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Stochastic quantisation of scalar QFT on noncommutative spaces

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CRC 1442 GEOMETRY: DEFORMATIONS AND RIGIDITY

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Quantum fields

Quantum field theory (QFT) is defined in terms of axioms of [Wightman 56], [Haag, Kastler 60], [Osterwalder, Schrader 74] or [Atiyah, Segal 89].

- Wightman and Osterwalder–Schrader agree that quantum fields ϕ are distributions. Haag–Kastler avoid quantum fields, Atiyah–Segal have different target.
- Non-linear constructs of quantum fields such as $\lambda\phi^n$ not naïvely defined; difficulties grow with dimension D of space(-time).

Stochastic quantisation [Parisi, Wu 81]

- Schwinger functions as moments of a measure are challenging.
- Pretend ergodicity and construct instead averages in a fictitious time t, for $t \to \infty$.
- Dynamics (here of $\lambda\phi^4$-model) governed by non-linear stochastic PDE

$$\partial_t \phi(t,x) = (\Delta - m^2)\phi(t,x) - \lambda{:}\phi^3(t,x){:} + \xi(t,x)$$

where t – fictitious time, Δ – Laplacian in D dimensions, ξ – white noise.

[Da Prato, Debussche 03] approach in $D = 2$ dimensions

Set $\phi = z + v$, where z solves heat equation $\partial_t z(t,x) = (\Delta - m^2)z(t,x) + \xi(t,x)$.

- Solution z is well understood; distribution of regularity $\frac{2-D}{2} - \epsilon$.
- PDE for remainder (without noise!), which is renormalised:

$$\partial_t v = (\Delta - m^2)v - \lambda(v^3 + 3v^2 z + 3v{:}z^2{:} + {:}z^3{:})$$

  where $:z^2 := z^2 - \mathbb{E}(z^2)$ and $:z^3 := z^3 - 3\mathbb{E}(z^2)z$ is normal ordering.
- One proves short-time existence+uniqueness by a fixed point argument and a priori bounds for long-time existence.

In $D > 2$ more work is necessary:

- regularity structures [Hairer 14]
- paracontrolled distributions [Gubinelli, Imkeller, Perkowski 15]
- plus "coming down from infinity" [Morrat, Weber 17]

Works up to $D = 4 - \epsilon$, but not $D = 4$, which is trivial [Aizenman, Duminil-Copin 21].

Aim

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We construct the $\lambda\phi^{*4}$-model on 2D Moyal space by stochastic quantisation.

- Main motivation: This model should exist even in dimension $D = 4$ (details in final part of this talk).
- Strategy: Work in matrix basis, adapt the [Da Prato, Debussche 03] approach to matrix setting and corresponding spaces of distributions.

Previous work:

- [Zhituo Wang 18] by multiscale loop vertex expansion (due to [Gurau, Rivasseau 14]); constructs free energy density for λ in a cardioid domain.
- [Chandra, Ferdinand 23] tensor models on torus; have translation invariance so that Besov spaces can be used.

Our construction is non-perturbative; works for any $\lambda > 0$.

$\lambda\phi^4$ on 2D Moyal space with harmonic propagation

Recall: Moyal product on Schwartz functions

$$(f \star g)(x) = \int_{\mathbb{R}^{2D}} \frac{dk\, dy}{(2\pi)^D}\, f(x + \tfrac{1}{2}\Theta k) g(x+y) e^{\mathrm{i}\langle k,y\rangle}$$

- There exists matrix basis $\{b_{mn}(x)\}$ satisfying $(b_{mn} \star b_{kl})(x) = \delta_{nk} b_{ml}(x)$ and $\int dx\, b_{mn}(x) = \sqrt{\det(2\pi\Theta)}\delta_{mn}$
- Turns $\phi^{\star n}$ into matrix product, integral into trace
- Kinetic term is band matrix, diagonalisable by Meixner polynomials.
- At RG-fixed point of 4D-model, it is exactly diagonal. We place ourselves at this point.

Stochastic quantisation equation

$$\partial_t \phi_{mn} = -A_{mn}\phi_{mn} - \lambda(\phi^3)_{mn} + \xi_{mn} \qquad \text{where}$$

- $A_{mn} = m + n + 1$ (up to irrelevant positive constants depending on mass and θ); we write $(\mathrm{H}v)_{mn} = A_{mn}v_{mn}$.
- ξ white noise (i.i.d. up to $\overline{\xi_{mn}} = \xi_{nm}$)

Banach spaces and techniques

At fixed time:

- H^α – sequences with norm $\|c(t)\|_{H^\alpha} = \big(\sum_{m,n=0}^{\infty} A_{mn}^{2\alpha}|c_{mn}(t)|^2\big)^{1/2}$
- M^p – sequences with norm $\|c(t)\|_{M^p} = \sup_{m,n} A_{mn}^p |c_{mn}(t)|$

Uniformly in $[0, T]$:

- $C_T H^\alpha$ – continuous sequences in $t \in [0, T]$ with norm $\sup_{t\in[0,T]} \|c(t)\|_{H^\alpha}$
- K_T^β – sequences in $t \in [0, T]$ with norm $\sup_{t\in[0,T]} t^\beta \|c(t)\|_{H^\beta} + \sup_{t\in[0,T]} \|c(t)\|_{H^0}$
- $C_T M^p$ – sequences in $t \in [0, T]$ with norm $\|c\|_{C_T M^p} = \sup_{t\in[0,T]} \|c\|_{M^p}$

Main techniques

- Schauder theory: heat operator improves regularity α by $1-\epsilon$
- Gaußian hypercontractivity: For ψ a homogeneous Wick power of order n and $1 < p < q < \infty$, one has $\|\psi\|_p \leq \|\psi\|_q \leq \big(\frac{q-1}{p-1}\big)^{n/2} \|\psi\|_p$
- Cauchy–Schwarz inequalities
- interpolation inequalities $\|\phi\|_{H^\alpha} \leq \|\phi\|_{H^{\alpha_1}}^{\beta} \|\phi\|_{H^{\alpha_2}}^{1-\beta}$ for $\alpha = \beta\alpha_1 + (1-\beta)\alpha_2$

The free field $\partial_t z_{mn} = -A_{mn} z_{mn} + \xi_{mn}$

collection of Ornstein-Uhlenbeck processes with correlation $\mathbb{E}[z_{mn}(s) z_{kl}(t)] = \frac{\delta_{ml}\delta_{nk}}{A_{mn}} e^{-A_{mn}|s-t|}$

- corresponding paths belong to $C_T H^{-1/2-\epsilon}$ and almost surely to $C_T M^{1/2-\epsilon}$
- $(\mathbb{E}[\|z\|^p_{C_T H^{-1/2-\epsilon}}])^{1/p} < \infty$ for large p
- gives logarithmically divergent $\mathbb{E}[(z^2)_{ml}] = \delta_{ml} \sum_{k=0}^{\infty} \frac{1}{m+k+1}$

Regularise noise $\xi^{(N)}$ and distribution $z^{(N)}$ via cut-off N of matrix size:

Proposition

$$:((z^{(N)})^2)_{ml}: \;= ((z^{(N)})^2)_{ml} - \mathbb{E}[((z^{(N)})^2)_{ml}] \qquad \text{and}$$

$$:((z^{(N)})^3)_{ml}: \;= ((z^{(N)})^3)_{ml} - \sum_{0\le k\le N} \mathbb{E}[((z^{(N)})^2)_{mk}] z^{(N)}_{kl} - \sum_{0\le k\le N} z^{(N)}_{mk} \mathbb{E}[((z^{(N)})^2)_{kl}]$$

Cauchy sequences in $L^p(\mathbb{P}, C_T H^{-1/2-\epsilon})$; limit defines distributions

Note that $\sum_{k,n=0}^{N} z^{(N)}_{kn} \mathbb{E}[z^{(N)}_{mk} z^{(N)}_{nl}]$ (expected in a commutative world) is not subtracted!

Renormalised remainder equation for $v = \phi - z$

$$\partial_t v = -\mathrm{H}v - \lambda(v^3 + vvz + vzv + zvv + v{:}z^2{:} + zvz + {:}z^2{:}v + {:}z^3{:})$$

- We cannot renormalise zvz by any reasonable operator, similarly as we couldn't include a corresponding term in ${:}z^3{:}$ [remark: For $-\Delta$ instead of H, this term produces UV/IR].
- Moreover, a naïve estimate $\|zvz\| \leq \|v\|\|z\|^2$ does not work! We need to take care of stochastic cancellations to control zvz; this is a main challenge in our work.

View $v \mapsto zvz$ as linear operator $\mathcal{N}_z : H^\alpha \to H^\beta$.

Lemma

$$\|\mathcal{N}_z(t)\|_{\mathcal{L}(H^\alpha,H^\beta)} \leq \Big(\sum_{i,j,k,l} \frac{1}{A_{ij}^{2\alpha} A_{kl}^{2\alpha}} \Big| \sum_{m,n} A_{mn}^{2\beta} z_{mk}(t) z_{ln}(t) z_{nj}(t) z_{im}(t) \Big|^2 \Big)^{1/4}$$

Proposition

For $\alpha = 1/2 - \epsilon$ and $\beta = 0 - \epsilon - \epsilon'$ we have $\mathbb{E}[\|\mathcal{N}_z(t)\|_{\mathcal{L}(H^\alpha,H^\beta)}] \leq C$ for time-independent C.

(amounts to check 105 terms)

Local existence

Local existence theorem

For any initial value $v(0) \in H^0$ there exists a random time T, which depends on the initial data $\|v(0)\|_{H^0}$ and on $z, {:}z^2{:}, {:}z^3{:}$, such that the renormalised remainder equation has a unique solution up to time T in the space $K_T^{1/2-\epsilon}$ almost surely.

Proof by fixed point method (Picard iteration), i.e. reformulation into integral equation

$$v(t) = e^{-\mathrm{H}t} v(0) - \lambda \int_0^t ds\, e^{-\mathrm{H}(t-s)} \big(v^3 + vvz + vzv + zvv + v{:}z^2{:} + zvz + {:}z^2{:}v + {:}z^3{:}\big)(s)$$

- Use previously established bounds and Schauder estimate for integral operator.
- After recentering, resulting map $\Psi : K_T^{1/2-\epsilon} \to K_T^{1/2-\epsilon}$ is contraction inside ball in $K_T^{1/2-\epsilon}$.

A priori estimate

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Start from initial data at $t = t_0$, run evolution up to time $t_1 = t_0 + T$, and iterate.

- There is the danger that, at every step, the time $T(t_n)$ shrinks and (t_n) has a finite limit. To rule this out one needs a priori estimates for $\|v(t)\|$.
- Such estimates are obtained by multiplying the remainder equation by v and controlling $\partial_t \|v\|^2$. Here this fails with the term $\|{:}z^3{:}v\|$.

We thus consider a second-order remainder $\phi = z + y + w$ where y solves $\partial_t y = -\mathrm{H}y - \lambda{:}z^3{:}$

Theorem

We have

$$\partial_t \|w\|^2_{H^0} + \|w\|^2_{H^{1/2}} + \lambda \|w^2\|^2_{H^0} \leq F[y,z]$$

where $F[y,z]$ (given by explicit formula) only depends on y and z and has time-independent stochastic moments of any order.

(decisive that $\lambda \geq 0$; strong coupling is better)

We follow the Krylov–Bogoliubov construction.

Theorem

- Let μ be the law of the initial value $\phi(0) = z(0) + v(0) \in H^{-1/2-\epsilon}$, where $v(0) \in H^0$ almost surely.
- Let $P_t^*\mu$ be the probability measure of the solution at time t for initial value μ.

There exists a sequence $\{t_k\}$ of times variables $t_k \to \infty$ such that the sequence of probability measures

$$\frac{1}{t_k}\int_0^{t_k} P_s^*\mu \, ds$$

has an invariant weak limit in the space $\mathcal{M}_1(H^{-1/2-\epsilon})$ of all probability measures on $H^{-1/2-\epsilon}$.

This measure gives us the Schwinger functions of the Euclidean QFT.

The hope for $D = 4$ dimensions

Long ago, with H. Grosse we studied these QFT models by Dyson-Schwinger equations.

- Matrix models have a formal $1/N$ expansion, where in this situation $N = \theta/4$ is the non-commutativity scale.
- The leading order consists of planar graphs / planar correlation functions.

Theorem [Grosse, W 09]

The planar 2-point function satisfies a closed non-linear equation

$$\Big(\zeta+\eta+M^2+\lambda\int_0^\infty dt\,\varrho_0(t) Z G^{(0)}(\zeta,t)\Big) Z G^{(0)}(\zeta,\eta) = 1+\lambda\int_0^\infty dt\,\varrho_0(t)\frac{Z(G^{(0)}(t,\eta)-G^{(0)}(\zeta,\eta))}{t-\zeta}$$

- M, Z are renormalisation parameters
- measure ϱ_0 encodes spectral dimension $d_{spec} = \inf\{p \mid \int_0^\infty \frac{\varrho_0(t)}{(1+t)^{p/2}} < \infty\}$
- $\varrho_0(t) = \sum_{k=0}^\infty \frac{1}{\mathcal{N}}\delta(t - \frac{k+1}{\mathcal{N}})$ for 2D Moyal,
  $\varrho_0(t) = \sum_{k=0}^\infty \frac{k+1}{\mathcal{N}^2}\delta(t - \frac{k+1}{\mathcal{N}})$ for 4D Moyal, $\mathcal{N} = \frac{\theta}{4}$

Solution

Theorem [Grosse, Hock, W 19]

The non-linear integral equation has the solution

$$G^{(0)}(\xi,\eta) = \frac{\xi+\eta}{(\xi+R^{-1}(\eta))(\eta+R^{-1}(\xi))} \exp\left(\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{R}} ds \left(\frac{d}{ds}\log\frac{\xi-R(\mathrm{i}s)}{\xi-\mathrm{i}s}\right)\log\frac{\eta-R(-\mathrm{i}s)}{\eta+\mathrm{i}s}\right)$$

with

- $R(z) = z - \lambda(-z)^{D/2}\int_0^\infty dt \frac{\varrho_\lambda(t)}{(t+1)^{D/2}(t+z)}$ for $D = 2[d_{spec}/2]$
- deformed measure $\varrho_0(R(t)) = \varrho_\lambda(t)$

Key step in [Panzer, W 18] for large-θ 2D-Moyal and $\varrho_0 = \chi_{[1,\Lambda^2]}$;
here $R(z) = z + \lambda\log(1+z)$ and R^{-1} expressed in terms of Lambert-W.

$D = 4$ Moyal space: $\varrho_0(t) = t$ [Grosse, Hock, W 19b]

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- $\varrho_\lambda(x) \equiv \varrho_0(R(x)) = R(x) = x - \lambda x^2 \int_0^\infty \frac{dt\, \varrho_\lambda(t)}{(1+t)^2(t+x)}$
- If $\varrho_\lambda(t) \sim \varrho_0(t) = t$, then $R(x)$ bounded above. Consequently, R^{-1} would not be globally defined: would be trivial! (sort of renormalon problem).
- Fredholm equation perturbatively solved by iterated integrals: Hyperlogarithms and $\zeta(2n)$ which can be summed to

$$R(z) \equiv \varrho_\lambda(z) = z \cdot {}_2F_1\left(\begin{matrix}\alpha_\lambda, 1-\alpha_\lambda \\ 2\end{matrix}\,\middle|\, -z\right) \qquad \alpha_\lambda = \begin{cases} \frac{1}{\pi}\arcsin(\lambda\pi) & \text{for } |\lambda| \leq \frac{1}{\pi} \\ \frac{1}{2} + \frac{\mathrm{i}}{\pi}\operatorname{arcosh}(\lambda\pi) & \text{for} \lambda \geq \frac{1}{\pi} \end{cases}$$

Corollary

The planar part of the non-linearity reduces the spectral dimension to $4 - \frac{2}{\pi}\arcsin(\lambda\pi)$ and thus avoids the triviality problem (in the planar sector).

All hope to construct the $\lambda\phi^{\star 4}$-model in four dimensions rests on this observation.

Topological recursion (TR)

Theorem [Branahl, Hock, W 20]

The function R identified before gives rise to a spectral curve of topological recursion

$$\Big(x(z) = R(z), \;\; y(z) = -R(-z), \;\; \omega_2^{(0)}(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2} + \frac{dz_1\, dz_2}{(z_1 + z_2)^2} \Big)$$

from which a family $\{\omega_n^{(g)}\}_{2g+n>2}$ of meromorphic differentials is recursively constructed.

- The part $\frac{dz_1\, dz_2}{(z_1+z_2)^2}$ is responsible for an extension to blobbed topological recursion.
- All moments of the measure can be obtained from $\omega_n^{(g)}$ as simple residue evaluations.
- Meromorphicity requires a meromorphic R; not the case for 2D and 4D Moyal.
- The strong entanglement $y(z) = -x(-z)$ is exceptional. In [Hock, W 21] we proved a formula which expresses $\omega_{n+1}^{(0)}(-z, z_1, \dots, z_n)$ in terms of $\omega_{m+1}^{(0)}(z, z_1, \dots, z_m)$ for $m \leq n$.
- [Hock 22] related this formula to x-y-symplectic duality of TR.
- Alexandrov, Bychkov, Dunin-Barkowski, Kazarian and Shadrin, with input from Hock, recently extended the x-y duality to Generalised topological recursion.

All planar n-point functions

For finite matrices we can recursively construct all moments of the measure $\frac{1}{Z}\exp(-N\,\mathrm{Tr}(E\Phi^2+\lambda/4\Phi^4))d\Phi$ as formal power series in $1/N$.

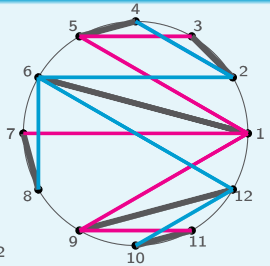
- For Moyal, $N\mapsto\frac{\theta}{4}$ is finite. Borel resummation seems hopeless.
- For Moyal, R has a branch cut, not isolated poles. Limits of colliding special points are delicate in TR. Only the planar sector can be solved exactly:

Theorem [de Jong, Hock, W 19]

Planar $2n$-point function $G^{(0)}(\zeta_1,\dots,\zeta_{2n})$ is sum of $\frac{1}{n}\binom{3n-2}{n-1}$ terms of the form

$$\frac{(-\lambda)^{n-1}\prod_1^n G^{(0)}(\zeta_i,\zeta_j)}{\prod_1^{2n-2}(\zeta_k-\zeta_l)}$$

- Pattern in bijection with nested Catalan tables
- Graphically described in terms of non-crossing chords with a pair of dual planar rooted trees in every pocket.



$2n=12$

$\langle(2,0,0),(1,1,0),(0),(0),(0),(1,0),(0)\rangle$

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Messages

1. Algebra (exact solution) alone is not enough; we need analysis.
2. Don't expand about free theory; this will trivial as in [Aizenman, Duminil-Copin 21].
3. We must expand about the planar theory for which we also have an exact solution.

Proposal: combine stochastic quantisation and exact solution of planar sector

Initial challenge: probabilistic interpretation of the planar sector

- Can't be a measure on matrix spaces, because this produces also non-planar moments.
- We see free probability as the right tool. It is probability on non-commutative spaces; the order matters. Large random matrices provide a model for free probability.
- Free probability connected to TR [Borot, Charbonnier, Garcia-Failde, Leid, Shadrin 21].
- Free probability distinguishes crossing and non-crossing structures, which correspond to non-planar vs. planar.

Project: Step 1+2

Step 1: Construct a non-commutative probability space $(\mathcal{A}, \varphi)$ whose functionals coincide with the planar moments of 4D $\lambda\phi^{\star 4}$:

- Cumulants of $\lambda\Phi^4$-matrix model are, analogously to free moments, given by non-crossing linear combinations of (the only non-zero) free cumulants $G^{(0)}(\zeta_i, \zeta_j)$.
- Free probability comes with measures. The semicircle law due to [Voiculescu 91] established the link to random matrix theory.

Step 2: Find an SPDE whose long-time averages coincide with the spatial averages (i.e. moments) of the free probability measure. There is some related work in free probability:

- [Biane, Speicher 98] defined stochastic integrals with respect to free Brownian motion.
- [Guionnet, Shlyakhtenko 09] considered SPDE $dX_t = d\xi_t - DV(X_t)dt$ for matrices and proved for locally-convex polynomial V existence and uniqueness of stationary distribution μ_V.

Project: Step 3

Merge free probability with Parisi-Wu stochastic quantisation, i.e. to produce the free probability measure as equilibrium limit of a free stochastic process

$$\partial_t z = -\mathrm{H} z - \mathcal{F}(z) + \xi \ .$$

- Solution z should have regularity that corresponds to dimension $4 - \frac{2}{\pi}\arcsin(\lambda\pi)$ when including non-linearity $\mathcal{F}$, but to dimension 4 for $\mathcal{F} = 0$.
- As preparation we must extend the work with Song and Weber to $4 - \epsilon$ dimensions. It seems that Da Prato–Debussche is not enouth; need to go to regularity structures.
- Remainder equation for $v = \phi - z$ produces only the non-planar sector from the renormalised planar sector z as input; it should be possible to control it.

If everything succeeds we can prove existence of the full $\lambda\phi^{\star 4}$ model on 4-dimensional Moyal space.