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Quantum fields on noncommutative geometries

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CRC 1442 GEOMETRY: DEFORMATIONS AND RIGIDITY

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Quantum fields

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Quantum field theory (QFT) is defined in terms of axioms of [Wightman 56], [Haag, Kastler 60], [Osterwalder, Schrader 74] or [Atiyah, Segal 89].

- Wightman and Osterwalder–Schrader agree that quantum fields Φ are distributions. Haag–Kastler avoid quantum fields, Atiyah–Segal have different target.
- Non-linear constructs of quantum fields such as $\lambda\Phi^n$ not naïvely defined.
- Difficulties to construct them grow with dimension D of space(-time).

Example: Stochastic quantisation [Parisi, Wu 81], here of $\lambda\Phi^4$-model

Euclidean QFT as equilibrium limit of statistical system coupled to thermal reservoir:

$$\partial_t\Phi(t,x) = (\Delta - m^2)\Phi(t,x) - {:}\lambda\Phi^3(t,x){:} + \xi(t,x)$$

where t – fictitious time, Δ – Laplacian in D dimensions, ξ – white noise.

- For $t \to \infty$, stochastic averages provide Schwinger functions of Euclidean QFT.

Quantum fields in dimension $D = 4$

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To construct $:\lambda\Phi^3:$, need to replace $\mathbb{R}^D$ by hypercubic lattice of length Λ and spacing a.
Then constuct sequence/net of distributions $:\lambda\Phi^3:_{a,\Lambda}$ which in some sense converges to $:\lambda\Phi^3:$

Triviality [Aizenman, Duminil-Copin 19]

The $\lambda\Phi^4$-QFT model in $D = 4$ does not exist; it is trivial.

- $:\lambda\Phi^3:_{a,\Lambda}$ needs regulator-dependent coupling constant $\lambda(a, \Lambda)$ which converges to zero for $(a \to 0, \Lambda \to \infty)$
- Already conjectured in early 80s [Aizenman 81; Fröhlich 82].
- Indication is positive β-function (understood as formal power series).

- Physical arguments (perturbative β-function is negative) support the conjecture that quantum Yang-Mills theory should exist in $D = 4$. Difficulty is confinement.
- Existence proof of YM_4 is one of the Millenium Prize problems.

It seems that non-linear $D = 4$ QFT examples tend to be trivial (e.g. $\lambda\Phi^4_4$, QED_4) or as difficult as Yang-Mills.

Quantum fields on non-commutative geometries

We relax rules of the game: Can we make sense of QFT on a noncommutative geometry?

- Motivated by compactification of M-theory on nc torus [Connes, Douglas, Schwarz 97].
- Also found in limiting regimes of String Theory [Schomerus 99; Seiberg, Witten 99].

We report on the considerable progress achieved since then.

Plan

- We follow the Euclidean approach via measures on spaces of distributions; its moments define candidate Schwinger functions.
- We cannot expect that these Schwinger functions satisfy reasonable axioms.
- Linear theory governed by spectral dimension of Laplace-type operator. Corresponding distributions conjectured to be as singular as on manifold of same spectral dimension.
- Aim is to learn how to build non-linear constructs of these distributions, defined by product in operator algebra. Works better than on manifolds!

The free Euclidean quantum field

$(\mathcal{A},\star)$ – Fréchet $*$-algebra, nuclear as vector space; $\mathcal{A}_*$ its subspace of self-adjoint elements.

Theorem [Bochner 32; Minlos 59]

Let $\mathcal{F}:\mathcal{A}_*\to\mathbb{C}$ with $\mathcal{F}(0)=1$ be continuous and of positive type:

$$\sum_{i,j=1}^{K} c_i\bar{c}_j\mathcal{F}(a_i-a_j)\geq 0 \text{ for any } a_i\in\mathcal{A}_*,\ c_i\in\mathbb{C}.$$

Then $\exists!$ Borel measure $d\mu$ on the dual $\mathcal{A}_*'$ with $\mathcal{F}(a)=\int_{\mathcal{A}_*'} e^{\mathrm{i}\Phi(a)}d\mu(\Phi)$.

For any inner product $C:\mathcal{A}_*\times\mathcal{A}_*\to\mathbb{R}$, called covariance, $\mathcal{F}(a):=\exp(-\frac{1}{2}C(a,a))$ is of positive type [Schur 1911] and (if continuous) defines $d\mu_C(\Phi)$.

- Consider Fréchet algebras which contain matrix units $e_{kl}\star e_{mn}=\delta_{lm}e_{kn}$, $(e_{kl})^*=e_{lk}$.
- For increasing sequence (E_k) of positive reals and parameter $\mathcal{N}$, we take covariance

$$C_E(e_{kl},e_{mn})=\frac{\delta_{kn}\delta_{lm}}{\mathcal{N}(E_k+E_l)}$$

Below, $d\mu_E(\Phi)$ denotes Bochner-Minlos measure associated with $\mathcal{F}(a):=\exp(-\frac{1}{2}C_E(a,a))$.

Product of distributions

So far we haven't used the product $\star$ in $\mathcal{A}$. This comes now, more precisely in the dual.

Regularity conjecture

It should be true that the support of $d\mu_E(\Phi)$ is not all of $\mathcal{A}'_*$, but reduced to a subspace determined by the spectral dimension $D = \inf\{p | \sum_{k=0}^{\infty} E_k^{-p/2} < \infty\}$.

- We want to make sense, for $\Phi \in \mathcal{A}'_*$, of

$$\mathrm{Tr}(\Phi^n) \text{ ``=''} \sum_{k_1,\dots,k_n=0}^{\infty} \Phi(e_{k_1 k_2})\Phi(e_{k_2 k_3}) \cdots \Phi(e_{k_{n-1} k_n})\Phi(e_{k_n k_1})$$

- Since $\sum_{k_1,\dots,k_n=0}^{\infty} e_{k_1 k_2} \otimes e_{k_2 k_3} \otimes \cdots \otimes e_{k_n k_1}$ is not Fréchet, $\mathrm{Tr}(\Phi^n)$ will not exist naïvely.

Renormalisation strategy

- Introduce cut-off $\sum_{k=0}^{\infty} \mapsto \sum_{k=0}^{\Lambda \mathcal{N}}$ in summation range and Λ-dependent parameters.
- Consider the resulting regulated measure and its moments.
- Adjust parameters so that dangerous moments are constant and others have a limit.

Moments and $1/\mathcal{N}$-expansion

Let $P(\Phi)$ be a polynomial in previous sense and Tr_Λ the regularised trace. We consider moments of

$$d\mu_{P,E}(\Phi) = \frac{1}{\mathcal{Z}} \exp(-\mathcal{N}\mathrm{Tr}_\Lambda(P(\Phi)))d\mu_E(\Phi).$$

- Viewed as moments of the Gaußian $d\mu_E(\Phi)$, these factorise into products of pairs. A pair is graphically represented as an edge; it contributes factor $\frac{1}{\mathcal{N}}$ and Kronecker δ's.
- $\mathcal{N}\mathrm{Tr}_\Lambda(\Phi^p)$ is graphically represented as p-valent vertex. Contributes factor $\mathcal{N}$.
- After resolving the Kronecker δ's, some summation over e_{kl}-matrix indices remain. We take a factor $\mathcal{N}$ out of every summation. Graphically they represent faces. Faces are labelled by matrix indices k, or better E_k.

Conclusion: $1/\mathcal{N}$-expansion

Every moment comes with a topological grading by the Euler characteristic $\chi_{g,n}$ of a genus-g Riemann surface (as formal power series in $\mathcal{N}^{-2}$)

$$\int_{\mathcal{A}'_*} d\mu_{P,E}(\Phi)\, \Phi(a_1)\cdots\Phi(a_n) = \sum_{g=0}^{\infty} \mathcal{N}^{\chi_{g,n}} \langle\Phi(a_1)\cdots\Phi(a_n)\rangle_{g,n}$$

Dyson-Schwinger equations and recursion

Dyson-Schwinger equations are identities between moments/cumulants obtained by integration by parts. They inherit the grading by the Euler characteristic.

- Cumulants represented as genus-g Riemann surface with boundary.
- Each boundary component carries external one-valent vertices which separate open faces of labels E_k, E_l. In an n-point function, in total n external vertices are distributed.
- The equations permit an extension to face labels $\zeta \in \mathbb{P}^1$.

Recursive structure

1. Non-linear equation for highest Euler characteristic: disk-amplitude with least number of one-valent vertices. Determines function y.
2. Algebraic recursion when increasing number of one-valent vertices at otherwise same topology. Combinatorial problem possibly connected to free probability.
3. Topological recursion [Eynard, Orantin 07] in decreasing Euler characteristic (for least number of vertices) starting from y and ramified covering $x : \Sigma \ni z \mapsto \zeta = x(z) \in \mathbb{P}^1$.

The Kontsevich matrix model

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Take $P(\Phi) = (\kappa_0 + \kappa_1 E + \kappa_2 E^2)\Phi + ((Z-1)E + \frac{1}{2}Zm_b^2)\Phi^2 + \frac{\lambda}{3}Z\Phi^3$.

- This is the [Kontsevich 92] model (matrix Airy function) with added counterterms.
- For simplicity, we focus on original formulation: $\mathcal{A} = M_N(\mathbb{C})$, $\mathcal{N} = N$ and $\kappa_i = Z - 1 = m_b^2 = 0$, $\lambda = \mathrm{i}$. See e.g. [Eynard, Orantin 07; Eynard 16].
- Relates generating series for intersection numbers on moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves to matrix Airy function from which KdV integrable hierarchy is deduced.

Main definition

$$\Big(\int_{H_N} d\mu_{\frac{\lambda}{3}\Phi^3,E}(\Phi)\;\Phi(e_{a_1a_1})\cdots\Phi(e_{a_na_n})\Big)_c - \delta_{n,1}\frac{NE_{a_1}}{2\lambda} := \sum_{g=0}^{\infty} N^{2-n-2g}\, W^{(g)}_{a_1,\dots,a_n}$$

as formal power series, all a_i pairwise different, $(\ )_c$ stands for "cumulant".

2. Algebraic formula for more complicated cumulants, e.g.

$$[N^{2-2g-n}]\Big(\int_{H_N} d\mu_{\frac{\lambda}{3}\Phi^3,E}(\Phi)\;\Phi(e_{a_1a_2})\Phi(e_{a_2a_3})\cdots\Phi(e_{a_na_1})\Big)_c = \sum_{k=1}^{n} W^{(g)}_{a_k}\prod_{l=1,l\neq k}^{n}\frac{1}{E_{a_k}^2 - E_{a_l}^2}$$

Dyson-Schwinger equations

Integration by parts establishes:

Dyson-Schwinger equations of Kontsevich model

$$\sum_{\substack{I_1 \uplus I_2 = \{a_1,\dots,a_n\} \\ g_1+g_2=g}} W^{(g_1)}_{a,I_1} W^{(g_2)}_{a,I_2} = E_a^2 \delta_{n,0}\delta_{g,0} - W^{(g-1)}_{a,a,a_1,\dots,a_n} - \frac{2}{N}\sum_{k=1}^{N} \frac{W^{(g)}_{k,a_1,\dots,a_n} - W^{(g)}_{a,a_1,\dots,a_n}}{E_k^2 - E_a^2}$$
$$- \sum_{j=1}^{n} \frac{\partial}{\partial E_{a_j}^2} \frac{W^{(g)}_{a_1,\dots,a_n} - W^{(g)}_{a_1,\dots,a_{j-1},a,a_{j+1},\dots,a_n}}{E_{a_j}^2 - E_a^2}$$

- Non-linear equation for $W^{(0)}_a$ if $g = n = 0$; solved by [Makeenko, Semenoff 91] $W^{(0)}_a = -\sqrt{E_a^2+c} + \frac{1}{N}\sum_{l=1}^{N} \frac{1}{\sqrt{E_l^2+c}(\sqrt{E_a^2+c}+\sqrt{E_l^2+c})}$ where $c = \frac{2}{N}\sum_{k=1}^{N}\frac{1}{\sqrt{E_k^2+c}}$.
- Counterterms in dimension $2 \leq D \leq 6$ achieve convergent sums.
- Affine equation for $W^{(g)}_{a_1,\dots,a_n}$ if $2g+n \geq 2$ with known inhomogeneity.

Complexification

Set $\sqrt{E_a^2+c} \mapsto z$, $\sqrt{E_{a_i}^2+c} \mapsto z_i$, $\varepsilon_k := \sqrt{E_k^2+c}$ and complexify DSE to system of equations

$$\sum_{\substack{I_1 \uplus I_2 = \{z_1,\dots,z_n\} \\ g_1+g_2=g}} \hat{W}^{(g_1)}_{|I_1|+1}(z, I_1)\hat{W}^{(g_2)}_{|I_2|+1}(z, I_2) + \hat{W}^{(g-1)}_{n+2}(z, z, z_1, \dots, z_n)$$
$$= (z^2 - c)\delta_{n,0}\delta_{g,0} - \frac{2}{N}\sum_{k=1}^{N} \frac{\hat{W}^{(g)}_{n+1}(\varepsilon_k, z_1, \dots, z_n) - \hat{W}^{(g)}_{n+1}(z, z_1, \dots, z_n)}{\varepsilon_k{}^2 - z^2}$$
$$- \sum_{j=1}^{n} \frac{\partial}{\partial z_j^2} \frac{\hat{W}^{(g)}_n(z_1, \dots, z_n) - \hat{W}^{(g)}_n(z_1, \dots, z_{j-1}, z, z_{j+1}, \dots, z_n)}{z_j^2 - z^2}$$

for meromorphic functions $\hat{W}^{(g)}_n(z_1, .., z_n)$ satisfying $W^{(g)}_{a_1,\dots,a_n} \equiv \hat{W}^{(g)}_n(\varepsilon_{a_1}, \dots, \varepsilon_{a_n})$.

- $\hat{W}^{(0)}_2(z, z_1) = \dfrac{1}{4zz_1(z+z_1)^2}$
- $\hat{W}^{(0)}_3(z_1, z_2, z_3) = \dfrac{1}{16(1-\hat{t}_3)z_1^3 z_2^3 z_3^3}$ where $\hat{t}_3 = -\frac{1}{N}\sum_{k=1}^{N}\frac{1}{\varepsilon_k^3}$

Linear and quadratic loop equations

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The complexified DSE imply inductively for $2g+n\geq 3$:

- $W_n^{(g)}(z_1,\dots,z_n)$ has poles only at $z_i=0$
- **Linear loop equation ($2g+n\geq 3$)**

$$W_n^{(g)}(z,z_2,\dots,z_n)+W_n^{(g)}(-z,z_2,\dots,z_n)=0$$

Use this and splitting of $\hat{W}_1^{(0)}$ and $\hat{W}_2^{(0)}$ into parts with $\pm z$ to rearrange DSE into

Quadratic loop equation ($2g+n\geq 3$)

$$\sum_{\substack{I_1\uplus I_2=\{z_2,\dots,z_n\}\\ g_1+g_2=g}} W_{|I_1|+1}^{(g_1)}(z,I_1)W_{|I_2|+1}^{(g_2)}(-z,I_2)+W_{n+1}^{(g-1)}(z,-z,z_2,\dots,z_n)$$
$$=\frac{1}{N}\sum_{k=1}^{N}\frac{W_n^{(g)}(\varepsilon_k,z_2,\dots,z_n)}{\varepsilon_k^2-z^2}+\sum_{j=2}^{n}\frac{\partial}{\partial z_j^2}\Big(\frac{W_{n-1}^{(g)}(z_2,\dots,z_n)}{z_j^2-z^2}\Big)$$

where $W_1^{(0)}(z)\equiv y(z):=z+\frac{1}{N}\sum_{k=1}^{N}\frac{1}{\varepsilon_k(\varepsilon_k-z)}$, $\quad W_2^{(0)}(z_1,z_2)=\frac{1}{4z_1z_2(z_1-z_2)^2}$

and $W_n^{(g)}=\hat{W}_n^{(g)}$ for $2g+n\geq 3$

Topological recursion

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[Eynard, Orantin 07] noticed that the non-linearity of many matrix models can be disentangled into initial data called the spectral curve and a universal recursion for meromorphic functions $W_n^{(g)}$ (or promoted to meromorphic differentials $\omega_n^{(g)}$).

Spectral curve

- Complex curve/Riemann surface Σ and two ramified coverings $x, y : \Sigma \to \mathbb{P}^1$. Polynomial equation $P(x, y) = 0$.
- Bergman kernel B: symmetric bidifferential on $\Sigma \times \Sigma$, with double pole on diagonal, no other pole, normalised.

Soon later many important examples other than matrix models were identified:

- Weil-Peterssen volumes of moduli spaces of bordered hyperbolic surfaces [Mirzakhani 07].
- ELSV formula, expresses simple Hurwitz numbers as integral of ψ- and λ-classes over $\overline{\mathcal{M}}_{g,n}$ [Bouchard, Mariño 07; Eynard, Mulase, Safnuk 09].
- Semisimple cohomological field theories [Dunin-Barkowski, Orantin, Shadrin, Spitz 14].

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Theorem

1. Given a spectral curve $(x, y : \Sigma \to \mathbb{P}^1, B)$, with technical assumptions.
2. Set $y =: W_1^{(0)}$, $B(z,w) =: W_2^{(0)}(z,w)dx(z)dx(w)$, $x^{-1}(x(z)) = \{\hat{z}^0 = z, \hat{z}^1, ..., \hat{z}^d\}$.
3. Assume that the following are holomorphic at any branch point of x:

$$L(x(z); z_2, ..., z_n) := \sum_{j=0}^{d} W_n^{(g)}(\hat{z}^j, z_2, ..., z_n)$$

$$Q(x(z); z_2, ..., z_n) = \sum_{j=0}^{d} \Big(\sum_{\substack{I_1 \uplus I_2 = \{z_2,...,z_n\} \\ g_1+g_2=g}} W^{(g_1)}_{|I_1|+1}(\hat{z}^j, I_1) W^{(g_2)}_{|I_2|+1}(\hat{z}^j, I_2) + W^{(g-1)}_{n+1,reg}(\hat{z}^j, \hat{z}^j, z_2, ..., z_n) \Big)$$

Then there is a formula which evaluates $W_n^{(g)}$ in terms of $W_{n'}^{(g')}$ with $2g'+n' < 2g+n$.

This formula is particularly simple (and universal!) under a projection property

$$W_n^{(g)}(z, z_2, ..., z_n) = \sum\nolimits_{\beta=\text{ramif.pts}} \mathrm{Res}_{q=\beta} \Big(\int_\beta^q W_2^{(0)}(z, .)dx(.) \Big) W_n^{(g)}(q, z_2, ..., z_n) dx(q)$$

Remarks

- Kontsevich model has $\zeta = x(z) = z^2 - c$ and $y(z) = W_1^{(0)}(z)$.
- Laurent expansion of $W_n^{(g)}(z_1, \ldots, z_n)$ near an n-tupel of ramification points can be expressed in terms of intersection numbers of ψ- and κ-classes on $\overline{\mathcal{M}}_{g,n}$ [Eynard 11].
- Absence of projection property gives blobbed topological recursion [Borot, Shadrin 15].
- Deformations of spectral curve express formal Baker-Akhiezer kernel in terms of $W_n^{(g)}$. Gives rise to formal KP τ-function [Eynard, Orantin 07; Borot, Eynard 12].

Recent highlight are functional relations [Borot, Charbonnier, Garcia-Failde, Leid, Shadrin 21] between two families $W_n^{(g)}$ and $W_n^{(g)\vee}$ of functions.

- $W_1^{(0)}[W_1^{(0)\vee}]$ is the R-transform [Voiculescu 86] between free moments & cumulants; $W_2^{(0)}[W_2^{(0)\vee}, W_1^{(0)\vee}]$ describes 2nd order freeness [Collins, Mingo, Śniady, Speicher 07].
- Conjecture: the functional relations describe the x-y swap of TR; proved later in [Hock 22; Bychkov, Dunin-Barkowski, Kazarian, Shadrin 22].

From the Kontsevich model to $\lambda\Phi^3$-QFT model on NCG

[Grosse, Steinacker 05/06] pointed out that the $\lambda\Phi^3$-QFT model on nc Moyal space

$$S(\Phi) = \int_{\mathbb{R}^D} dx \left(\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{m^2}{2}\Phi^2 + \frac{\Omega^2}{2}(\tilde{x}^\mu\Phi)(\tilde{x}_\mu\Phi) + \frac{\lambda}{3}\Phi\star\Phi\star\Phi \right)$$

can be mapped for $\Omega = 1$ to the Kontsevich model (here $[x^\mu, x^\nu]_\star = \mathrm{i}\Theta^{\mu\nu}$, $\tilde{x}_\mu = 2\Theta^{-1}_{\mu\nu}x^\nu$). They identified the renormalisation procedure in dimension $D \in \{2, 4, 6\}$.

- Topological expansion parameter $\mathcal{N} = \frac{\theta}{4}$ depends on noncommutativity scale θ.
- Planar limit $\mathcal{N} \to \infty$ means infinite noncommutativity. Translation invariance restored; get Schwinger functions of Euclidean QFT on $\mathbb{R}^D$ [Grosse, Sako, W 16/17].
- 2-point function has Osterwalder–Schrader positivity in $D = 4$ and $D = 6$, not in $D = 2$. Higher n-point function loose OS positivity because of $\hat{W}_2^{(0)}(z_1, z_2) = \frac{1}{4z_1z_2(z_1+z_2)^2}$.

Lesson: keep θ finite, keep higher genus contributions and try to resum the series in $\mathcal{N}^{-2g}$.

Recent progress in [Eynard, Garcia-Failde, Giacchetto, Gregori, Lewański 23], connects to resurgence.

The $\lambda\Phi^4$-model

Take $P(\Phi) = \frac{\lambda}{4}\Phi^4$, shift $E_a \mapsto E_a + \frac{1}{2}M^2$ where M depends on cut-off.

- If D_{spec} is spectral dimension of $\{E_k\}$, then $D := 2[\frac{D_{spec}}{2}] \in \{0,2,4\}$.
- Genus-expanded & field-renormalised 2-point function

$$G^{(g)}_{|ab|} = \frac{1}{Z(\Lambda)}[\mathcal{N}^{-1-2g}] \int d\mu^{\Lambda}_{(\lambda/4)\Phi^4,E}(\Phi)\ \Phi(e_{ab})\Phi(e_{ba})$$

- Dyson-Schwinger eq. for $G^{(0)}_{|ab|}$ extends to complexified $G^{(0)}(\zeta,\eta)$ with $G^{(0)}(E_a,E_b) = G^{(0)}_{|ab|}$:

Theorem [Grosse, W 09]

The planar two-point function satisfies the closed non-linear equation

$$\Big(\zeta+\eta+M^2+\frac{\lambda}{\mathcal{N}}\sum_{k=0}^{\Lambda^D\mathcal{N}} ZG^{(0)}(\zeta,E_k)\Big)ZG^{(0)}(\zeta,\eta) = 1+\frac{\lambda}{\mathcal{N}}\sum_{k=0}^{\Lambda^D\mathcal{N}} \frac{ZG^{(0)}(E_k,\eta)-ZG^{(0)}(\zeta,\eta)}{E_k-\zeta}$$

Alternatively, setting $\varrho_0(t) = \frac{1}{\mathcal{N}}\sum_{k=0}^{\Lambda^D\mathcal{N}}\delta(t-E_k)$, get non-linear integral equation

$$\Big(\zeta+\eta+M^2+\lambda\int_0^\infty dt\ \varrho_0(t)ZG^{(0)}(\zeta,t)\Big)ZG^{(0)}(\zeta,\eta) = 1+\lambda\int_0^\infty dt\ \varrho_0(t)\frac{Z(G^{(0)}(t,\eta)-G^{(0)}(\zeta,\eta))}{t-\zeta}$$

Solution

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Theorem [Panzer, W 18 for $\varrho_0 = 1$, Grosse, Hock, W 19a]

1. Ansatz $G^{(0)}(\zeta,\eta) = \dfrac{e^{\mathcal{H}_\zeta[\tau_\eta(\bullet)]}\sin\tau_\eta(\zeta)}{Z\lambda\pi\varrho_0(\zeta)}$, $\quad \mathcal{H}_\zeta[f] := \frac{1}{\pi}\fint_0^{\Lambda^2}\frac{dp\, f(p)}{p-\zeta}$ finite Hilbert transf.
2. $\tau_\eta(\zeta) = \lim_{\epsilon\searrow 0}\operatorname{Im}\log\left(\eta - R_D(-m^2 - R_D^{-1}(\zeta + \mathrm{i}\epsilon)\right)$ for m – renormalised mass
3. $$R_D(z) = z - \lambda(-z)^{D/2}\int_0^\infty \frac{dt\;\varrho_\lambda(t)}{(m^2+t)^{D/2}(t+m^2+z)}$$
4. ϱ_λ is implicit solution of $\varrho_0(R_D(\zeta)) = \varrho_\lambda(\zeta)$.

Then the non-linear integral equation for $G^{(0)}(\zeta,\eta)$ holds identically.

- Proof: [Cauchy 1831] residue theorem, [Lagrange 1770] inversion theorem, [Bürmann 1799] formula.
- $\varrho_0(t) \equiv 1$ (2D Moyal, $m = 1$) in terms of Lambert-W satisfying $W(z)e^{W(z)} = z$.

$D = 4$ Moyal space: $\varrho_0(t) = t$ [Grosse, Hock, W 19b]

- $\varrho_\lambda(x) \equiv \varrho_0(R_4(x)) = R_4(x) = x - \lambda x^2 \int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(m^2+t)^2(t+x)}$
- If $\varrho_\lambda(t) \sim \varrho_0(t) = t$, then $R_4(x)$ bounded above. Consequently, R_4^{-1} would not be globally defined: triviality!
- Fredholm equation perturbatively solved by iterated integrals: Hyperlogarithms and $\zeta(2n)$ which can be summed to

$$R_4(x) \equiv \varrho_\lambda(x) = x \cdot {}_2F_1\left(\begin{matrix}\alpha_\lambda, 1-\alpha_\lambda \\ 2\end{matrix}\middle| -\frac{x}{m^2}\right) \qquad \alpha_\lambda = \begin{cases} \frac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda| \leq \frac{1}{\pi} \\ \frac{1}{2} + \mathrm{i}\frac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{f or } \lambda \geq \frac{1}{\pi} \end{cases}$$

- Gives non-perturbative integral representation for $G^{(0)}(\xi, \eta)$.

Corollary

The planar part of the non-linearity reduces the spectral dimension to $4 - 2\frac{\arcsin(\lambda\pi)}{\pi}$ and thus avoids the triviality problem (in the planar sector).

All hope to construct the $\lambda\Phi^4$-model in four dimensions rests on this observation.

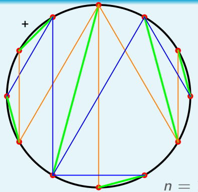
All planar cumulants

Planar cumulants $G^{(0)}_{a_1\dots,a_n} = \frac{1}{Z^{n/2}}[\mathcal{N}^{1-n}] \int_{\mathcal{A}'_*} \Big(\int d\mu_{\lambda\Phi^4,E}(\Phi)\Phi(e_{a_1a_2})\Phi(e_{a_2a_3})\cdots\Phi(e_{a_na_1})\Big)_c$,

extend to $G^{(0)}(\zeta_1,..,\zeta_n)$

Theorem [de Jong, Hock, W 19]

$G^{(0)}(\zeta_1,\dots,\zeta_n)$ is sum of $\frac{2}{n}\binom{\frac{3n}{2}-2}{\frac{n}{2}-1}$ terms of the form $\dfrac{(-\lambda)^{n/2-1}\prod_1^{n/2} G^{(0)}(\zeta_i,\zeta_j)}{\prod_1^{n-2}(\zeta_k-\zeta_l)}$



- Pattern in bijection with nested Catalan tables
- Graphically described in terms of non-crossing chords with a pair of dual planar rooted trees in every pocket.

Link to free probability?

Expectation values of powers of large random matrices show freeness (crossings suppressed).

- Cumulants of $\lambda\Phi^4$-model are, analogously to free moments, given by non-crossing linear combinations of (the only non-zero) free cumulants $G^{(0)}(\zeta_i,\zeta_j)$.
- Is this more than an analogy?

The genus expansion

We thus succeeded in constructing the planar sector of the $\lambda\Phi^4$-QFT model on a particular 4-dimensional noncommutative geometry.

Main message

Don't perturb the linear theory; this fails as in [Aizenman, Duminil-Copin 19].
Take it together with the planar part of the non-linearity! Only NCG can do this.

- But we do not have quantitative estimates for error between full theory and planar sector.
- One would expect that the difference is $\mathcal{O}(1/\mathcal{N}^2)$. There are refinements of Dyson-Schwinger techniques [Guionnet 17] which could achieve this.
- Alternatively, one can try to control the cumulants to any genus and establish Borel summability of the genus expansion via resurgence as for $\lambda\Phi^3$ in [EGFGGL 23].

We describe some modest (but already difficult) steps in this dírection. They concern an $N \times N$-matrix model where N can be large but finite. Limit $N \to \infty$ is currently out of reach.

Finite matrices

Consider the partition function $\mathcal{Z}_{(\lambda/4)\Phi^4,E} := \int_{H_N} d\Phi\; e^{-N\,\mathrm{Tr}(E\Phi^2+\frac{\lambda}{4}\Phi^4)}$ on $\mathcal{A} = M_N(\mathbb{C})$.
Let $(e_1, \dots, e_d)$ be the pairwise different eigenvalues of E with multiplicities $(r_1, \dots, r_d)$.

Theorem [Schürmann, W 19]

A solution of the non-linear equation for $G^{(0)}(\zeta,\eta)$ can be implicitly found in the form $G^{(0)}(x(z),x(w)) =: \mathcal{G}^{(0)}(z,w)$ with $x(z) = z - \frac{1}{N}\sum_{k=1}^{N}\frac{\varrho_k}{\varepsilon_k+z}$, $x(\varepsilon_k) = e_k$ and $x'(\varepsilon_k)\varrho_k = r_k$:

$$\mathcal{G}^{(0)}(z,w) = \frac{P_1^{(0)}(x(z),x(w))}{(x(z)+y(w))(x(w)+y(z))}$$ where $\boxed{y(z) = -x(-z)}$ and

$$P_1^{(0)}(x(z),x(w)) = \frac{\prod_{u\in x^{-1}(\{x(w)\})}(x(z)+y(u))}{\prod_{k=1}^{d}(x(z)-x(\varepsilon_k))} \equiv P_1^{(0)}(x(w),x(z))$$

Main definition [Branahl, Hock W 20]

For pairwise different $a_1, \dots, a_n$, set $W^{(g)}_{a_1,\dots,a_n} := [N^{2-2g-n}]\frac{(-1)^n\partial^n \log \mathcal{Z}_{(\lambda/4)\Phi^4,E}}{\partial E_{a_1}\cdots\partial E_{a_n}} + \frac{\delta_{g,0}\delta_{n,2}}{(E_{a_1}-E_{a_2})^2}$ for $2g+n\geq 2$, and complexify to $W_n^{(g)}(z_1,\dots,z_n)$. Moreover, $W_1^{(0)}(z) = y(z)$.

Linear and quadratic loop equations for $g = 0$

Extract from DSE (which relate $W_n^{(g)}$ to auxiliary functions) the lin./quad. loop equations:

Proposition [Hock, W 21; Hock, W 23]

The functions $W^{(0)}_{|I|+1}$ satisfy for $\emptyset \neq I = \{u_1, \dots, u_n\}$ the global linear loop equations

$$\sum_{k=0}^{d} W^{(0)}_{|I|+1}(\hat{z}^k, I) = \frac{\delta_{|I|,1}}{(x(z)-x(u_1))^2} - \sum_{j=1}^{|I|} \frac{\partial}{\partial x(u_j)} D_{I\setminus u_j}\Big(\frac{1}{x(z)+y(u_j)}\Big)$$

and the global quadratic loop equations

$$\frac{1}{2} \sum_{I_1 \uplus I_2 = I} \sum_{k=0}^{d} W^{(0)}_{|I_1|+1}(\hat{z}^k, I_1) W^{(0)}_{|I_2|+1}(\hat{z}^k, I_2)$$

$$= \sum_{j=1}^{|I|} \frac{\partial}{\partial x(u_j)} D_{I\setminus u_j}\Big(\frac{x(u_j)}{x(z)+y(u_j)}\Big) - \frac{1}{N}\sum_{k=1}^{d} \frac{r_k W^{(0)}_{|I|+1}(\varepsilon_k, I)}{x(z)-x(\varepsilon_k)} + \sum_{j=1}^{|I|} \frac{\partial}{\partial x(u_j)} \frac{W^{(0)}_{|I|}(I)}{x(z)-x(u_j)},$$

for $D_{\{u_1,\dots,u_n\}} = \prod_{j=1}^{n} D_{u_j}$ and derivations $D_u W^{(g)}_m(z_1, \dots, z_m) = W^{(g)}_{m+1}(z_1, \dots, z_m, u)$, $D_u x(z) = 0$

Projection property does not hold: blobbed topological recursion [Borot, Shadrin 15]

Linear and quadratic loop equations for $g = 1$

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Proposition [Hock, W 23]

The genus-1 meromorphic functions $W^{(1)}_{|I|+1}(z, I)$ satisfy the linear loop equation

$$\sum_{k=0}^{d} W^{(1)}_{|I|+1}(\hat{z}^k, I) = -D^0_I \frac{1}{8(x(z)-x(0))^3}$$
$$-\sum_{j=1}^{|I|} \frac{\partial}{\partial x(u_j)} D_{I\setminus u_j}\Big\{ \frac{W^{(0)reg}_2(u_j,u_j)}{(x(z)+y(u_j))^3} - \frac{W^{(1)}_1(u_j)}{(x(z)+y(u_j))^2}$$
$$-\frac{1}{2(x(z)+y(u_j))^2}\frac{\partial^2}{\partial(x(u_j))^2}\frac{1}{(x(z)+y(u_j))}\Big\}$$

and . . .

Linear and quadratic loop equations for $g = 1$

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Proposition [Hock, W 23]

...the quadratic loop equation

$$\frac{1}{2}\sum_{\substack{g_1+g_2=1\\ I_1\uplus I_2=I}}\sum_{k=0}^{d} W^{(g_1)}_{|I|+1}(\hat{z}^k, I_1)W^{(g_2)}_{|I|+1}(\hat{z}^k, I_2) + \frac{1}{2}\sum_{k=0}^{d} W_2^{(0)reg}(\hat{z}^k,\hat{z}^k, I)$$

$$= \frac{1}{6}\sum_{j=1}^{|I|}\frac{\partial^2}{\partial(x(u_j))^2}\Big(D_{I\setminus u_j}\frac{1}{(x(z)+y(u_j))^3}\Big) - D_I^0\frac{1}{8(x(z)-x(0))^2} + x(z)D_I^0\frac{1}{8(x(z)-x(0))^3}$$

$$+\sum_{j=1}^{|I|}\frac{\partial}{\partial x(u_j)}\Big[x(u_j)D_{I\setminus u_j}\Big\{\frac{W_2^{(0)reg}(u_j,u_j)}{(x(z)+y(u_j))^3} - \frac{W_1^{(1)}(u_j)}{(x(z)+y(u_j))^2} - \frac{1}{2(x(z)+y(u_j))^2}\frac{\partial^2}{\partial(x(u_j))^2}\frac{1}{(x(z)+y(u_j))}\Big\}\Big]$$

$$-\frac{1}{N}\sum_{l=1}^{d}\frac{W^{(1)}_{|I|+1}(\varepsilon_l, I)}{x(z)-x(\varepsilon_l)} + \sum_{j=1}^{|I|}\frac{\partial}{\partial x(u_j)}\frac{W^{(1)}_{|I|}(I)}{x(z)-x(u_j)}\,.$$

Final remarks

- The global linear and quadratic loop equations give explicit recursion formulae for $W_n^{(g)}$ (so far for $g \leq 1$).
- Original blobbed TR [Borot, Shadrin 15] defined for local curves; this leaves large freedom (called 'blobs') in $W_n^{(g)}$. Validity of local loop equations is clear.
- It would be interesting to know whether matricial QFT-models other than $(\lambda\Phi^3, \lambda\Phi^4)$ admit a similar formulation. A hint:

Theorem [Borot, W 23]

Let $P \in \mathbb{C}(\mathbb{R})$ be such that $e^{-E_{min}x^2 - P(x)}$ has finite moments, let $d\mu_E$ be as before. Then $\mathcal{Z}(\boldsymbol{t}) = \int_{H_N} d\mu_E(\Phi)\, \exp\big(\mathrm{Tr}\big(-P(\Phi) + \sum_{k=0}^{\infty} t_{2k+1}\Phi^{2k+1}\big)\big)$ is a BKP τ-function.

- In particular, $\exists$ infinitely many quadratic relations between moments, e.g. (for P even)

$$0 = \big\langle(\mathrm{Tr}(\Phi))^6\big\rangle + 15\big\langle(\mathrm{Tr}(\Phi))^4\big\rangle\big\langle(\mathrm{Tr}(\Phi))^2\big\rangle - 5\big\langle(\mathrm{Tr}(\Phi))^3\mathrm{Tr}(\Phi^3)\big\rangle$$
$$- 15\big\langle\mathrm{Tr}(\Phi)\mathrm{Tr}(\Phi^3)\big\rangle\big\langle(\mathrm{Tr}(\Phi))^2\big\rangle - 5\big\langle(\mathrm{Tr}(\Phi^3))^2\big\rangle + 9\big\langle\mathrm{Tr}(\Phi^5)\mathrm{Tr}(\Phi)\big\rangle$$