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BKP-relations in Kontsevich model with arbitrary potential

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Kontsevich model

Consider the probability measure $d\mu_\lambda(M) = \frac{dM \exp(-\mathrm{Tr}(\frac{1}{2}\Lambda M^2))}{\int_{\mathcal{H}_N} dM \exp(-\mathrm{Tr}(\frac{1}{2}\Lambda M^2))}$ on space $\mathcal{H}_N$ of Hermitian $N \times N$-matrices, where $\Lambda \in \mathcal{H}_N$ be positive. In his investigation of intersection numbers on $\overline{\mathcal{M}}_{g,n}$, [Kontsevich 92] was led to introduce $Z(\Lambda) := \int_{\mathcal{H}_N} d\mu_\Lambda(M) \exp(\frac{\mathrm{i}}{6}\mathrm{Tr}(M^3))$.

1. Shift $M \mapsto M - \mathrm{i}\Lambda$ to turn ΛM^2 into $\Lambda^2 M$
2. Change of variables $M = U \,\mathrm{diag}(m_1, \dots, m_N) U^\dagger$ with $U \in \mathcal{U}(N)$ and
   $$dM = \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{k=1}^N k!} (\Delta(m_1, \dots, m_N))^2 \frac{d\mu_H(U)}{\mathrm{vol}(\mathcal{U}(N))} \prod_{k=1}^N dm_k$$
3. Itzykson–Zuber–Harish-Chandra formula
   $$\int_{\mathcal{U}(N)} \frac{d\mu_H(U)}{\mathrm{vol}(\mathcal{U}(N))} \exp(AUBU^\dagger) = \Big(\prod_{k=1}^{N-1} k!\Big) \frac{\det(e^{a_i b_j})}{\Delta(a_1,\dots,a_N)\Delta(b_1,\dots,b_N)}$$

Then $Z(\Lambda) \equiv Z(s_1, s_3, ..)$ becomes formal power series in $s_{2i+1} = (2i-1)!! \sum_{k=1}^N \lambda_k^{-2i-1}$ ('times'); its coefficients provide intersection numbers on $\overline{\mathcal{M}}_{g,n}$.

- In fact, $Z(s_1, s_3, ..)$ is τ-function of KdV integrable hierarchy.
- Moments of a related measure computable by topological recursion.

Generalised Kontsevich model

$$Z(\Lambda, V) := \int_{\Lambda+\mathcal{H}_N} dM \exp(-\mathrm{Tr}(V(M) - V(\Lambda) - (M-\Lambda)V'(\Lambda)))$$

- Shift $M = \Lambda + \tilde{M}$ produces mixed terms $\mathrm{Tr}(\tilde{M}\Lambda\tilde{M}\Lambda)$ etc if $\deg(V) > 3$
- $V(M) = \frac{1}{r+1}M^{r+1}$ leads to r-KdV integrable hierarchy [Adler, van Moerbeke 92] (times with label divisible by r are absent)
- Topological recursion for any V [Belliard, Charbonnier, Eynard, Garcia-Failde 21]; uses several type of graphs (one for ciliated maps)
- Encodes intersection numbers with Witten's r-spin class
- Generating series of ciliated maps in terms of r-spin intersection numbers

Kontsevich model with any potential

We are interested in another generalisation to $Z(\Lambda, V) = \int_{\mathcal{H}_N} d\mu_\Lambda(M) \exp(\mathrm{Tr}(V(M)))$ where V is continuous and such that $e^{-\lambda_{\min} x^2/2 + V(x)} dx$ has finite moments.

- ΛM^2 cannot be removed anymore. Not the generalised Kontsevich model.
- We directly change variables and use IZHC.
- This works also for the original case $V(x) = x^3$ and leads to an expression of $Z(\Lambda, M^3)$ in terms of Schur-Q functions [Mironov, Morozov 20].
- Schur-Q functions play a key rôle in the BKP integrable hierarchy. [Alexandrov 20] converts it back to KdV. Time variables $s_1, s_3, s_5, \ldots$ as before.

Our result goes in another direction: We polynomially deform the potential to

$$Z(\Lambda, V, \boldsymbol{t}) = \int_{\mathcal{H}_N} d\mu_\Lambda(M) \exp\left(\mathrm{Tr}\left(V(M) + \sum_{i=0}^{\infty} t_{2i+1} M^{2i+1}\right)\right)$$

where $t_1, t_3, t_5, \ldots$ are formal parameters unrelated to Λ.

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Theorem [Borot, W 23]

For fixed Λ, the formal power series

$$Z(\Lambda, V, \boldsymbol{t}) = \int_{\mathcal{H}_N} d\mu_\Lambda(M) \exp\Big(\mathrm{Tr}\Big(V(M) + \sum_{i=0}^{\infty} t_{2i+1} M^{2i+1}\Big)\Big)$$

is a BKP τ-function with respect to the times $\boldsymbol{t}$, i.e. it satisfies the Hirota bilinear equation

$$Z(\Lambda,V,\boldsymbol{t})Z(\Lambda,V,\tilde{\boldsymbol{t}}) = \operatorname*{Res}_{z=0}\Big(\frac{dz}{z}\, e^{\sum_{k\geq 0} z^{2k+1}(t_{2k+1}-\tilde{t}_{2k+1})} Z\big(\Lambda,V,\boldsymbol{t}-2[z^{-1}]\big)\, Z\big(\Lambda,V,\tilde{\boldsymbol{t}}+2[z^{-1}]\big)\Big),$$

where $[z^{-1}] = \big(\frac{1}{z}, \frac{1}{3z^3}, \frac{1}{5z^5}, \dots\big)$.

Quadratic relations

The Hirota equation imples an infinite hierarchy of relations between moments

$$M_{2\ell_1+1,\dots,2\ell_n+1} = \frac{\int_{\mathcal{H}_N} d\mu_\Lambda(M)\, e^{\mathrm{Tr}(-\frac{1}{2}\Lambda M^2 + V(M))}\, \mathrm{Tr}(M^{2\ell_1+1}) \cdots \mathrm{Tr}(M^{2\ell_n+1})}{\int_{\mathcal{H}_N} d\mu_\Lambda(M)\, e^{\mathrm{Tr}(-\frac{1}{2}\Lambda M^2 + V(M))}}$$

For even $V(M) = V(-M)$, the first equations read

$$\begin{aligned}
0 =& M_{1^6} + 15 M_{1^4} M_{1,1} - 5 M_{3,1^3} - 15 M_{3,1} M_{1,1} - 5 M_{3,3} + 9 M_{5,1} \,, \\
0 =& M_{1^8} + 28 M_{1^6} M_{1,1} + 35 (M_{1^4})^2 + 7 M_{3,1^5} + 70 M_{3,1^3} M_{1,1} + 35 M_{3,1} M_{1^4} \\
&- 35 M_{3,3} M_{1,1} - 70 (M_{3,1})^2 - 21 M_{5,1^3} - 63 M_{5,1} M_{1,1} - 42 M_{5,3} + 90 M_{7,1} \,.
\end{aligned}$$

Their form is independent of N, Λ, V (if even), although the individual moments depend crucially on these data.
For $V = 0$ one has in terms of $p_k = \mathrm{Tr}(\Lambda^{-k})$:

$$\begin{aligned}
&M_{1,1} = p_1, && M_{3,1} = 3p_1^2, && M_{1^4} = 3p_1^2, && M_{1^6} = 15 p_1^3, \\
&M_{3,1^3} = 6p_3 + 9p_1^3, && M_{3,3} = 3p_3 + 12 p_1^3, && M_{5,1} = 5p_3 + 10 p_1^3.
\end{aligned}$$

Remarks on the proof

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- Jacobian $(\Delta(\boldsymbol{m})^2)$ and $\frac{1}{\Delta(\boldsymbol{m}^2)}$ from IZHC cancel to $\prod_{i<j} \frac{m_j-m_i}{m_j+m_i} = \mathrm{Pf}\big(\frac{m_j-m_i}{m_j+m_i}\big)$.
- [de Bruijn 55] integration formula would permit to express an integral of a Pfaffian times a determinant as Pfaffian of an integral. But Fubini requires integrability on $\mathbb{R}^N$ which is does not hold for singular kernels $\big(\frac{m_j-m_i}{m_j+m_i}\big)$. Replacement: Cauchy principal values

Lemma

Let $\rho > 0$ be a measurable on $\mathbb{R}$ and (f_n) be $\mathcal{C}^{N-1}$-functions on $\mathbb{R}_+$ with polynomially bounded derivatives. Let $S(x,y) = \frac{\tilde{S}(x,y)}{x+y}$ where $\tilde{S}$ is measurable on $\mathbb{R}^2$ such that $\int_{\mathbb{R}^2} |\tilde{S}(x,y)x^k y^l|\, \rho(x)\rho(y)\, dx\, dy < +\infty$ for all $k,l \in \mathbb{Z}_{\geq 0}$. Then, for N even

$$\int_{\mathbb{R}^N} \mathop{\mathrm{Pf}}_{1\leq i,j\leq N} \big(S(x_i,x_j)\big) \det_{\substack{0\leq m\leq N-1\\ 1\leq j\leq N}} \big(f_m(x_j^2)\big) \prod_{i=1}^n \rho(x_i) dx_i$$

$$= N! \mathop{\mathrm{Pf}}_{0\leq m,n\leq N-1} \Big(⨍_{\mathbb{R}^2} S(x,y) f_m(x^2) f_n(y^2)\, \rho(x)\rho(y)\, dx\, dy \Big),$$

where $⨍ = \lim_{\epsilon\to 0} \int_{|x+y|>\epsilon}$ and the integrand in the lhs is integrable.

The Pfaffian formula

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Theorem

For even N, we have

$$Z(\Lambda.V, \boldsymbol{t}) = \frac{\sqrt{\prod_{1\leq i,j\leq N}(\lambda_i+\lambda_j)}}{2^{\frac{N^2}{2}}(2\pi)^{\frac{N}{2}}\prod_{n=1}^{N-1} n!} \operatorname*{Pf}_{0\leq m,n\leq N-1}\big(K_{m,n}(\boldsymbol{t})\big), \qquad \text{where}$$

$$K_{m,n}(\boldsymbol{t}) = \fint_{\mathbb{R}^2} \frac{x-y}{x+y} F_m(x)F_n(y)\, e^{V(x)+V(y)+\sum_{m=0}^{\infty} t_{2m+1}(x^{2m+1}+y^{2m+1})} dx\, dy,$$

$$F_n(x) = \frac{(-2)^n n!}{\Delta(\boldsymbol{\lambda})} \det\big(\lambda_i^0 \,|\, \lambda_i^1 \,|\, \cdots \,|\, \lambda_i^{n-1} \,|\, e^{-\frac{1}{2}\lambda_i x^2} \,|\, \lambda_i^{n+1} \,|\, \cdots \,|\, \lambda_i^{N-1}\big)$$

Here, $\fint = \lim_{\epsilon\to 0}\int_{|x+y|\geq\epsilon}$ denotes the Cauchy principal value integral.

- We show next: a Pfaffian with such an integral kernel is a BKP τ-function.
- Needs the same Cauchy principal value integral; fails without symmetric regularisation.

Neutral fermions

- Consider fermionic operators $(\phi_j)_{j\in\mathbb{Z}}$ with anticommutor $\{\phi_j,\phi_k\}=(-1)^j\delta_{j+k,0}$ and vacuum $|0\rangle$ with $\phi_{-j}|0\rangle=0=\langle 0|\phi_j$ for $j>0$ and $\langle 0|\phi_0|0\rangle=0$. Let $\phi(x):=\sum_{j\in\mathbb{Z}}x^j\phi_j$
- Gives pair expectation values $\langle 0|\phi_j\phi_k|0\rangle=(-1)^k\delta_{j,-k}$ if $k>0$, $\langle 0|\phi_j\phi_0|0\rangle=\frac{1}{2}\delta_{j,0}$ and $\langle 0|\phi_j\phi_k|0\rangle=0$ if $k<0$.
- Vacuum expectations need radial ordering; If all $|x_i|$ pairwise distinct, then

$$\langle 0|\phi(x_1)\cdots\phi(x_N)|0\rangle := (-1)^{\mathrm{sign}(\pi)}\langle 0|\phi(x_{\pi(1)})\cdots\phi(x_{\pi(n)})|0\rangle$$
$$\text{if}\qquad |x_{\pi(1)}|>|x_{\pi(2)}|>\cdots>|x_{\pi(N)}|\,.$$

- $\langle 0|\phi(x_1)\phi(x_2)|0\rangle=\frac{1}{2}\frac{x_1-x_2}{x_1+x_2}$, convergent power series in $\frac{x_1}{x_2}$ for $|x_1|<|x_2|$, in $\frac{x_2}{x_1}$ for $|x_2|<|x_1|$
- Wick's theorem: For pairwise different $|x_i|$ one has for N even

$$\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_N)|0\rangle=\mathop{\mathrm{Pf}}_{1\leq k,l\leq N}\Big(\langle 0|\phi(x_k)\phi(x_l)|0\rangle\Big)=\frac{1}{2^{\frac{N}{2}}}\prod_{1\leq k<l\leq N}\frac{x_k-x_l}{x_k+x_l}\,,$$

and $\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_N)|0\rangle=0$ for N odd.

BKP flow

- Source operators $J_m = \frac{1}{2}\sum_{j\in\mathbb{Z}}(-1)^j\big(\phi_{-j-m}\phi_j - \langle 0|\phi_{-j-m}\phi_j|0\rangle\big)$ for $m \geq 0$
- Gives $J_{2m} \equiv 0$ and $[J_{2m+1}, J_{2n+1}] = 0$ and thus infinite family of commuting BKP flows

$$\gamma(\boldsymbol{t}) := e^{\sum_{m=0}^{\infty} J_{2m+1}t_{2m+1}} \text{ for } \boldsymbol{t} = (t_1, t_3, t_5, \dots), \quad \gamma(\boldsymbol{t})\phi(x) = e^{\sum_{m\geq 0} x^{2m+1}t_{2m+1}}\phi(x)\gamma(\boldsymbol{t}).$$

- For pairwise distinct $|x_1|, \dots, |x_N|$, let $\tau(\boldsymbol{t}; x_1, \dots, x_N) := \langle 0|\gamma(\mathrm{t})\phi(x_1)\cdots\phi(x_N)|0\rangle$.
- For $[z^{-1}] := (\frac{1}{z}, \frac{1}{3z^3}, \frac{1}{5\cdot z^5}, \dots)$ an exercise shows:

$$e^{\sum_{m\geq 0} z^{2m+1}t_{2m+1}}\tau(\boldsymbol{t} - 2[z^{-1}]; x_1, \dots, x_N) = 2\langle 0|\gamma(\boldsymbol{t})\phi(z)\phi(x_1)\cdots\phi(x_N)\phi_0|0\rangle\ .$$

Convergent for distinct $|x_i|$; analytic in $R < |z| < \tilde{R}$ if $\max_i |x_i| < R$ and $|t_{2m+1}| < \frac{1}{\tilde{R}^{2m+1}}$

- Multiply by a second copy $\boldsymbol{t} \mapsto \tilde{\boldsymbol{t}}$, $z \mapsto -z$, $x_i \mapsto x_i'$ and evaluate $\mathrm{Res}_{z=0}\frac{dz}{z}\dots$ by a contour integral in $R < z| < \tilde{R}$. On the rhs, anticommute $\phi(z)$ to the right.

One term is of the form of the BKP equation, but there are $2N$ additional terms.

Finishing

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Every additional term is antisymmetric in one pair x_j, x_j'. We remove it by a symmetric integration. Convergence requires to avoid the fat diagonals. Let

$$D^N_{R,\epsilon} := \{\boldsymbol{x} \in \mathbb{R}^N \mid \max_i |x_i| \leq R \text{ and } \min_{i<j} ||x_i| - |x_j|| \geq \epsilon\}, \quad \mathbb{1}^N_{R,\epsilon} \text{ its indicator function.}$$

Lemma

Let $h_1, \dots, h_N$ be $\mathcal{C}^1$ on $\mathbb{R}$ and h_i, h_i' be polynomially bounded. Let ρ be a positive even function that admits moments of any order. Then, as formal power series in $t_{2m+1}, \tilde{t}_{2m+1}$,

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \mathrm{Res}_{z=0} \frac{dz}{z} e^{\sum_{m\geq 0} z^{2m+1}(t_{2m+1}-\tilde{t}_{2m+1})} \tau(\boldsymbol{t} - 2[z^{-1}]; \boldsymbol{x}) \tau(\tilde{\boldsymbol{t}} + 2[z^{-1}]; \boldsymbol{x}') \\
&\qquad \times \mathbb{1}^{2N}_{\infty,\epsilon}(\boldsymbol{x}, \boldsymbol{x}') \prod_{i=1}^N h_i(x_i) h_i(x_i') \rho(x_i) \rho(x_i') dx_i\, dx_i' \\
&= \int_{\mathbb{R}^{2N}} \tau(\boldsymbol{t}; \boldsymbol{x}) \tau(\tilde{\boldsymbol{t}}; \boldsymbol{x}')\, \mathbb{1}^{2N}_{\infty,\epsilon}(\boldsymbol{x}, \boldsymbol{x}') \prod_{i=1}^N h_i(x_i) h_i(x_i') \rho(x_i) \rho(x_i') dx_i\, dx_i'.
\end{aligned}$$

Limit $\epsilon \to 0$

- We insert Wick's theorem, split h_i into even/odd parts and rearrange to integral over $x_i, x_i' > 0$. Symmetric ϵ-regularisation is crucial!
- Bound uniform in ϵ can be established, which justifies dominated convergence and Fubini.

Theorem

$$\operatorname*{Res}_{z=0}\Big(\frac{dz}{z}e^{\sum_{m\geq 0} z^{2m+1}(t_{2m+1}-\tilde{t}_{2m+1})}\mathcal{T}(\boldsymbol{t}-2[z^{-1}])\mathcal{T}(\tilde{\boldsymbol{t}}+2[z^{-1}])\Big) = \mathcal{T}(\boldsymbol{t})\mathcal{T}(\tilde{\boldsymbol{t}})\,, \quad \text{where}$$

$$\mathcal{T}(\boldsymbol{t}) = \operatorname*{Pf}_{1\leq i,j\leq N}\Big(\fint_{\mathbb{R}^2} H_{i,j}(\boldsymbol{t};x,y)\rho(x)\rho(y)dx\,dy\Big),$$

$$H_{i,j}(\boldsymbol{t};x,y) = e^{\sum_{m\geq 0} t_{2m+1}(x^{2m+1}+y^{2m+1})}\cdot\frac{1}{2}\frac{x-y}{x+y}\cdot h_i(x)h_j(y)\,.$$

The partition function $Z(\Lambda, V, \boldsymbol{t})$ of the Kontsevich model with any potential V is of that form: $\rho(x) = e^{-\frac{1}{2}\lambda_{\min}x^2}$, $h_i x) = e^{-\frac{1}{2}(\lambda_i-\lambda_{\min})x^2+V(x)}$ (up to irrelevant scalar factor)

Fourier transform

Consider the Fourier transform and its expansion ($X \in \mathcal{H}_N$)

$$F(X) := \frac{\int_{\mathcal{H}_N} dM\ e^{\mathrm{Tr}(-\frac{1}{2}N\Lambda M^2 + NV(M) + \mathrm{i}MX)}}{\int_{\mathcal{H}_N} dM\ e^{\mathrm{Tr}(-\frac{1}{2}N\Lambda M^2 + NV(M))}}$$

$$= \exp\Big(\sum_{p=1}^{\infty} \frac{\mathrm{i}^p}{pN^{p-1}} \sum_{k_1,\dots,k_p=1}^{N} G^{(0)}_{k_1k_2\dots,k_p} X_{k_1k_2}X_{k_2k_3}\cdots X_{k_pk_1} + N\text{-subleading}\Big),$$

where $G^{(0)}_{k_1k_2\dots,k_p} = G^{(0)}_{k_2\dots,k_pk_1}$ are cyclic.

- n-th moment obtained by n repeated differentiations with respect to $X_{i_kj_k}$:

$$\frac{\int_{\mathcal{H}_N} dM\ M_{i_1j_1}\cdots M_{i_nj_n}\ e^{\mathrm{Tr}(-\frac{1}{2}N\Lambda M^2 + NV(M))}}{\int_{\mathcal{H}_N} dM\ e^{\mathrm{Tr}(-\frac{1}{2}N\Lambda M^2 + NV(M))}}$$

$$= \sum_{\pi\in\mathcal{P}(n)} \prod_{\beta=(r_1,\dots,r_{|\beta|})\in\pi} \Big(\frac{1}{N^{|\beta|-1}|\beta|} \sum_{\sigma\in\mathcal{S}(|\beta|)} G^{(0)}_{i_{r_{\sigma(1)}}\dots i_{r_{\sigma(|\beta|)}}} \prod_{h=1}^{|\beta|} \delta_{i_{r_{\sigma(h)}},j_{r_{\sigma(h+1)}}}\Big).$$

Planar moments

- To get traces, for partition $(n_1, n_2, .., n_k)$ of $n = n_1 + ... + n_k$, introduce permutation $\gamma_{n_1...n_k} = (1, 2, .., n_1)(n_1, n_1+1, ..., n_1+n_2)\dots(n_1+...+n_{k+1}, n_1+...+n_{k+1}+1, ..., n)$.
- Moments of $\mathrm{Tr}(M^{n_1})\cdots\mathrm{Tr}(M^{n_k})$ obtained by multiplication with $\prod_{l=1}^{n}\delta_{j_l, i_{\gamma_{n_1...n_k}(l)}}$ and summation over all indices. Asks for cycles of $\gamma_{n_1..n_k}\pi$:

$$\frac{1}{N^k}\frac{\int_{\mathcal{H}_N} dM\ \mathrm{Tr}(M^{n_1})\cdots\mathrm{Tr}(M^{n_k})e^{\mathrm{Tr}(-\frac{1}{2}N\Lambda M^2+NV(M))}}{\int_{\mathcal{H}_N} dM\ e^{\mathrm{Tr}(-\frac{1}{2}N\Lambda M^2+NV(M))}}$$

$$= \sum_{\pi\in\mathcal{S}(n)}\frac{N^{2-2k-2g_\pi}}{N^{\#(\gamma_{n_1...n_k}\pi)}}\sum_{i_1,\dots,i_{\#(\gamma_{n_1...n_k}\pi)}=1}^{N}\Big(\prod_{c\in\pi} G^{(0)}_{\underline{i}_{[c:\gamma_{n_1...n_k}\pi]}}\Big).$$

- Notation: Label the cycles of $\gamma_{n_1...n_k}\pi$ in lexicographic order and denote by $[r:\gamma_{n_1...n_k}\pi]$ the label of the unique cycle of $\gamma_{n_1...n_k}\pi$ which contains r. Then, for a cycle $c = (l_1, l_2, ..., l_{|c|})$ we let $(\underline{i}_{[c:\gamma_{n_1...n_k}\pi]}) := (i_{[l_1:\gamma_{n_1...n_k}\pi]}, i_{[l_2:\gamma_{n_1...n_k}\pi]}, \dots, _{[l_{|c|}:\gamma_{n_1...n_k}\pi]})$.
- g_π minimal genus of surface upon which graph of π relative to $\gamma^{-1}_{n_1...n_k}$ can be embedded.

First BKP equation for even V

$$
\begin{aligned}
0 = \frac{45}{N^5} \sum_{i_1,i_2,i_3,i_4,i_5=1}^{N} & \big(G^{(0)}_{i_1 i_1 i_2 i_3 i_4 i_5} - G^{(0)}_{i_1 i_2 i_3 i_1 i_4 i_5}\big) \\
+ \frac{45}{N^5} \sum_{i_1,i_2,i_3=1}^{N} & \Big(G^{(0)}_{i_1 i_1 i_2 i_1 i_1 i_3} - G^{(0)}_{i_1 i_1 i_1 i_1 i_2 i_3} + G^{(0)}_{i_1 i_1 i_2 i_3 i_1 i_2} - G^{(0)}_{i_1 i_1 i_2 i_1 i_2 i_3} + G^{(0)}_{i_1 i_1 i_2 i_1 i_3 i_2} - G^{(0)}_{i_1 i_1 i_3 i_2 i_1 i_2} \\
& + G^{(0)}_{i_1 i_1 i_2 i_2 i_1 i_3} - G^{(0)}_{i_1 i_1 i_1 i_2 i_2 i_3} + G^{(0)}_{i_1 i_1 i_2 i_1 i_3 i_3} - G^{(0)}_{i_1 i_1 i_1 i_2 i_3 i_3} + G^{(0)}_{i_1 i_1 i_2 i_3 i_2 i_3} - \frac{1}{3} G^{(0)}_{i_1 i_2 i_3 i_1 i_2 i_3} - \frac{2}{3} G^{(0)}_{i_1 i_1 i_2 i_2 i_3 i_3}\Big) \\
+ \frac{45}{N^4} \sum_{i_1,i_2,i_3,i_4=1}^{N} & \Big(G^{(0)}_{i_1 i_1} G^{(0)}_{i_1 i_2 i_3 i_4} - G^{(0)}_{i_1 i_2} G^{(0)}_{i_1 i_1 i_3 i_4} + G^{(0)}_{i_2 i_3} G^{(0)}_{i_1 i_1 i_3 i_4} + G^{(0)}_{i_2 i_3} G^{(0)}_{i_1 i_1 i_4 i_3} - G^{(0)}_{i_1 i_2} G^{(0)}_{i_1 i_3 i_2 i_4} - G^{(0)}_{i_4 i_4} G^{(0)}_{i_1 i_1 i_2 i_3}\Big) \\
+ \frac{45}{N^4} \sum_{i_1,i_2=1}^{N} & \Big(G^{(0)}_{i_1 i_2} G^{(0)}_{i_1 i_1 i_2 i_2} - G^{(0)}_{i_1 i_1} G^{(0)}_{i_1 i_1 i_2 i_2} - G^{(0)}_{i_1 i_1} G^{(0)}_{i_1 i_2 i_2 i_2} + G^{(0)}_{i_1 i_1} G^{(0)}_{i_1 i_1 i_1 i_2} - G^{(0)}_{i_1 i_2} G^{(0)}_{i_1 i_1 i_1 i_1} + G^{(0)}_{i_1 i_1} G^{(0)}_{i_2 i_2 i_2 i_2}\Big) \\
+ \frac{45}{N^3} \sum_{i_1,i_2,i_3=1}^{N} & \Big(G^{(0)}_{i_1 i_1} G^{(0)}_{i_1 i_2} G^{(0)}_{i_2 i_3} - \frac{1}{3} G^{(0)}_{i_1 i_2} G^{(0)}_{i_2 i_3} G^{(0)}_{i_3 i_1} - 2 G^{(0)}_{i_1 i_1} G^{(0)}_{i_1 i_2} G^{(0)}_{i_3 i_3} + \frac{4}{3} G^{(0)}_{i_1 i_1} G^{(0)}_{i_2 i_2} G^{(0)}_{i_3 i_3}\Big)
\end{aligned}
$$

Quartic potential

Take $V(x) = -\frac{1}{4}x^4$. By previous work we know how to find any $1/N$-expanded moment.
Key step is the 2-point function (assuming $k \neq l$)

$$\sum_{g=0}^{\infty} N^{-2g} G_{kl}^{(g)} = N \frac{\int_{\mathcal{H}_N} dM \; M_{kl} M_{lk} e^{-\mathrm{Tr}(\Lambda N M^2 + \frac{1}{4} N M^4)}}{\int_{\mathcal{H}_N} dM \; e^{-\mathrm{Tr}(\Lambda N M^2 + \frac{1}{4} N M^4)}} .$$

Using integration by parts one can prove a quadratic equation for $G_{kl}^{(9)}$ alone [Grosse, W 09]:

$$\Big(\lambda_k + \lambda_l + \frac{1}{N}\sum_{n=1}^{N} G_{kn}^{(0)}\Big) G_{kl}^{(0)} = 1 + \frac{1}{N}\sum_{n=1}^{N} \frac{G_{nl}^{(0)} - G_{kl}^{(0)}}{\lambda_n - \lambda_k}$$

Complexify free variables $\lambda_k \mapsto \zeta$, $\lambda_l \mapsto \eta$ and introduce $G^{(0)}(\zeta,\eta)$ with $G^{(0)}(\lambda_k,\lambda_l) = G_{kl}^{(0)}$

$$\Big(\zeta + \eta + \frac{1}{N}\sum_{n=1}^{N} G^{(0)}(\zeta,\lambda_n)\Big) G^{(0)}(\zeta,\eta) = 1 + \frac{1}{N}\sum_{n=1}^{N} \frac{G^{(0)}(\lambda_n,\eta) - G^{(0)}(\zeta,\eta)}{\lambda_n - \zeta} .$$

Complexified equation permits tools from complex analysis. Decisive step was solution [Panzer-W 18] of $(N \to \infty)$-limit to integral equation combined with equidistant λ_i

Ansatz [Schürmann, W 19], building on [Panzer, W 18] and [Grosse, Hock, W 19]

Assume there is a ramified cover $x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with

1. x has degree $N+1$ and maps some domain $\mathcal{U}$ bijectively to neighbourhood of $\lambda_1, \dots, \lambda_N$
2. $\zeta = x(z)$, $\eta = x(w)$, $\lambda_k = x(\varepsilon_k)$, $G^{(0)}(\zeta, \eta) = \mathcal{G}^{(0)}(z, w)$
3. $$\boxed{x(z) + \frac{1}{N}\sum_{n=1}^{N} \mathcal{G}^{(0)}(z, \varepsilon_n) + \frac{1}{N}\sum_{n=1}^{N} \frac{1}{x(\varepsilon_n) - x(z)} = -x(-z)}$$

Gives $(x(w) - x(-z))\mathcal{G}^{(0)}(z, w) = 1 + \dfrac{1}{N}\displaystyle\sum_{n=1}^{N} \frac{\mathcal{G}^{(0)}(\varepsilon_k, w)}{x(\varepsilon_k) - x(z)}$

Solution

With techniques for Lagrange interpolation polynomials [Schechter 59] it is *easy* to establish:

Theorem [Schürmann, W 19]

- $x(z) = z - \frac{1}{N}\sum_{k=1}^{N} \frac{1}{x'(\varepsilon_k)(z+\varepsilon_k)}$ where $\lambda_k = x(\varepsilon_k)$
- $\mathcal{G}^{(0)}(z,w) = \frac{P_1^{(0)}(x(z),x(w))}{(x(z)+y(w))(x(w)+y(z))}$ where $\boxed{y(z) = -x(-z)}$ and

  $$P_1^{(0)}(x(z),x(w)) = \frac{\prod_{u\in x^{-1}(\{x(w)\})}(x(z)+y(u))}{\prod_{k=1}^{N}(x(z)-x(\varepsilon_k))}$$
- $\mathcal{G}^{(0)}(z,w)=\mathcal{G}^{(0)}(w,z)$
- The ansatz ③ is identically satisfied!

In [Branahl, Hock, W 20] extended to $\omega_{g,n}$ satisfying blobbed topological recursion.

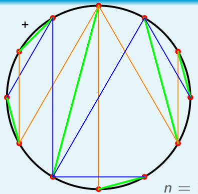
All planar cumulants

Planar cumulants $G^{(0)}_{a_1\dots a_n}$ extend to $\mathcal{G}^{(0)}(z_1,\dots,z_n)$:

Theorem [de Jong, Hock, W 19]

$\mathcal{G}^{(0)}(z_1,\dots,z_n)$ is sum of $\frac{2}{n}\binom{\frac{3n}{2}-2}{\frac{n}{2}-1}$ terms of the form $\frac{(-1)^{n/2-1}\prod_1^{n/2}\mathcal{G}^{(0)}(z_i,z_j)}{\prod_1^{n-2}(x(z_k)-x(z_l))}$

- Pattern in bijection with nested Catalan tables
- Graphically described in terms of non-crossing chords with a pair of dual planar rooted trees in every pocket.



In BKP equation, difference of 6-point functions has 14 terms which partly cancel:

$$G^{(0)}_{i_0 i_1 i_2 i_3 i_4 i_5} - G^{(0)}_{i_0 i_5 i_4 i_1 i_2 i_3} = \frac{\det\begin{pmatrix} G^{(0)}_{i_0 i_1} & G^{(0)}_{i_0 i_3} & G^{(0)}_{i_0 i_5} \\ G^{(0)}_{i_2 i_1} & G^{(0)}_{i_2 i_3} & G^{(0)}_{i_2 i_5} \\ G^{(0)}_{i_4 i_1} & G^{(0)}_{i_4 i_3} & G^{(0)}_{i_4 i_5} \end{pmatrix}}{(\lambda_{i_0}-\lambda_{i_2})(\lambda_{i_2}-\lambda_{i_4})(\lambda_{i_1}-\lambda_{i_5})(\lambda_{i_3}-\lambda_{i_5})}$$

Questions

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- For quartic potential, is the BKP equation combinatorially true? (independent of the precise form of the 2-point function)
- Or do we get a cubic differential equation for $\mathcal{G}^{(0)}(z_i, z_j)$?
- If so, can we verify this equation from the explicit solution?
- (Very difficult:) Conversely, does the hierarchy of BKP equations and the Catalan structure determine the 2-point function?
  Or is the Catalan structure compatible with other (even) potentials?
- Is there a solution strategy for Kontsevich matrix models with potentials other than x^3 and x^4?
- Is there (blobbed) topological recursion?