $x \in 2^\omega$, see [1], [14], and [4]. Our classes will always be *definable* ones. Our notations about forcing are standard (see [8] and [7]). For the general theory of forcing, see [13], and for Jensen's theory of subcomplete forcing, see [10]. For Revised Countable Support (RCS) iteration, see [22] and also [9]. For notions of large cardinals, see [12] or [21]. We say that $0^\sharp$ exists if there exists an iterable premouse of the form $(L_\alpha, \in, U)$ where $U \neq \emptyset$, see e.g. [21]. We can define $0^\sharp$ in Z_2. In Z_2, $0^\sharp$ exists if and only if

$$\exists x \in \omega^\omega \, (x \text{ codes a countable iterable premouse}),$$

which is a Σ^1_3 statement.

The notion of remarkable cardinals was introduced by the second author in [19].

Definition 2.1. ([19]) A cardinal κ is *remarkable* if and only if for all regular cardinals $\theta > \kappa$ there are $\pi, M, \bar{\kappa}, \sigma, N$, and $\bar{\theta}$ such that the following hold: $\pi : M \to H_\theta$ is an elementary embedding, M is countable and transitive, $\pi(\bar{\kappa}) = \kappa$, $\sigma : M \to N$ is an elementary embedding with critical point $\bar{\kappa}$, N is countable and transitive, $\bar{\theta} = M \cap Ord$ is a regular cardinal in $N, \sigma(\bar{\kappa}) > \bar{\theta}$, and $M = H^N_{\bar{\theta}}$, i.e. $M \in N$ and $N \models M$ is the set of all sets which are hereditarily smaller than $\bar{\theta}$.

Definition 2.2. ([19]) Let κ be an inaccessible cardinal. Let G be $Col(\omega, < \kappa)$-generic over V, let $\theta > \kappa$ be a cardinal, and let $X \in [H^{V[G]}_\theta]^\omega \cap V[G]$. We say that X *condenses remarkably* if $X = ran(\pi)$ for some elementary

$$\pi : (H^{V[G \cap H^V_\alpha]}_\beta, \in, H^V_\beta, G \cap H^V_\alpha) \to (H^{V[G]}_\theta, \in, H^V_\theta, G)$$

where $\alpha = crit(\pi) < \beta < \kappa$ and β is a regular cardinal in V.

Lemma 2.3. ([19]) *A cardinal κ is remarkable if and only if for all regular cardinals $\theta > \kappa$ we have that*

$$\Vdash^{V}_{Col(\omega,<\kappa)} \text{"}\{X \in [H^{V[\dot{G}]}_{\check{\theta}}]^{\omega} \cap V[\dot{G}] : X \text{ condenses remarkably } \} \text{ is stationary."}$$

From Lemma 2.3, κ is remarkable in L if and only if for any L-cardinal $\mu \geq \kappa$, for any G which is $Col(\omega, <\kappa)$-generic over L, we have $L[G] \models$ "$S_\mu = \{X \prec L_\mu | X$ is countable and $o.t.(X \cap \mu)$ is an L-cardinal$\}$ is stationary."

All the following facts on remarkable cardinals are from [19]: every remarkable cardinal is remarkable in L; every remarkable cardinal κ is n-ineffable for every $n < \omega$; if $0^\sharp$ exists, then every Silver indiscernible is remarkable in L; if there exists a ω-Erdös cardinal, then there exist $\alpha < \beta < \omega_1$ such that $L_\beta \models$ "$ZFC + \alpha$ is remarkable."

3. The strength of Harrington's Principle over higher order arithmetic

3.1. The strength of Z_2+ Harrington's Principle.

Theorem 3.1. *Z_2+ HP is equiconsistent with ZFC.*

Proof. It is easy to see that Z_2+ HP implies $L \models ZFC$.

We now show that $Con(ZFC)$ implies $Con(Z_2+$ HP$)$. We assume that L is a minimal model of ZFC, i.e.,

$$\text{there is no } \alpha \text{ such that } L_\alpha \models ZFC. \tag{3.1}$$

Let G be $Col(\omega, < Ord)$-generic over L. Then $L[G] \models Z_2$. In $L[G]$, we may pick some $A \subseteq Ord$ such that $V = L[A]$ and if $\lambda \geq \omega$ is an L-cardinal, then $A \cap [\lambda, \lambda+\omega)$ codes a well ordering of $(\lambda^+)^L$. By (3.1) we will then have that for all $\alpha \geq \omega$,

$$L_{\alpha+1}[A \cap \alpha] \models \alpha \text{ is countable.} \tag{3.2}$$

By (3.2) there exists then a canonical sequence $(c_\alpha | \alpha \in Ord)$ of pairwise almost disjoint subset of ω such that c_α is the $L_{\alpha+1}[A \cap \alpha]$-least subset of ω such that c_α is almost disjoint from every member of $\{c_\beta | \beta < \alpha\}$. Do almost disjoint forcing to code A by a real (i.e., a subset of ω) x such that for any $\alpha \in Ord, \alpha \in A \Leftrightarrow |x \cap c_\alpha| < \omega$. (Cf. e.g. [2, §1.2].) This forcing is *c.c.c.* Note that $L[A][x] = L[x]$ and $L[x] \models Z_2$.

We claim that HP holds in $L[x]$. It suffices to show that if α is x-admissible, then α is an L-cardinal. Suppose α is x-admissible but is not an L-cardinal. Let λ be the largest L-cardinal $< \alpha$. Note that we can define $A \cap \alpha$ over $L_\alpha[x]$. Since $A \cap [\lambda, \lambda + \omega) \in L_\alpha[x]$ and $A \cap [\lambda, \lambda + \omega)$ codes a well ordering of $(\lambda^+)^L$, we have $(\lambda^+)^L \in L_\alpha[x]$, as α is x–admissible. But $(\lambda^+)^L > \alpha$. Contradiction! So $L[x] \models Z_2+$ HP. □

3.2. The strength of Z_3+ Harrington's Principle.

Theorem 3.2. *The following two theories are equiconsistent:*
(1) Z_3+ HP.
(2) $ZFC+$ there exists a remarkable cardinal.

Proof. We first prove that Z_3+ HP implies $L \models ZFC+$ there exists a remarkable cardinal. Assume Z_3+ HP. It is easy to verify that $L \models ZFC$. We now want to show that ω_1^V is remarkable in L. Suppose $L \models \theta > \omega_1^V$ is regular, and set $\eta = \theta^{+L}$. Let $x \in 2^\omega$ witness HP, and let G be $Col(\omega, < \omega_1^V)$-generic over V.

Let $f : [L_\theta[G]]^{<\omega} \to L_\theta[G]$, $f \in L[G]$, and let $X \prec L_\eta[x][G]$ be such that $|X| = \omega, \{\omega_1, \theta, f\} \subseteq X$. Let $\tau : L_{\bar\eta}[x][G \cap L_\alpha[x]] \cong X$ be the collapsing map, where $\alpha = crit(\tau)$, $\tau(\alpha) = \omega_1^V$, and $\tau(\bar f) = f$. As $\bar\eta$ is x-admissible, $\bar\eta$ is an L-cardinal by the choice of x as witnessing HP, and hence $\beta = o.t.(X \cap \theta) = \tau^{-1}(\theta)$ is a regular L-cardinal. Therefore, $X \cap L_\theta[G]$ condenses remarkably. By absoluteness, there is *in* $L[G]$ some elementary $\bar\tau\colon L_{\bar\eta}[G \cap L_\alpha] \to L_\eta[G]$ such that $\bar\tau(\beta) = \theta$ and $\bar\tau(\bar f) = f$. I.e., in $L[G]$, there is some $X \in [H_\theta^{L[G]}]^\omega \cap L[G]$ which condenses remarkably and is closed under f. Hence ω_1^V is remarkable in L by Lemma 2.3.

We now prove that the consistency of (2) implies the consistency of (1).

We assume that $L \models$ "$ZFC + \kappa$ is a remarkable cardinal" and

(3.3) there is no α such that $L_\alpha \models$ "$ZFC + \kappa$ is a remarkable cardinal."

In what follows, we shall write S_μ for

$$\{X \in [L_\mu]^\omega | X \prec L_\mu \text{ and } o.t.(X \cap \mu) \text{ is an } L\text{-cardinal }\},$$

as defined in the respective models of set theory which are to be consiederd.

Let G be $Col(\omega, < \kappa)$-generic over L. Since κ is remarkable in L, $L[G] \models$ "S_μ is stationary for any L-cardinal $\mu \geq \kappa$." Let H be $Col(\kappa, < Ord)$-generic over $L[G]$. Note that $Col(\kappa, < Ord)$ is countably closed. Standard arguments give that

(3.4) $L[G][H] \models Z_3 + S_\mu$ is stationary for all L-cardinals $\mu \in Card^L \setminus (\kappa + 1)$.

In $L[G][H]$, we may pick some $B \subseteq Ord$ such that $V = L[B]$ and if $\lambda \geq \omega_1$ is an L-cardinal, then $B \cap [\lambda, \lambda + \omega_1)$ codes a well ordering of $(\lambda^+)^L$. By (3.3) we will then have that for all $\alpha \geq \omega_1$,

(3.5) $$L_{\alpha+1}[B \cap \alpha] \models Card(\alpha) \leq \aleph_1.$$

By (3.5), there exists then a canonical sequence $(C_\alpha | \alpha \in Ord)$ of pairwise almost disjoint subsets of ω_1 such that C_α is the $L_{\alpha+1}[B \cap \alpha]$-least subset of ω_1 such that C_α is almost disjoint from every member of $\{C_\beta | \beta < \alpha\}$. Do almost disjoint forcing to code B by some $A \subset \omega_1$ such that for any $\alpha \in Ord$, $\alpha \in B \Leftrightarrow |A \cap C_\alpha| < \omega_1$. This forcing is countably closed and has the *Ord-c.c.* Note that $L[B][A] = L[A]$ and $L[A] \models Z_3$. Also,

(3.6) $L[A] \models$ "S_μ is stationary for any L-cardinal $\mu \geq \kappa$."

Suppose $\alpha > \omega_1$ is A-admissible, but α is not an L-cardinal. Let λ be the largest L-cardinal $< \alpha$. Note that $\lambda + \omega_1 < \alpha$ and we can compute $B \cap \alpha$ over $L_\alpha[A]$. Hence $B \cap [\lambda, \lambda + \omega_1) \in L_\alpha[A]$, and $B \cap [\lambda, \lambda + \omega_1)$ codes a well–ordering of λ^{+L}. So $\lambda^{+L} < \alpha$, as α is A–admissible. Contradiction! We have shown that in $L[A]$,

(3.7) every A–admissible ordinal above ω_1 is an L–cardinal.

Now over $L[A]$ we do reshaping as follows. (Cf. e.g. [2, §1.3] on the original reshaping forcing.)

Definition 3.3. Define $p \in \mathbb{P}$ if and only if $p : \alpha \to 2$ for some $\alpha < \omega_1$ and $\forall \xi \leq \alpha \, \exists \gamma \, (L_\gamma[A \cap \xi, p \restriction \xi] \models$ "ξ is countable" and every $(A \cap \xi)$-admissible $\lambda \in [\xi, \gamma]$ is an L-cardinal).

It is easy to check the extendability property of $\mathbb{P}$: $\forall p \in \mathbb{P} \, \forall \alpha < \omega_1 \, \exists q \leq p \, (dom(q) \geq \alpha)$. Note that $|\mathbb{P}| = \aleph_1$, as CH holds true in $L[A]$.

We now vary an argument from [23], cf. also [17], to show the following.

Claim 3.4. $\mathbb{P}$ is ω-distributive.

Proof. Let $p \in \mathbb{P}$ and $\vec{D} = (D_n|n \in \omega)$ be a sequence of open dense sets. Take $\nu > \omega_1$ such that $\vec{D} \in L_\nu[A]$ and $L_\nu[A]$ is a model of a reasonable fragment of ZFC^-. By (3.7) we have that

(3.8) $$L_\mu[A] \models \text{“every } A\text{-admissible ordinal} \geq \omega_1 \text{ is an } L\text{-cardinal,”}$$

where $\mu = (\nu^+)^L$. By (3.6) we can pick X such that $\pi : L_{\bar{\mu}}[A \cap \delta] \cong X \prec L_\mu[A]$, $|X| = \omega$, $\{p, \mathbb{P}, A, \vec{D}, \omega_1, \nu\} \subseteq X$, $\bar{\mu}$ is an L-cardinal, and $\pi(\delta) = \omega_1$, $\delta = crit(\pi)$. Note that (3.8) yields that $L_{\bar{\mu}}[A \cap \delta] \models$ “every $A \cap \delta$-admissible ordinal $\geq \delta$ is an L-cardinal”. Since $\bar{\mu}$ is an L-cardinal, we have that

(3.9) $$\text{every } A \cap \delta\text{-admissible } \lambda \in [\delta, \bar{\mu}] \text{ is an } L\text{-cardinal.}$$

This is the key point. Let $\pi(\bar{\nu}) = \nu, \pi(\bar{\mathbb{P}}) = \mathbb{P}$ and $\pi(\bar{D}) = \vec{D}$ with $\bar{D} = (\bar{D}_n|n \in \omega)$.

By (3.5) we may let $(E_i|i < \delta) \in L_{\bar{\mu}}[A \cap \delta]$ be an enumeration of all clubs in δ which exist in $L_{\bar{\nu}}[A \cap \delta]$. Let E be the diagonal intersection of $(E_i|i < \delta)$. Note that $E \setminus E_i$ is bounded in δ for all $i < \delta$. In $L[A]$, let us pick a strictly increasing sequence $(\epsilon_n|n < \omega)$ such that $\{\epsilon_n|n < \omega\} \subseteq E$ and $(\epsilon_n|n < \omega)$ is cofinal in δ.

We want to find a $q \in \mathbb{P}$ such that $q \leq p$, $dom(q) = \delta$, $L_{\bar{\mu}}[A \cap \delta, q] \models$ “δ is countable,” and $q \in \bar{D}_n$ for all $n \in \omega$. For this we construct a sequence $(p_n|n \in \omega)$ of conditions such that $p_0 = p$, $p_{n+1} \leq p_n$ and $p_{n+1} \in \bar{D}_n = D_n \cap L_{\bar{\nu}}[A \cap \delta]$ for all $n \in \omega$. Also we construct a sequence $\{\delta_n|n \in \omega\}$ of ordinals. Suppose $p_n \in L_{\bar{\nu}}[A \cap \delta]$ is given. Let $\gamma = dom(p_n)$. Note that $\gamma < \delta$ since $p_n \in L_{\bar{\nu}}[A \cap \delta]$. Now we work in $L_{\bar{\nu}}[A \cap \delta]$. By extendability, for all ξ with $\gamma \leq \xi < \delta$ we may pick some $p^\xi \leq p_n$ such that $p^\xi \in \bar{D}_n$, $dom(p^\xi) > \xi$, and for all limit ordinals λ with $\gamma \leq \lambda \leq \xi$ we have $p^\xi(\lambda) = 1$ if and only if $\lambda = \xi$. There exists $C \in L_{\bar{\nu}}[A \cap \delta]$ which is a club in δ such that for all $\eta \in C, \xi < \eta$ implies $dom(p^\xi) < \eta$.

Now we work in $L_{\bar{\mu}}[A \cap \delta]$. We may pick some $\eta \in E$, $\eta \geq \epsilon_n$, such that $E \setminus C \subseteq \eta$. Let $p_{n+1} = p^\eta$ and $\delta_n = \eta$. Note that $p_{n+1} \leq p_n$ and $p_{n+1} \in \bar{D}_n$. Also $dom(p_{n+1}) < min(E \setminus (\delta_n + 1))$ so that for all limit ordinals $\lambda \in E \cap (dom(p_{n+1}) \setminus dom(p_n))$, we have $p_{n+1}(\lambda) = 1$ if and only if $\lambda = \delta_n$.

Now let $q = \bigcup_{n \in \omega} p_n$. We need to check that $q \in \mathbb{P}$. Note that $dom(q) = \delta$. By (3.9) it suffices to check that $L_{\bar{\mu}}[A \cap \delta, q] \models \delta$ is countable. From the construction of the p_n's we have $\{\lambda \in E \cap (dom(q) \setminus dom(p))|\lambda$ is a limit ordinal and $q(\lambda) = 1\} = \{\delta_n|n \in \omega\}$, which is cofinal in δ, as $\delta_n \geq \epsilon_n$ for all $n < \omega$. Recall that $E \in L_{\bar{\mu}}[A \cap \delta, q]$. So $\{\delta_n|n \in \omega\} \in L_{\bar{\mu}}[A \cap \delta, q]$ witnesses that δ is countable in $L_{\bar{\mu}}[A \cap \delta, q]$. □

The proof of Claim 3.4 can be adapted to show that $\mathbb{P}$ is stationary preserving, cf. [17].

Forcing with $\mathbb{P}$ adds some $F : \omega_1 \to 2$ such that for all $\alpha < \omega_1$ there exists γ such that $L_\gamma[A \cap \alpha, F \upharpoonright \alpha] \models \alpha$ is countable and every $(A \cap \alpha)$-admissible $\lambda \in [\alpha, \gamma]$ is an L-cardinal; for each $\alpha < \omega_1$ let α^* be the least such γ. Let $D = A \oplus F$. We may assume that for any L-cardinal $\lambda < \omega_1^V$, D restricted to odd ordinals in $[\lambda, \lambda + \omega)$ codes a well ordering of the least L-cardinal $> \lambda$. By Claim 3.4, $L[A][F] = L[D] \models Z_3$.

Now we do almost disjoint forcing over $L[D]$ to code D by a real x. There exists a canonical sequence $(x_\alpha|\alpha < \omega_1)$ of pairwise almost disjoint subset of ω such that x_α is the $L_{\alpha^*}[D \cap \alpha]$-least subset of ω such that x_α is almost disjoint from every member of $\{x_\beta|\beta < \alpha\}$. Almost disjoint forcing adds a real x such that for all

$\alpha < \omega_1$, $\alpha \in D$ if and only if $|x_\alpha \cap x| < \omega$. The forcing has the *c.c.c.*, and thus $L[D][x] = L[x] \models Z_3$.

We finally claim that $L[x] \models \mathsf{HP}$. Suppose α is x-admissible. We show that α is an L-cardinal. If $\alpha \geq \omega_1$, then α is also A-admissible and hence is an L-cardinal by (3.7). Now we assume that $\alpha < \omega_1$ and α is not an L-cardinal. Let λ be the largest L–cardinal $< \alpha$. Recall that for $\xi < \omega_1$, $\xi^* > \xi$ is least such that $L_{\xi^*}[A \cap \xi, F \restriction \xi] \models \xi$ is countable. Every $(D \cap \xi)$-admissible $\lambda' \in [\xi, \xi^*]$ is an L–cardinal.

Case 1: For all $\xi < \lambda + \omega$, $\xi^* < \alpha$. Then $D \cap (\lambda + \omega)$ can be computed inside $L_\alpha[x]$. But then, as α is x–admissible, the ordinal coded by D restricted to the odd ordinals in $[\lambda, \lambda + \omega)$, namely the least L–cardinal $> \lambda$, is in $L_\alpha[x]$, so that $\lambda^{+L} < \alpha$. Contradiction!

Case 2: Not Case 1. Let $\xi < \lambda + \omega$ be least such that $\xi^* \geq \alpha$. Then $D \cap \xi$ can be computed inside $L_\alpha[x]$. As α is x–admissible, α is thus $(D \cap \xi)$–admissible also. But all $(D \cap \xi)$-admissibles $\lambda' \in [\xi, \xi^*]$ are L–cardinals, so that α is an L–cardinal by $\xi < \alpha \leq \xi^*$. Contradiction!

We have shown that $L[x] \models Z_3 + \mathsf{HP}$. □

Corollary 3.5. $Z_3 + \mathsf{HP}$ *does not imply* $0^\sharp$ *exists.*

3.3. $Z_4 +$ **Harrington's Principle implies** $0^\sharp$ **exists.** We construe the following as part of the folklore, cf. [6].

Theorem 3.6. (Z_4) HP *implies* $0^\sharp$ *exists.*

Proof. Let $x \in 2^\omega$ witness HP. Now we work in $L[x]$. Take $\beta > \omega_2$ big enough such that β is x–admissible and ${}^\omega L_\beta[x] \subseteq L_\beta[x]$. Take $X \prec L_\beta[x]$ such that $\omega_2 \in X$, $|X| = \omega_1$, and $X^\omega \subseteq X$. Let $j : L_\theta[x] \cong X \prec L_\beta[x]$ be the collapsing map. Note that $\omega_1 \leq \theta < \omega_2$, θ is x–admissible, and $L_\theta[x]$ is closed under ω-sequences. Let $\kappa = crit(j)$. Define $U = \{A \subseteq \kappa \mid A \in L \wedge \kappa \in j(A)\}$. Since θ is an L-cardinal by the choice of x as witnessing HP, $(\kappa^+)^L \leq \theta < \omega_2$. Therefore, U is an L-ultrafilter on κ.

Let $\alpha = (\kappa^+)^L$. Consider the structure $(L_\alpha, \in, U)$ which is a premouse. Since $L_\theta[x]$ is closed under ω-sequences from $L_\theta[x], U$ is countably complete.[2] So $(L_\alpha, \in, U)$ is iterable. Hence $0^\sharp$ exists. □

So in Z_4, HP is equivalent to $0^\sharp$ exists. In fact in Z_2, $0^\sharp$ exists implies HP. By Corollary 3.5 and Theorem 3.6, we have Z_4 is the minimal system in higher order arithmetic to show that HP and $0^\sharp$ exists are equivalent with each other.

4. Strengthenings of Harrington's Principle over higher order arithmetic

Recall the hypothesis on $\varphi(-)$ as stated in Definition 1.5: $\varphi(-)$ is a Σ_2–formula in the language of set theory such that, provably in ZFC: for all α, if $\varphi(\alpha)$, then α is an inaccessible cardinal and $L \models \varphi(\alpha)$. Let us give some examples of such $\varphi(-)$: κ is inaccessible, Mahlo, weakly compact, Π^n_m-indescribable, totally indescribable, n-subtle, n-ineffable, totally ineffable cardinal, α-iterable ($\alpha < \omega_1^L$), and α-Erdös cardinal ($\alpha < \omega_1^L$). However, κ being reflecting, unfoldable, or remarkable cannot be expressed in a Σ_2 fashion.

[2] I.e. if $\{X_n | n \in \omega\} \subseteq U$, then $\bigcap_{n\in\omega} X_n \neq \emptyset$.

Definition 4.1. Let $\varphi(-)$ be as in Definition 1.5. Let δ be an inaccessible cardinal or $\delta = Ord$. We say that δ is *φ–Mahlo* iff $\{\alpha < \delta | \varphi(\alpha)\}$ is stationary in δ. We say that δ is *2–φ–Mahlo* iff $\{\alpha < \delta | \varphi(\alpha) \wedge \{\beta < \alpha | \varphi(\beta)\}$ is stationary in $\alpha\}$ is stationary in δ.

Notice that we do not require a φ–Mahlo or a 2–φ–Mahlo to satisfy $\varphi(-)$.

4.1. **The strength of $Z_2 +$ $\mathsf{HP}(\varphi)$.**

Theorem 4.2. *Let $\varphi(-)$ be as in Definition 1.5. The following theories are equiconsistent.*

(1) $Z_2 +$ $\mathsf{HP}(\varphi)$, and
(2) ZFC $+ Ord$ is φ–Mahlo.

Proof. Let us first suppose (1), and let $x \in 2^\omega$ be as in $\mathsf{HP}(\varphi)$. There is a club class of x–admissibles, so that $\{\alpha | L \models \varphi(\alpha)\}$ contains a club. Hence $L \models$ "ZFC + $\{\alpha \in Ord | \varphi(\alpha)\}$ is stationary." This shows (2) in L.

Let us now suppose (2). We force over L. Let $S = \{\alpha \in Ord \mid \varphi(\alpha)\}$. Let G be $Col(\omega, < Ord)$-generic over L. Then $L[G] \models Z_2$, and in $L[G]$, S is still stationary, because $Col(\omega, < Ord)$ has the Ord–c.c. We can thus shoot a club through S via $\mathbb{P} = \{p | p$ is a closed set of ordinals and $p \subseteq S\}$. Let H be $\mathbb{P}$–generic over $L[G]$. Standard arguments give that $\mathbb{P}$ is ω-distributive, which implies that $L[G][H] \models Z_2$. Let $C \subseteq S$ be the club added by H. We may pick $A \subseteq Ord$ such that $L[G][H] = L[A]$.

We need to reshape A as follows.[3] Let $p \in \mathbb{R}$ iff $p \colon \alpha \to 2$ for some ordinal α such that for all $\xi \leq \alpha$,

$$L_{\xi+1}[A \cap \xi, p \restriction \xi] \models \xi \text{ is countable.}$$

We claim that $\mathbb{R}$ is ω–distributive. To see this, let $(D_n | n < \omega)$ be a, say, Σ_m–definable sequence of open dense classes, and let $p \in \mathbb{R}$. Let E be the class of all β such that $L_\beta[G][H] \prec_{\Sigma_{m+5}} L[G][H]$ and p as well as the parameters defining $(D_n | n < \omega)$ are all in $L_\beta[G][H]$. E is club, and we may let α be the ω^{th} element of E. Then $E \cap \alpha$ is Σ_{m+6}–definable over $L_\alpha[G][H]$ and cofinal in α, so that α has cofinality ω in $L_{\alpha+1}[G][H]$. A much simplified variant of the argument from Claim 3.4, which we will leave as an exercise to the reader, then produces some $q \in \mathbb{R}$ with $q \leq p$, $q \colon \alpha \to 2$, and $q \in \bigcap_{n<\omega} D_n$.

Let K be $\mathbb{R}$–generic over $L[G][H]$. In $L[G][H][K]$, we may then pick some $B \subseteq Ord$ such that $L[G][H][K] = L[B]$, if $\lambda \in C \setminus (\omega + 1)$, then $B \cap [\lambda, \lambda + \omega)$, restricted to the odd ordinals, codes a well ordering of $\min(C \setminus (\lambda + 1))$, and for all $\alpha \geq \omega$,

$$L_{\alpha+1}[B \cap \alpha] \models \alpha \text{ is countable.} \tag{4.1}$$

We may now continue as in the proof of Theorem 3.1.

We do standard almost disjoint forcing to add a real x such that if $(c_\alpha | \alpha \in Ord)$ is the canonical sequence of pairwise almost disjoint subsets of ω given by (4.1), then for any $\alpha \in Ord$, $\alpha \in B \Leftrightarrow |x \cap c_\alpha| < \omega$. In particular, $L[B][x] = L[x]$. This forcing is *c.c.c.*, so that also $L[x] \models Z_2$.

We claim that in $L[x]$, $\mathsf{HP}(\varphi)$ holds true. It suffices to show that if α is x-admissible, then $\alpha \in C$. Suppose α is x-admissible but $\alpha \notin C$. Let λ be the largest element of C such that $\lambda < \alpha$. Note that we can define $B \cap \alpha$ over $L_\alpha[x]$. Since

[3] In the proof of Theorem 3.1 there was no need for reshaping due to (3.2).

$B \cap [\lambda, \lambda+\omega) \in L_\alpha[x]$ and $B \cap [\lambda, \lambda+\omega)$, restricted to the odd ordinals, codes a well ordering of $\min(C \setminus (\lambda+1))$, we have $\min(C \setminus (\lambda+1)) \in L_\alpha[x]$, because α is x–admissible. But $\min(C \setminus (\lambda+1)) > \alpha$. Contradiction! So $L[x] \models Z_2 +$ $\mathsf{HP}(\varphi)$. □

4.2. **The strength of $Z_3 +$ $\mathsf{HP}(\varphi)$.**

Definition 4.3. ([10])

(1) Let N be transitive. N is *full* if and only if $\omega \in N$ and there is γ such that $L_\gamma(N) \models ZFC^-$ and N is regular in $L_\gamma(N)$, i.e., if $f : x \to N, x \in N$, and $f \in L_\gamma(N)$, then $ran(f) \in N$.
(2) Let $\mathbb{B}$ be a complete Boolean algebra. Let $\delta(\mathbb{B})$ be the smallest cardinality of a set which lies dense in $\mathbb{B} \setminus \{0\}$.
(3) Let $N = L^A_\gamma = (L_\gamma[A], \in, A \cap L_\gamma[A])$ be a model of ZFC^-. Let $X \cup \{\delta\} \subseteq N$. Define $C^N_\delta(X) =$ the smallest $Y \prec N$ such that $X \cup \{\delta\} \subseteq Y$.

Definition 4.4. ([10, p. 31]) Let $\mathbb{B}$ be a complete Boolean algebra. $\mathbb{B}$ is a *subcomplete* forcing if and only if for sufficiently large cardinals θ we have: $\mathbb{B} \in H_\theta$ and for any ZFC^- model $N = L^A_\tau$ such that $\theta < \tau$ and $H_\theta \subseteq N$ we have: Let $\sigma : \bar{N} \to N$ where $\bar{N}$ is countable and full. Let $\sigma(\bar{\theta}, \bar{s}, \bar{\mathbb{B}}) = \theta, s, \mathbb{B}$ where $\bar{s} \in \bar{N}$. Let $\bar{G}$ be $\bar{\mathbb{B}}$-generic over $\bar{N}$. Then there is $b \in \mathbb{B} \setminus \{0\}$ such that whenever G is $\mathbb{B}$-generic over V with $b \in G$, there is $\sigma' \in V[G]$ such that

(a) $\sigma' : \bar{N} \to N$,
(b) $\sigma'(\bar{\theta}, \bar{s}, \bar{\mathbb{B}}) = \theta, s, \mathbb{B}$,
(c) $C^N_\delta(ran(\sigma')) = C^N_\delta(ran(\sigma))$ where $\delta = \delta(\mathbb{B})$,
(d) $\sigma'" \bar{G} \subseteq G$.

By [10], cf. also [9], subcomplete forcings add no reals and are closed under Revised Countable Support (RCS) iterations subject to the usual constraints (see [10, Theorem 3, p. 56]). In the following, we give some examples of forcing notions which are subcomplete that will be used in this paper.

The set $\omega_2^{<\omega}$ of monotone finite sequences in ω_2 is a tree ordered by inclusion. Namba forcing is the collection of all subtrees $T \neq \emptyset$ of $\omega_2^{<\omega}$ with a unique stem, stem(T), such that every element of T is compatible with stem(T), and every element extending stem(T) has ω_2 immediate successors in T. The order is defined by: $T \leq \bar{T}$ if and only if $T \subseteq \bar{T}$. If G is generic for Namba forcing, then $S = \bigcup \bigcap G$ is a cofinal map of ω into ω_2^V. We call any such S a Namba sequence. Namba forcing is stationary set preserving and adds no reals if CH holds.

Fact 4.5. ([10], Lemma 6.2) Assume CH. Then Namba forcing is subcomplete.

Definition 4.6. Suppose κ is a cardinal or $\kappa = Ord$. Define $Club(\kappa, S) = \{p | p\colon \alpha + 1 \to S$ for some $\alpha < \kappa$ and p is increasing and continuous$\}$. The extension relation is defined by: $p \leq q$ if and only if $p \supseteq q$.

The forcing $Club(\omega_1, S)$ has been used in the proof of Thorem 3.1. If G is $Club(\omega_1, S)$-generic, then $\bigcup G : \omega_1 \to S$ is increasing, continuous and cofinal in S.

Fact 4.7. ([10, Lemma 6.3]) Let $\kappa > \omega_1$ be a regular cardinal. Let $S \subseteq \kappa$ be a stationary set. Then $Club(\omega_1, S)$ is subcomplete.

Lemma 4.8. ([3, Lemma 18.6]) *Suppose CH holds and $S \subseteq \omega_2$ is such that $\{\alpha \in S \cap cf(\omega_1) |$ there exists $C \subseteq S \cap \alpha$ such that C is a club in $\alpha\}$ is stationary. Then $Club(\omega_2, S)$ is ω_1–distributive.*

Theorem 4.9. *The following two theories are equiconsistent:*

(1) $ZFC+$ there is a remarkable cardinal κ with $\varphi(\kappa)$ + Ord is 2–φ–Mahlo.
(2) Z_3 + $HP(\varphi)$.

Proof. We first prove that (2) implies that (1) holds in L. As $\mathsf{HP}(\varphi)$ implies HP, Theorem 3.2 gives that Z_3+ $\mathsf{HP}(\varphi)$ implies $L \models ZFC + \omega_1^V$ is remarkable. Let $x \in 2^\omega$ witness $\mathsf{HP}(\varphi)$. As ω_1^V is x–admissible, $\varphi(\omega_1^V)$ holds true in L.

There is a club of x–admissibles, so that we may pick some club $C \subseteq \{\alpha \in Ord \mid L \models \varphi(\alpha)\}$. Suppose D is a club in L. Pick α in $C \cap D$ of cofinality ω_1 such that α is a limit point of $C \cap D$. Since $\alpha \in C, L \models \varphi(\alpha)$. We want to see that $\{\beta < \alpha \mid L \models \varphi(\beta)\}$ is stationary in L. Let $E \subseteq \alpha$ in L be a club in α. Note that $E \cap C \cap \alpha \neq \emptyset$. If $\beta \in E \cap C \cap \alpha$, then $L \models \varphi(\beta)$. Hence Ord is 2–φ–Mahlo in L.

Now we show that consistency of (1) implies consistency of (2). We force over L. Suppose that (1) holds in L.

Let H be $Col(\omega, <\kappa)$-generic over L.

Claim 4.10. $\{\alpha < \kappa \colon L \models \varphi(\alpha)\}$ is stationary in $L[H]$.

Proof. We work in $L[H]$. Let $C \subset \kappa = \omega_1^{L[H]}$ be club, and let $L_\theta \models \varphi(\kappa)$, where $\theta > \kappa$ is regular. As κ is remarkable, there is some $\sigma \colon L_{\bar\theta}[H \cap L_\alpha] \to L_\theta[H]$ such that $\alpha = crit(\sigma)$, $\sigma(\alpha) = \kappa$, $C \in \operatorname{ran}(\sigma)$, and $\bar\theta$ is a regular cardinal in L. By elementarity, $L_{\bar\theta} \models \varphi(\alpha)$, which implies that $L \models \varphi(\alpha)$, as φ is Σ_2. But $\alpha \in C$. □

Let H be $Col(\omega, <\kappa)$-generic over L. Over $L[H]$, we define a class RCS-iteration $((P_\alpha, \dot{Q}_\alpha)|\alpha \in Ord)$ as follows. We let $P_0 = \emptyset, P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$ for $\alpha \in Ord$ and for limit ordinal α we let P_α be the revised limit (Rlim) of $((P_\beta, \dot{Q}_\beta)|\beta \in \alpha)$. The definition of Q_α splits into three cases as follows.

Let

(0) $S_0 = \{\alpha | L \models \neg\varphi(\alpha)\}$,
(1) $S_1 = \{\alpha | L \models \varphi(\alpha)$, but $\{\beta < \alpha | \varphi(\beta)\}$ is not stationary in L $\}$, and
(2) $S_2 = \{\alpha | L \models \varphi(\alpha)$, and $\{\beta < \alpha | \varphi(\beta)\}$ is stationary in L $\}$.

Case 0. If $\alpha \in S_0$, then let $Q_\alpha = Col(\omega_1, 2^{\omega_1})$ which collapses 2^{ω_1} to ω_1 by countable conditions.

Case 1. If $\alpha \in S_1$, then let $Q_\alpha =$ Namba forcing.

Case 2. If $\alpha \in S_2$, then let $Q_\alpha = Club(\omega_1, S_1 \cap \alpha)$.

Note that if $L \models \varphi(\alpha)$, then $L^{Col(\omega,<\kappa)*P_\alpha} \models \alpha = \omega_2$ since $Col(\omega, <\kappa) * P_\alpha$ has the α-c.c. This also implies that $S_1 \cap \alpha$ is stationary in $L^{Col(\omega,<\kappa)*P_\alpha}$. Moreover, in $L^{Col(\omega,<\kappa)*P_\alpha}$, $S_1 \cap \alpha$ consists of points of cofinality of ω. So it makes sense to shoot a club subset of α with order type ω_1 through $S_1 \cap \alpha$.

Finally let $\mathbb{P}$ be the revised limit of $((P_\alpha, \dot{Q}_\alpha)|\alpha \in Ord)$. By Facts 4.5 and 4.7 and by [10, Theorem 3, p. 56], P_α is subcomplete for all $\alpha \in Ord$. Standard arguments give us that $\mathbb{P}$ has the Ord-c.c. Hence $\mathbb{P}$ does not add reals and ω_1 is preserved. Let G be $\mathbb{P}$-generic over $L[H]$. $L[H, G] \models Z_3$. The following is stated for the record.

Claim 4.11. In $L[H][G]$, if $\alpha \in S_1$, then $cf(\alpha) = \omega$, and if $\alpha \in S_2$, then $cf(\alpha) = \omega_1$ and there is a club in α of order type ω_1 contained in $S_1 \cap \alpha$.

For each L-cardinal $\mu > \omega_1$, we again let $S_\mu = \{X \prec L_\mu | X$ is countable and $o.t.(X \cap \mu)$ is an L-cardinal$\}$, as being defined in the respective models of set theory which are to be considered.

The following proof shows that subcomplete forcings preserve the stationarity of S_μ.

Claim 4.12. In $L[H,G]$, for each L-cardinal $\mu > \omega_1$, S_μ as defined in $L[H,G]$ is stationary.

Proof. Fix an L-cardinal $\mu > \omega_1$. Suppose S_μ is not stationary in $L[G,H]$. Then there are $p \in P_\alpha$ and $\tau \in L[H]^{P_\alpha}$ for some α such that $p \Vdash^{P_\alpha}_{L[H]}$ "$\tau : [\check{\mu}]^{<\omega} \to \check{\mu}$ and there is no countable $X \subseteq \check{\mu}$ such that X is closed under τ and $o.t.(X)$ is an L-cardinal." Let μ^* be an L–cardinal which is bigger than μ. Let $\sigma : N \to L_{\mu^*}[H]$ where N is countable, transitive and full, such that P_α, p, μ, $\tau \in N$. Let $\sigma(\bar{P}, \delta, \bar{p}, \bar{\mu}, \bar{\tau}) = P_\alpha, \omega_1, p, \mu, \tau$. Let us write $N = L_\gamma[H \restriction \delta]$.

Because κ was remarkable in L, cf. Lemma 2.3, may assume that N was picked in such a way that γ is an L-cardinal. Let $\bar{G}$ be $\bar{P}$-generic over $L_\gamma[H \restriction \delta]$ with $\bar{p} \in \bar{G}$. Since P_α is subcomplete, by the definition of subcompleteness, there is $p^* \in P_\alpha$, $p^* \leq p$, such that whenever G^* is P_α-generic over $L[H]$ with $p^* \in G^*$, then there is $\sigma' \in L[H][G^*]$ such that $\sigma' : L_\gamma[H \restriction \delta][\bar{G}] \to L_\mu[H][G^*]$ and $\sigma'(\bar{P}, \delta, \bar{p}, \bar{\mu}, \bar{\tau}) = P_\alpha, \omega_1, p, \mu, \tau$.

Since $p \in G^*$, there is no countable $X \subseteq \mu$ such that X is closed under τ^{G^*} and $o.t.(X)$ is an L-cardinal. But $ran(\sigma') \cap \mu$ is countable, closed under τ^{G^*} and $o.t.(ran(\sigma') \cap \mu) = \gamma$ is an L-cardinal. Contradiction! □

We now let $\mathbb{Q} = Club(Ord, S_1 \cup S_2)$. The proof of the following Claim imitates the proof of Lemma 4.8.

Claim 4.13. $\mathbb{Q}$ is ω_1–distributive.

Proof. In $L[H,G]$, S_2 is stationary and CH holds. Suppose $\vec{D} = (D_i | i < \omega_1)$ is a, say Σ_m–, definable sequence of open dense classes. Pick $M \prec_{\Sigma_{m+5}} V$ such that M contains the parameters needed in the definition of $\vec{D}$, $M^\omega \subseteq M$, and $M \cap Ord \in S_2$.

Let us write $\delta = M \cap Ord$. By Claim 4.11, we may pick some $C \subseteq S_1 \cap \delta$, a club in δ. Now we can simultaneously build a descending sequence $(p_i | i \leq \omega_1)$ with $p_0 = p$ and a continuous tower $(M_i | i \leq \omega_1)$ of countable elementary substructures of M with $M_{\omega_1} = M$ such that for all $i < \omega_1$ we have:

(a) $p_i \in M_{i+1}$,
(b) $p_{i+1} \in D_i$ and $p_{i+1}(\max(\mathrm{dom}(p_{i+1}))) > \sup(M_i \cap Ord)$,
(c) $\sup(M_i \cap Ord) \in C$, and
(d) if $i < \omega_1$ is a limit ordinal, then $p_i \restriction \max(\mathrm{dom}(p_i)) = \bigcup_{j<i} p_j$ and hence $p_i(\max(\mathrm{dom}(p_i))) = \sup(M_i \cap Ord) \in C$.

Then $p_{\omega_1} \leq p$ and $p_{\omega_1} \in \bigcap_{i<\omega_1} D_i$. □

Let I be $\mathbb{Q}$-generic over $L[H,G]$, and let $C \subseteq S_1 \cup S_2$ be the club added by I. By Claim 4.13, $L[H,G,I] \models Z_3$. As in the proof of Theorem 3.2, we can pick $B \subseteq Ord$ such that $L[H,G,I] = L[B]$ and for any $\alpha \in C$, B restricted to the odd ordinals in $[\alpha, \alpha + \omega_1)$ codes a well ordering of $\min(C \setminus (\alpha + 1))$.

We now reshape as follows.[4]

Definition 4.14. Define $p \in \mathbb{S}$ if and only if $p : \alpha \to 2$ for some α and for any $\xi \leq \alpha, L_{\xi+1}[B \cap \xi, p \restriction \xi] \models |\xi| \leq \omega_1$.

[4]In the proof of Theorem 3.2 there was no need for reshaping at this point due to (3.3).

Claim 4.15. $\mathbb{S}$ is ω_1-distributive.

Proof. Let $\vec{D} = (D_i | i < \omega_1)$ be a sequence of open dense subclass of $\mathbb{S}$. Let $p \in \mathbb{S}$. We want to find p_{ω_1} such that $p_{\omega_1} \in \bigcap_{i<\omega_1} D_i$ and $p_{\omega_1} \leq p$. Say $\vec{D}$ is Σ_m-definable in $L[B]$ with parameters $\bar{s}$. Let $(\beta_i | i \leq \omega_1)$ the the first $\omega_1 + 1$ many β such that $L_\beta \prec_{\Sigma_{m+5}} L[B]$ and $\omega_1 + 1 \cup \{\bar{s}\} \subseteq L_\beta[B]$. For every $i \leq \omega_1$, $(\beta_j | j < i)$ is Σ_{m+6}–definable over $L_{\beta_i}[B]$ and hence $(\beta_j | j < i) \in L_{\beta_i+1}[B]$. So for $i \leq \omega_1, L_{\beta_i+1}[B] \models \beta_i$ is singular.

Now we define $(p_i | i \leq \omega_1)$ by induction as follows. Let $p_0 = p$. Given $p_n \in \mathbb{S}$, take $p_{n+1} \in \mathbb{S}$ such that $p_{n+1} \in D_n \cap X_{n+1}, p_{n+1} \leq p_n$ and $dom(p_{n+1}) \geq \beta_n$. Let $p_{\omega_1} = \bigcup_{i<\omega_1} p_i$. Note that $dom(p_{\omega_1}) = \beta_{\omega_1}$, $p_{\omega_1} \in \mathbb{S}$, in fact $p_{\omega_1} \in \bigcap_{i<\omega_1} D_i$, and $p_{\omega_1} \leq p$. □

By forcing with $\mathbb{S}$ over $L[H, G, I]$, we get $\bar{B} \subseteq Ord$ such that for any $\alpha \in Ord$, $L_{\alpha+1}[B \cap \alpha, \bar{B} \cap \alpha] \models |\alpha| \leq \omega_1$. Let $E = B \oplus \bar{B}$. Of course, $L[E] \models Z_3$, and for any $\alpha \in Ord$, $L_{\alpha+1}[E \cap \alpha] \models |\alpha| \leq \omega_1$. We also have that for all $\alpha \in C$, E restricted to the odd ordinals in $[\alpha, \alpha + \omega_1)$ codes a well ordering of $\min(C \setminus (\alpha + 1))$.

By Claims 4.13 and 4.15, $L[H, G]$ and $L[E]$ have the same sets. Therefore, trivially, Claim 4.12 is still true with $L[E]$ replacing $L[H, G]$.

Exactly as in the proof of Theorem 3.2 we can do almost disjoint forcing to add $A \subseteq \omega_1$ to code E. Note that $L[E][A] = L[A]$ and the forcing we use to add A is countably closed and *Ord-c.c.*. Since $L[E] \models Z_3$, $L[A] \models Z_3$. By the countable closure, Claim 4.12 is still true with $L[A]$ replacing $L[H, G]$.

By the same argument as in Theorem 3.2 we can show that if $\alpha > \omega_1$ is A-admissible then $\alpha \in C$, and hence $L \models \varphi(\alpha)$. By our hypothesis on κ, $L \models \varphi(\kappa)$, so that if fact if $\alpha \geq \omega_1$ is A-admissible then $L \models \varphi(\alpha)$.

Now we do reshaping over $L[A]$ as follows.

Definition 4.16. Define $p \in \mathbb{R}$ if and only if $p : \alpha \to 2$ for some $\alpha < \omega_1$ and $\forall \xi \leq \alpha \, \exists \gamma \, (L_\gamma[A \cap \xi, p \upharpoonright \xi] \models$ "ξ is countable" and if $\lambda \in [\xi, \gamma]$ is $(A \cap \xi)$-admissible, then $L \models \varphi(\lambda))$.

Claim 4.17. $\mathbb{R}$ is ω-distributive.

Proof. Recall that for each L-cardinal $\mu > \omega_1$, we defined $S_\mu = \{X \prec L_\mu | X$ is countable and $o.t.(X \cap \mu)$ is an L-cardinal $\}$. We shall use the fact that in $L[A]$, S_μ as defined in $L[A]$ is stationary.

In fact, essentially the same argument as in the proof of Claim 3.4 shows that $\mathbb{R}$ is ω–distributive. In the following we only point out the place we use φ is Σ_2 in our argument.

Let $p \in \mathbb{R}$ and $\vec{D} = (\bar{D}_n | n \in \omega)$ be a sequence of open dense sets. Pick large enough L-cardinal μ such that $\vec{D} \in L_\mu[A]$ and $L_\mu[A] \models$ "if $\alpha \geq \omega_1$ is A-admissible, then $L \models \varphi(\alpha)$". As S_μ is stationary, we can pick X such that $\pi : L_{\bar{\mu}}[A \cap \delta] \cong X \prec L_\mu[A], |X| = \omega, \{p, \mathcal{P}, A, \vec{D}, \omega_1, \nu\} \subseteq X$ and $\bar{\mu}$ is an L-cardinal where $\pi(\delta) = \omega_1 (\delta = X \cap \omega_1)$. Note that by elementarity, $L_{\bar{\mu}}[A \cap \delta] \models$ "if $\alpha \geq \delta$ is $A \cap \delta$-admissible, then $L \models \varphi(\alpha)$". Suppose $\alpha \in [\delta, \bar{\mu})$ is $A \cap \delta$-admissible. Then $L_{\bar{\mu}} \models \varphi(\alpha)$. Since $\bar{\mu}$ is an L-cardinal and φ is Σ_2, $L \models \varphi(\alpha)$. The rest of the arguments are the same as in the proof of Claim 3.4. □

Using Claim 4.10, a simple variant of the previous proof also shows the following.

Claim 4.18. $\{\alpha < \kappa \colon L \models \varphi(\alpha)\}$ is stationary in $L[A]^{\mathbb{R}}$.

Forcing with $\mathbb{R}$ adds $F : \omega_1 \to 2$ such that for all $\alpha < \omega_1$ there exists γ such that $L_\gamma[A \cap \alpha, F \restriction \alpha] \models \alpha$ is countable and every $(A \cap \alpha)$-admissible $\lambda \in [\alpha, \gamma]$ satisfies that $L \models \varphi(\lambda)$. Using Claim 4.10, we may force over $L[A, F]$ and shoot a club C^* through $\{\alpha < \kappa \colon L \models \varphi(\alpha)\}$ in the standard way. Let $D = A \oplus F \oplus C^*$. We may assume that for $\lambda \in C^*$, D restricted to odd ordinals in $[\lambda, \lambda + \omega)$ codes a well ordering of $\min(C^* \setminus (\lambda + 1))$. Since $\mathbb{R}$ and the club shooting adding C^* are ω–distributive, it is easy to see that $L[D] \models Z_3$.

Now we work in $L[D]$. Do almost disjoint forcing to code D by a real x. This forcing is *c.c.c.* Note that $L[D][x] = L[x]$, and $L[x] \models Z_3$.

Now we work in $L[x]$. Suppose α is x-admissible. We show that $L \models \varphi(\alpha)$. If $\alpha \geq \omega_1$, then α is also A-admissible and hence $L \models \varphi(\alpha)$. Now we assume that $\alpha < \omega_1$ and $L \not\models \varphi(\alpha)$. Then $\alpha \notin C^*$. Let $\lambda < \alpha$ be the largest element of C^* which is smaller than α and $\bar{\lambda} = \min(C \setminus (\alpha + 1)) > \alpha$. For every $\xi < \omega_1$, let $\xi^* > \xi$ be least such that $L_{\xi^*}[A \cap \xi, F \restriction \xi] \models \xi$ is countable. By the properties of F, every $(D \cap \xi)$-admissible $\lambda' \in [\xi, \xi^*]$ satisfies $L \models \varphi(\lambda')$.

Case 1: For all $\xi < \lambda + \omega$, $\xi^* < \alpha$. Then $D \cap (\lambda + \omega)$ can be computed inside $L_\alpha[x]$. But then, as α is x–admissible, the ordinal coded by D restricted to the odd ordinals in $[\lambda, \lambda + \omega)$, namely $\bar{\lambda}$, is in $L_\alpha[x]$, so that $\bar{\lambda} < \alpha$. Contradiction!

Case 2: Not Case 1. Let $\xi < \lambda + \omega$ be least such that $\xi^* \geq \alpha$. Then $D \cap \xi$ can be computed inside $L_\alpha[x]$. As α is x–admissible, α is thus $(D \cap \xi)$–admissible also. But all $(D \cap \xi)$-admissibles $\lambda' \in [\xi, \xi^*]$ satisfy $L \models \varphi(\lambda')$, so that $L \models \varphi(\alpha)$ by $\xi < \alpha \leq \xi^*$. Contradiction!

We have shown that $L[x] \models Z_3 + \mathsf{HP}(\varphi)$. □

Corollary 4.19. *$Z_3 + \mathsf{HP}(\varphi)$ does not imply $0^\sharp$ exists.*

By Theorem 3.6, $Z_4 + \mathsf{HP}(\varphi)$ implies $0^\sharp$ exists. As a corollary, Z_4 is the minimal system of higher order arithmetic to show that HP, $\mathsf{HP}(\varphi)$, and $0^\sharp$ exists are equivalent with each other.

Hugh Woodin conjectures that "$Det(\Sigma^1_1)$ implies $0^\sharp$ exists" can be proven in Z_2.

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