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Blobbed topological recursion of the $\lambda\Phi^4$-matrix model

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CRC 1442 GEOMETRY: DEFORMATIONS AND RIGIDITY

MM Mathematics Münster Cluster of Excellence

- No interacting QFT-model in 4 dimensions is in sight. Either 4D models are too difficult (Yang-Mills, millenium prize problem), or trivial ($\lambda\phi^4_4$ [Aizenman, Duminil-Copin 19]).
- Quantum field theories on noncomutative geometries provide a new class of 4D QFT-models to try. They violate symmetry axioms, but renormalisation is very similar.
- The simplest one is the $\lambda\Phi^4$-model on noncommutative Moyal space. Has deformation parameter θ. D-dim Moyal space (D even) is Fréchet-isomorphic to the space of infinite matrices with rapidly decaying entries.
- In Euclidean approach, have (formal) measure

$$d\mu_\lambda(\Phi)\text{``:=''}\frac{1}{\mathcal{Z}}d\mu_0(\Phi)\;\exp\Big(-\frac{\lambda}{4}\mathcal{N}\,\mathrm{Tr}(\Phi^4)\Big)\;,\qquad \mathcal{N}:=\Big(\frac{\theta}{4}\Big)^{D/2}\;.$$

- $d\mu_0$ is Gaußian, defined by covariance. Simplest choice is

$$\langle\Phi_{kl}\Phi_{mn}\rangle=\int d\mu_0(\Phi)\;\Phi_{kl}\Phi_{mn}=\frac{\delta_{kn}\delta_{lm}}{\mathcal{N}(E_k+E_l)}\;,\qquad E_k>0$$

- Sequence (E_k) defines spectral dimension $\delta=\inf(p>0\;:\;\sum_{k=0}^\infty E_k^{-p/2}<\infty)$.

Since the product of distributions is not defined, we meet various divergences of QFT.

- Restrict all sums $\sum_{k=0}^{\infty} \mapsto \sum_{k=0}^{\Lambda^D \mathcal{N}}$. Shift covariance to

$$\int d\mu_0^{\Lambda}(\Phi)\, \Phi_{kl}\Phi_{mn} = \frac{\delta_{kn}\delta_{lm}}{\mathcal{N}(E_k + E_l + M^2(\Lambda))}$$

- Rescale n-point functions by $Z(\Lambda)^{-n/2}$, e.g. 2-point functions

$$G_{|ab|} = \frac{\mathcal{N}}{Z(\Lambda)} \int d\mu_\lambda^{\Lambda}(\Phi)\, \Phi_{ab}\Phi_{ba}\,, \qquad G_{|a|b|} = \frac{\mathcal{N}^2}{Z(\Lambda)} \int d\mu_\lambda^{\Lambda}(\Phi)\, \Phi_{aa}\Phi_{bb}\,.$$

- Remark: no renormalisation of λ necessary (β-function $= 0$, asymptotic safety)

Standard QFT procedure

For finite Λ, identify $M(\Lambda), Z(\Lambda)$ such that limit $\lim_{\Lambda\to\infty} G_{|ab|}$ exists and show that this turns all moments/cumulants finite for $\Lambda \to \infty$.

Equations of motion

Lemma [Schürmann, W 19]

The Fourier transform $\mathcal{Z}(f) := \int d\mu_\lambda^\Lambda(\Phi)\, e^{\mathrm{i}\Phi(f)}$ of the measure satisfies

1. $$\frac{1}{\mathrm{i}}\frac{\partial \mathcal{Z}(f)}{\partial f_{ab}} = \frac{\mathrm{i} f_{ba}\mathcal{Z}(f)}{\mathcal{N}(E_a+E_b+M^2)} - \frac{\lambda}{\mathrm{i}^3(E_a+E_b+M^2)}\sum_{k,l=0}^{\Lambda^D\mathcal{N}} \frac{\partial^3\mathcal{Z}(f)}{\partial f_{ak}\partial f_{kl}\partial f_{lb}}$$

2. $$\frac{1}{\mathcal{N}}\frac{\partial \mathcal{Z}(f)}{\partial E_a} = \sum_{k=0}^{\Lambda^D\mathcal{N}} \frac{\partial^2\mathcal{Z}(f)}{\partial f_{ak}\partial f_{ka}} + \Big(\frac{1}{\mathcal{N}}\sum_{k=0}^{\Lambda^D\mathcal{N}} ZG_{|ak|} + \frac{1}{\mathcal{N}^2} ZG_{|a|a|}\Big)\mathcal{Z}(f)$$

Corollary (Ward-Takahashi identity [Disertori, Gurau, Magnen, Rivasseau 06])

3. $$-\mathcal{N}\sum_{k=0}^{\Lambda^D\mathcal{N}}(E_a-E_b)\frac{\partial^2\mathcal{Z}(f)}{\partial f_{ak}\partial f_{kb}} = \sum_{k=0}^{\Lambda^D\mathcal{N}}\Big(f_{ka}\frac{\partial\mathcal{Z}(f)}{\partial f_{kb}} - f_{bk}\frac{\partial\mathcal{Z}(f)}{\partial f_{ak}}\Big)$$

Dyson-Schwinger equations

The equation of motion 1 induces Dyson-Schwinger equations between moments. Thereby an n-point function is expressed in terms of $n+2$-point functions. Using 3 and 2 one can avoid this. For the 2-point function one finds

$$
\begin{aligned}
(E_a+E_b)ZG_{|ab|} &= 1+\frac{\lambda}{\mathcal{N}}\sum_{\substack{k=0\\ k\neq a}}^{\Lambda^D\mathcal{N}}\frac{ZG_{|kb|}-ZG_{|ab|}}{E_k-E_a}+\frac{\lambda}{\mathcal{N}^2}\frac{ZG_{|b|b|}-ZG_{|a|b|}}{E_b-E_a} \\
&\quad -ZG_{|ab|}\Big(\frac{\lambda}{\mathcal{N}}\sum_{k=0}^{\Lambda^D\mathcal{N}}ZG_{|ak|}+\frac{\lambda}{\mathcal{N}^2}ZG_{|a|a|}\Big)+\frac{Z\lambda}{\mathcal{N}}\frac{\partial G_{|ab|}}{\partial E_a}
\end{aligned}
$$

Complexification

- The dependence of $G_{|ab|}$ on the matrix indices a, b is of the form of an evaluation $G_{|ab|} = G(E_a, E_b)$ of a function G of two complex variables at E_a, E_b.
- $G(\zeta, \eta)$ still depends on summation variables E_k. Differentiating wrt some E_c is also an evaluation at E_c of another function $-\mathcal{N}\frac{\partial}{\partial E_c} G(\zeta, \eta) = T(E_c \| \zeta, \eta|)$.
- Similarly for $G_{|a|b|} = G(E_a|E_b)$.

Complexified Dyson-Schwinger equation

$$
\begin{aligned}
(\zeta + \eta + M^2) Z G(\zeta, \eta) &= 1 + \frac{\lambda}{\mathcal{N}} \sum_{k=0}^{\Lambda^D \mathcal{N}} \frac{Z G(E_k, \eta) - Z G(\zeta, \eta)}{E_k - \zeta} + \frac{\lambda}{\mathcal{N}^2} \frac{Z G(\eta|\eta) - Z G(\zeta|\eta)}{\eta - \zeta} \\
&\quad - Z G(\zeta, \eta) \Big(\frac{\lambda}{\mathcal{N}} \sum_{k=0}^{\Lambda^D \mathcal{N}} Z G(\zeta, E_k) + \frac{\lambda}{\mathcal{N}^2} Z G(\zeta|\zeta) \Big) - \frac{\lambda}{\mathcal{N}^2} Z T(\zeta \| \zeta, \eta|)
\end{aligned}
$$

The genus expansion

We approach the solution of such equations in a formal genus expansion

$$G(\zeta,\eta) = \sum_{g=0}^{\infty} \mathcal{N}^{-2g} G^{(g)}(\zeta,\eta)\,, \qquad G(\zeta|\eta) = \sum_{g=0}^{\infty} \mathcal{N}^{-2g} G^{(g)}(\zeta|\eta)\,,$$

$$T(\xi\|\zeta,\eta|) = \sum_{g=0}^{\infty} \mathcal{N}^{-2g} T^{(g)}(\xi\|\zeta,\eta|)$$

together with the convention that $\frac{1}{\mathcal{N}}$ in front of a summation is neutral.

These series have zero radius of convergence! Making sense of them via Borel resummation is a main challenge for the future. Connection to to resurgence?

Theorem [Grosse, W 09]

The planar two-point function satisfies the closed non-linear equation

$$\Big(\zeta+\eta+M^2+\frac{\lambda}{\mathcal{N}}\sum_{k=0}^{\Lambda^D\mathcal{N}} ZG^{(0)}(\zeta,E_k)\Big)ZG^{(0)}(\zeta,\eta) = 1+\frac{\lambda}{\mathcal{N}}\sum_{k=0}^{\Lambda^D\mathcal{N}} \frac{ZG^{(0)}(E_k,\eta)-ZG^{(0)}(\zeta,\eta)}{E_k-\zeta}$$

Integral equation

We can arrange the two-point function of a large family of matrix and QFT models with quartic interaction into the integral equation

$$\Big(\zeta+\eta+M^2+\lambda\int_0^\infty dt\,\varrho_0(t) Z G^{(0)}(\zeta,t)\Big) Z G^{(0)}(\zeta,\eta) = 1+\lambda\int_0^\infty dt\,\varrho_0(t)\frac{Z(G^{(0)}(t,\eta)-G^{(0)}(\zeta,\eta))}{t-\zeta}$$

- large-θ 4D Moyal: $\varrho_0(t)=t\chi_{[0,\Lambda^2]}(t)$ and $M=M(\Lambda)$, $Z=Z(\Lambda)$
- large-θ 2D Moyal: $\varrho_0(t)=\chi_{[0,\Lambda^2]}(t)$ and $M=M(\Lambda)$, $Z=1$
- $N\times N$ matrix model $\varrho(t)=\frac{1}{N}\sum_{k=1}^d r_k\delta(t-e_k)$, $r_1+\ldots+r_d=N$, $M=0$, $Z=1$

The spectral dimension becomes $d_{spec} := \inf\big(p>0 \;:\; \int_0^\infty \frac{\varrho_0(t)}{(1+t)^{p/2}}<\infty\big)$

- In [Panzer, W 18] we solved the case $\varrho_0(t)=1$. Key step was to extrapolate a computer algebra evaluation of iterated integrals.
- In [Grosse, Hock, W 19a] we succeeded in solving the integral equation for any Hölder-continuous measure ϱ_0 of spectral dimension $d_{spec}<6$.

Solution

Theorem [Panzer-W 18 for $\varrho_0 = 1$, Grosse-Hock-W 19a]

1. Ansatz $G^{(0)}(\zeta, \eta) = \dfrac{e^{\mathcal{H}_\zeta[\tau_\eta(\bullet)]} \sin \tau_\eta(\zeta)}{Z\lambda\pi\varrho_0(\zeta)}$, $\quad \mathcal{H}_\zeta[f] := \frac{1}{\pi} ⨍_0^{\Lambda^2} \frac{dp\, f(p)}{p-\zeta}$ finite Hilbert transf.
2. $\tau_\eta(\zeta) = \lim_{\epsilon \searrow 0} \operatorname{Im} \log\big(\eta - R_D(-m^2 - R_D^{-1}(\zeta + \mathrm{i}\epsilon)\big)$
3. $$\boxed{R_D(z) = z - \lambda(-z)^{D/2} \int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(m^2+t)^{D/2}(t+m^2+z)}} \qquad D = 2[\tfrac{\delta}{2}]$$
4. ϱ_λ is implicit solution of $\varrho_0(R_D(\zeta)) = \varrho_\lambda(\zeta)$.

- Proof: [Cauchy 1831] residue theorem, [Lagrange 1770] inversion theorem, [Bürmann 1799] formula.
- $\varrho_0(t) \equiv 1$ (2D Moyal, $m = 1$) in terms of Lambert-W satisfying $W(z)e^{W(z)} = z$.

$D = 4$ Moyal space: $\varrho_0(t) = t$ [Grosse-Hock-W 19b]

- $\varrho_\lambda(x) \equiv \varrho_0(R_4(x)) = R_4(x) = x - \lambda x^2 \int_0^\infty \frac{dt\, \varrho_\lambda(t)}{(m^2+t)^2(t+x)}$
- If $\varrho_\lambda(t) \sim \varrho_0(t) = t$, then $R_4(x)$ bounded above. Consequently, R_4^{-1} would not be globally defined: triviality!
- Fredholm equation perturbatively solved by iterated integrals: Hyperlogarithms and $\zeta(2n)$ which can be summed to

$$R_4(x) \equiv \varrho_\lambda(x) = x \cdot {}_2F_1\left(\begin{matrix}\alpha_\lambda, 1-\alpha_\lambda \\ 2\end{matrix}\middle| -\frac{x}{m^2}\right) \qquad \alpha_\lambda = \begin{cases} \frac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda| \leq \frac{1}{\pi} \\ \frac{1}{2} + \mathrm{i}\frac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{for } \lambda \geq \frac{1}{\pi} \end{cases}$$

Corollary

The interaction alters the spectral dimension to $4 - 2\frac{\arcsin(\lambda\pi)}{\pi}$ and thus avoids the triviality problem (in the planar sector).

Gives non-perturbative integral representation for $G^{(0)}(\xi, \eta)$.

Finite matrices $\varrho_0(t) = \frac{1}{N}\sum_{k=1}^d r_k \delta(t-e_k)$

Suppose there is a rational function R of degree $d+1$, with simple pole at ∞ of residue -1 and

$$R(z) + \frac{\lambda}{N}\sum_{k=1}^d r_k G^{(0)}(R(z), e_k) + \frac{\lambda}{N}\sum_{k=1}^d \frac{r_k}{e_k - R(z)} = -R(-z)\,.$$

Setting $\zeta = R(z)$, $\eta = R(w)$ and $G^{(0)}(\zeta,\eta) = \mathcal{G}^{(0)}(z,w)$ and chosing $\varepsilon_k \in R^{-1}(e_k)$, the non-linear equation becomes

$$(R(w) - R(-z))\mathcal{G}^{(0)}(z,w) = 1 + \frac{\lambda}{N}\sum_{k=1}^d r_k \frac{\mathcal{G}^{(0)}(\varepsilon_k, w)}{R(\varepsilon_k) - R(z)}$$

Theorem [Schürmann, W 19]

1. $R(z) = z - \frac{\lambda}{N}\sum_{k=1}^d \frac{\varrho_k}{\varepsilon_k + z}$ where $R(\varepsilon_k) = E_k$ and $R'(\varepsilon_k)\varrho_k = r_k$.
2. $\mathcal{G}^{(0)}(z,w) = \dfrac{P_1^{(0)}(R(z),R(w))}{(R(z)-R(-w))(R(w)-R(-z))}$ where

$$P_1^{(0)}(R(z),R(w)) = \frac{\prod_{u\in R^{-1}(\{w\})}(R(z)-R(-u))}{\prod_{k=1}^d (R(z)-R(\varepsilon_k))} \equiv P_1^{(0)}(R(w),R(z))$$

Higher genus and auxiliary functions [Branahl, Hock, W 20]

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The DSE for 2-point functions contain $T^{(g)}(E_a\|\zeta,\eta|) := -N\frac{\partial}{\partial E_a}G^{(g)}(\zeta,\eta)$ and $T^{(g)}(E_a\|\zeta|\eta|) := -N\frac{\partial}{\partial E_a}G^{(g)}(\zeta|\eta)$. To get equations for them we differentiate again, and so on. Another combination appears:

$$\Omega^{(g)}_{q_1,\dots,q_n} := \frac{(-N)^{n-1}\partial^{n-1}\big(\frac{1}{N}\sum_{k=1}^N G^{(g)}_{|kq_1|} + G^{(g-1)}_{|q_1|q_1|}\big)}{\partial E_{q_2}\cdots\partial E_{q_n}} + \frac{\delta_{g,0}\delta_{n,2}}{(E_{q_1}-E_{q_2})^2}$$

- Complexify all these auxiliary functions and pass to preimages

$$\tilde{\Omega}^{(g)}_n(R(z_1),..,R(z_n)) =: \Omega^{(g)}_n(z_1,\dots,z_n),$$
$$T^{(g)}(R(z_1),..,R(z_n)\|R(w_1),R(w_2)|) =: \mathcal{T}^{(g)}(z_1,\dots,z_n\|w_1,w_2|),$$
$$T^{(g)}(R(z_1),..,R(z_n)\|R(w_1)|R(w_2)|) =: \mathcal{T}^{(g)}(z_1,\dots,z_n\|w_1|w_2|)$$

  with $\Omega^{(g)}_n(\varepsilon_{q_1},\dots,\varepsilon_{q_n}) = \Omega^{(g)}_{q_1,\dots,q_n}$ and similarly for the T.
- Pass to meromorphic differentials
  $\omega^{(g)}_n(z_1,\dots,z_n) = \lambda^{2-2g-n}\Omega^{(g)}_n(z_1,\dots,z_m)\prod_{k=1}^n dR(z_k)$.

Contact with topological recursion

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A lengthy calculation gives $\omega_2^{(0)}(z_1,z_2) = \frac{dz_1\,dz_2}{(z_1-z_2)^2} + \frac{dz_1\,dz_2}{(z_1+z_2)^2}$ and then

$$\omega_3^{(0)}(u_1,u_2,z) = -\sum_{i=1}^{2d} \frac{\Big(\frac{1}{(u_1-\beta_i)^2}+\frac{1}{(u_1+\beta_i)^2}\Big)\Big(\frac{1}{(u_2-\beta_i)^2}+\frac{1}{(u_2+\beta_i)^2}\Big)du_1\,du_2\,dz}{R'(-\beta_i)R''(\beta_i)(z-\beta_i)^2}$$

$$+\Big[d_{u_1}\Big(\frac{\omega_2^{(0)}(u_2,u_1)}{(dR)(u_1)}\frac{dz}{R'(-u_1)(z+u_1)^2}\Big)+u_1\leftrightarrow u_2\Big]$$

$$\omega_1^{(1)}(z) = \sum_{i=1}^{2d}\frac{dz}{R'(-\beta_i)R''(\beta_i)}\Big\{-\frac{1}{8(z-\beta_i)^4}+\frac{\frac{1}{24}x_{1,i}}{(z-\beta_i)^3}+\frac{(x_{2,i}+y_{2,i}-x_{1,i}y_{1,i}-x_{1,i}^2-\frac{6}{\beta_i^2})}{48(z-\beta_i)^2}$$

$$-\frac{dz}{8(R'(0))^2z^3}+\frac{R''(0)dz}{16(R'(0))^3z^2}$$

where $\beta_{1,\dots,2d}$ solve $dR(\beta_i)=0$ (ramification pnts), $x_{n,i} := \frac{R^{(n+2)}(\beta_i)}{R''(\beta_i)}$, $y_{n,i} := \frac{(-1)^n R^{(n+1)}(-\beta_i)}{R'(-\beta_i)}$

Observation [Branahl, Hock, W 20]

The blue terms correspond to topological recursion for $x(z)=R(z)$ & $y(z)=-R(-z)$, the magenta terms signal an extension to blobbed topological recursion [Borot-Shadrin 15].

Details: Dyson-Schwinger equations

In terms of

$$\mathcal{T}^{(g)}(u_1,\dots,u_n\|v,z|) =: \partial_{x(u_1)}\cdots\partial_{x(u_1)} U^{(g)}_{n+1}(v,z;u_1,\dots,u_n)$$

$$\mathcal{T}^{(g)}(u_1,\dots,u_n\|v|z|) =: \partial_{x(u_1)}\cdots\partial_{x(u_1)} V^{(g)}_{n+1}(v,z;u_1,\dots,u_n)$$

$$\Omega^{(g)}_{n+1}(z,u_1,\dots,u_n) =: \partial_{x(u_1)}\cdots\partial_{x(u_1)} W^{(g)}_{n+1}(z;u_1,\dots,u_n)$$

have coupled system ❶, ❷, ❸ of Dyson-Schwinger equations between $(U^{(g)}_n, V^{(g)}_n, W^{(g)}_n)$.

❶

$$W^{(g)}_{|I|+1}(z;I) = \frac{\delta_{|I|,1}\delta_{g,0}}{x(z)-x(u_1)} + \frac{\delta_{|I|,0}\delta_{g,0}}{\lambda}\Big(x(z)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{x(\varepsilon_k)-x(z)}\Big)$$

$$+\frac{1}{N}\sum_{l=1}^{d} r_l U^{(g)}_{|I|+1}(z,\varepsilon_l;I) - \sum_{j=1}^{|I|} U^{(g)}_{|I|}(z,u_i;I\backslash u_i) + V^{(g-1)}_{|I|+1}(z,z;I)\,.$$

- Here and below, $I=\{u_1,\dots,u_n\}$ collects spectator variables $u_j\in\hat{\mathbb{C}}$.
- Have $x(z)=R(z)$ and $y(z)=-R(-z)$.
- There are $d+1$ preimages $R^{-1}(R(z))=\{z=\hat{z}^0,\hat{z}^1,\dots,\hat{z}^d\}$ of $R(z)$.
- The other equations are better rearranged:

Auxiliary functions partially symmetric in preimages

Introducing

$$H^{(g)}_{|I|+1}(x(v);z;I) := \delta_{g,0}\delta_{I,\emptyset} - \frac{\lambda}{N}\sum_{l=1}^{d}\frac{r_l U^{(g)}_{|I|+1}(z,\varepsilon_l;I)}{x(v)-x(\varepsilon_l)} + \lambda\sum_{j=1}^{|I|}\frac{U^{(g)}_{|I|}(z,u_j;I\backslash u_j)}{x(v)-x(u_j)} - \lambda\frac{V^{(g-1)}_{|I|+1}(z,z;I)}{x(v)-x(z)}\,,$$

$$M^{(g)}_{|I|+1}(x(v);z;I) := -\frac{\lambda}{N}\sum_{l=1}^{d}\frac{r_l V^{(g)}_{|I|+1}(z,\varepsilon_l;I)}{x(v)-x(\varepsilon_l)} + \lambda\sum_{j=1}^{|I|}\frac{V^{(g)}_{|I|}(z,u_j;I\backslash u_j)}{x(v)-x(u_j)} - \lambda\frac{U^{(g)}_{|I|+1}(z,z;I)}{x(v)-x(z)}\,,$$

then (sums below are $\sum' = \sum_{I_1\uplus I_2=I,\ g_1+g_2=g,\ (g_1,I_1)\neq(0,\emptyset)}$)

2.
$$H^{(g)}_{|I|+1}(x(v);z;I) = (x(z)+y(v))U^{(g)}_{|I|+1}(v,z;I) + \lambda\sum{}' W^{(g_1)}_{|I_1|+1}(v;I_1)U^{(g_2)}_{|I_2|+1}(v,z;I_2)$$
$$+\lambda\frac{\partial}{\partial x(s)}U^{(g-1)}_{|I|+2}(v,z;I\cup s)\Big|_{s=v} + \lambda\frac{V^{(g-1)}_{|I|+1}(v,z;I)}{x(z)-x(v)}\,,$$

3.
$$M^{(g)}_{|I|+1}(x(v);z;I) = (x(v)+y(v))V^{(g)}_{|I|+1}(v,z;I) + \lambda\sum{}' W^{(g_1)}_{|I_1|+1}(v;I_1)V^{(g_2)}_{|I_2|+1}(v,z;I_2)$$
$$+\lambda\frac{\partial}{\partial x(s)}V^{(g-1)}_{|I|+2}(v,z;I\cup s)\Big|_{s=v} + \lambda\frac{U^{(g)}_{|I|+1}(v,z;I)}{x(z)-x(v)}\,.$$

Auxiliary functions completely symmetric in preimages

GEOMETRY: DEFORMATIONS AND RIGIDITY

Set

$$P^{(g)}_{|I|+1}(x(v),x(z);I) := \delta_{|I|,0}\delta_{g,0}x(z) - \frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k H^{(g)}_{|I|+1}(x(v);\varepsilon_k;I)}{x(z)-x(\varepsilon_k)} + \lambda\sum_{j=1}^{|I|}\frac{H^{(g)}_{|I|}(x(v);u_j;I\setminus u_j)}{x(z)-x(u_j)}$$
$$+\frac{\lambda\delta_{|I|,1}\delta_{g,0}}{x(v)-x(u_1)} + \delta_{|I|,0}\delta_{g,0}\Big(x(v)-\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{x(v)-x(\varepsilon_k)}\Big)\,,$$

$$Q^{(g)}_{|I|+1}(x(v),x(z);I) := -\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{M^{(g)}_{|I|+1}(x(v);\varepsilon_k;I)}{x(z)-x(\varepsilon_k)} + \lambda\sum_{j=1}^{|I|}\frac{M^{(g)}_{|I|+1}(x(v);u_j;I\backslash u_j)}{x(z)-x(u_j)}\,,$$

then

$$P^{(g)}_{|I|+1}(x(v),x(z);I) = (x(v)+y(z))H^{(g)}_{|I|+1}(x(v);z;I) + \lambda\sum W^{(g_1)}_{|I_1|+1}(z;I_1)H^{(g)}_{|I_2|+1}(x(v);z;I_2)$$
$$+\lambda\frac{\partial}{\partial x(s)}H^{(g-1)}_{|I|+2}(x(v);z;I\cup s)\Big|_{s=z} + \lambda\frac{M^{(g-1)}_{|I|+1}(x(v);z;I)}{x(v)-x(z)}\,,$$

$$Q^{(g)}_{|I|+1}(x(v),x(z);I) = (x(z)+y(z))M^{(g)}_{|I|+1}(x(v);z;I) + \lambda\sum W^{(g_1)}_{|I_1|+1}(z;I_1)M^{(g_2)}_{|I_2|+1}(x(v);z;I_2)$$
$$+\lambda\frac{\partial}{\partial x(s)}M^{(g-1)}_{|I|+2}(x(v);z;I\cup s)\Big|_{s=z} + \lambda\frac{H^{(g)}_{|I|+1}(x(v);z;I)}{x(v)-x(z)}\,.$$

Remarks

1. $U^{(g)}_{|I|+1}(v,z;I) = U^{(g)}_{|I|+1}(z,v;I)$, $V(v,z;I) = V(z,v;I)$,
$P^{(g)}_{|I|+1}(x(v),x(z);I) = P^{(g)}_{|I|+1}(x(z),x(v);I)$, $Q^{(g)}_{|I|+1}(x(v),x(z);I) = Q^{(g)}_{|I|+1}(x(z),x(v);I)$
are symmetric! Exceptional consequence of $y(z) = -x(-z)$!

2. Decoupling of $(\hat{U},\hat{H},\hat{P})$ from $(\hat{V},\hat{M},\hat{Q})$ with (C_h are the Catalan numbers)

$$\begin{pmatrix} \hat{U}^{(g)}(v,z;I) \\ \hat{H}^{(g)}(x(v);z;I) \\ \hat{P}^{(g)}(x(v),x(z);I) \end{pmatrix} = \begin{pmatrix} U^{(g)}(v,z;I) \\ H^{(g)}(x(v);z;I) \\ P^{(g)}(x(v),(z);I) \end{pmatrix} + \sum_{h=0}^{g-1} \frac{(-1)^h \lambda^{2h+1} C_h}{(x(v)-x(z))^{4h+2}} \begin{pmatrix} V^{(g-h-1)}(v,z;I) \\ M^{(g-h-1)}(x(v);z;I) \\ Q^{(g-h-1)}(x(v),x(z);I) \end{pmatrix}$$

plus $(U,H,P) \leftrightarrow (V,M,Q)$ but with $\sum_{h=0}^{g-1} \mapsto -\sum_{h=0}^{g}$.

3. These equations are the analogue of the single equation

$$\begin{aligned}(y(w)-y(z))H^{(g),TR}_{n+1}(y(w);z;I) + P^{(g),TR}_{n+1}(y(w);x(z);I) \\ = -\sum W^{(g_2),TR}_{|I_1|+1}(z;I_1)H^{(g_2),TR}_{|I_2|+1}(y(w);z;I_2) - \partial_{x(z')}H^{(g-1),TR}_{n+2}(y(w);z;z',I)\Big|_{z'=z}\end{aligned}$$

discovered by [Chekhov, Eynard, Orantin 06] for the Hermitian 2-matrix model. From this and $H^{(g),TR}_{n+1}(y(w);z;I) = \frac{1}{y(w)}W^{(g),TR}_{n+1}(z;I) + \mathcal{O}(y(w)^{-2})$ they proved TR.

Solution for $g = 0$

- Loop insertion operators D_u and $D_{\{u_1,\dots,u_n\}} = D_{u_1} \cdots D_{u_n}$ as convenient abbreviation

$$D_u(x(z)) = 0, \quad D_u y(z) = \lambda W_2^{(0)}(z;u), \quad D_u W^{(g)}_{|I|+1}(z;I) = W^{(g)}_{|I|+2}(z;I \cup u),$$

and $D_u \hat{F}^{(g)}_{|I|+1}(v,z;I) = \hat{F}^{(g)}_{|I|+2}(v,z;I\cup u)$ for $F \in \{U,H,P,V,M,Q\}$.

- Equations for $g = 0$ turn into

$$D_I \log P_1^{(0)}(x(v),x(z)) = D_I \log(x(v)+y(z)) + D_I \log H_1^{(0)}(x(v);z)$$

Proposition [Hock, W 23]

$$D_I \log H_1^{(0)}(x(v);z) = \sum_{k=1}^{d} D_I \log(x(v)+y(\hat{z}^k)) + F^{(0)}_{|I|+1}(x(v);x(z);I)\ ,$$

$$D_I \log P_1^{(0)}(x(v),x(z)) = \sum_{k=0}^{d} D_I \log(x(v)+y(\hat{z}^k)) + F^{(0)}_{|I|+1}(x(v);x(z);I)$$

with $F^{(0)}_{|I|+1}(x(v);x(z);I) = \sum_{j=1}^{|I|} D_{I\setminus u_j} \dfrac{\lambda}{(x(v)-x(u_j))(x(z)+y(u_j))}$ (from eq. for $H^{(0)}$)

Linear and quadratic loop equations for $g = 0$

Comparing the obvious expansion of $P^{(0)}_{|I|+1}(x(v), x(z); I)$ at $x(v) = \infty$ with the one resulting from the defining equation and $H^{(g)}_{|I|+1}(x(v); z; I) = -\frac{\lambda}{x(v)} W^{(g)}_{|I|+1}(z; I) + \mathcal{O}((x(v))^{-2})$ gives:

Proposition

The functions $W^{(0)}_{|I|+1}$ satisfy for $I \neq \emptyset$ the global linear loop equations

$$\sum_{k=0}^{d} W^{(0)}_{|I|+1}(\hat{z}^k; I) = \frac{\delta_{|I|,1}}{(x(z) - x(u_1))} - \sum_{j=1}^{|I|} D_{I \backslash u_j}\Big(\frac{1}{x(z) + y(u_j)}\Big)$$

and the global quadratic loop equations

$$\sum_{k=0}^{d} y(\hat{z}^k) W^{(0)}_{|I|+1}(\hat{z}^k; I) + \frac{\lambda}{2} \sum_{\substack{I_1 \uplus I_2 = I \\ I_1, I_2 \neq \emptyset}} \sum_{k=0}^{d} W^{(0)}_{|I_1|+1}(\hat{z}^k; I_1) W^{(0)}_{|I_2|+1}(\hat{z}^k; I_2)$$

$$= \sum_{j=1}^{|I|} D_{I \backslash u_j}\Big(\frac{x(u_j)}{x(z)+y(u_j)}\Big) - \frac{\lambda}{N} \sum_{k=1}^{d} \frac{r_k W^{(0)}_{|I|+1}(\varepsilon_k; I)}{x(z) - x(\varepsilon_k)} + \lambda(1-\delta_{|I|,1}) \sum_{j=1}^{|I|} \frac{W^{(0)}_{|I|+1}(u_j; I \backslash u_j)}{x(z) - x(u_j)} + \frac{\delta_{|I|,1} y(u_1)}{x(z) - x(u_1)}.$$

Discussion

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Linear and quadratic loop equations are in some sense the heart of topological recursion.

- $$\sum_{k=0}^{d} W^{(g)}_{|I|+1}(\hat{z}^k; I) \text{ and } \sum_{k=0}^{d}\Big(W^{(g-1)reg}_{|I_2|+1}(\hat{z}^k,\hat{z}^k;I) + \sum_{\substack{I_1\uplus I_1=I\\ g_1+g_2=g}} W^{(g_1)}_{|I_1|+1}(\hat{z}^k;I_1)W^{(g_2)}_{|I_2|+1}(\hat{z}^k;I_2)\Big)$$
  are required to be holomorphic in neighbourhood of the ramification points of x, where $W^{(0)}_1(q;\emptyset) := y(q)$ [Borot, Eynard, Orantin 13].
- If $W^{(g)}_{|I|+1}(z;I)dx(z)$ has (for $2g+|I|>1$) poles only at ramification points of x, then the recursion kernel representation of TR follows.
- Pole condition seems artificial. [Borot, Shadrin 15] suggested to require just the linear & quadratic loop equations (locally), nothing else. Then the part with poles at ramification points is still be given by the TR formula, but there can be something else, called blobs.

In our case, the linear and quadratic loop equations hold globally, they contain precise information about all poles of $W^{(g)}_{|I|+1}(z;I)dx(z)$. We get a recursion for the whole $\omega^{(g)}_n$, not only for the part with poles at ramification points.

Solution for $\hat{Q}^{(0)}$

By a similar method applied to

$$\hat{Q}_1^{(0)}(x(v),x(z)) = -\frac{\lambda(x(v)+x(z)-2x(0))}{2(x(v)-x(z))^2}\sqrt{\frac{P_1^{(0)}(x(v),x(v))P_1^{(0)}(x(z),x(z))}{(x(v)-x(0))(x(z)-x(0))}}$$

(found in [Schürmann, W 19], here $y(z)=-x(-z)$ is essential) one finds:

Proposition

$$D_I \log \hat{Q}_1^{(0)}(x(v),x(z))$$
$$= \frac{1}{2} D_I \log P_1^{(0)}(x(v),x(v)) + \frac{1}{2} D_I \log P_1^{(0)}(x(z),x(z))$$
$$-\frac{1}{2}\sum_{l=1}^{|I|}\frac{(-1)^{l-1}}{l}\Big(\frac{2^{l+1}}{(x(v)+x(z)-2x(0))^l}-\frac{1}{(x(v)-x(0))^l}-\frac{1}{(x(z)-x(0))^l}\Big)\sum_{\substack{I_1\uplus\ldots\uplus I_l=I\\ I_1,\ldots,I_l\neq\emptyset}}\prod_{i=1}^{l} D^0_{I_i}x(0)\,,$$

where $D^0_{I_i}x(0)$ is uniquely determined by holomorpicity of $D_I \log \hat{Q}_1^{(0)}(x(v),x(z))$ at $v=0$.

Theorem [Hock, W 23]: $\hat{P}^{(1)}_{|I|+1}(x(v),x(z);I)$

$$
\begin{aligned}
D_I \frac{\hat{P}^{(1)}_1(x(v),x(z))}{P^{(0)}_1(x(v),x(z))} &= \sum_{k=0}^{d} D_I \frac{\lambda W^{(1)}_1(\hat{z}^k)}{x(v)+y(\hat{z}^k)} + \frac{\lambda^2}{2} \sum_{\substack{j,k=0\\ j\neq k}}^{d} D_I \frac{\Omega^{(0)}_2(\hat{z}^j,\hat{z}^k)}{(x(v)+y(\hat{z}^j))(x(v)+y(\hat{z}^k))} \\
&- \sum_{j=1}^{|I|} \frac{1}{(x(v)-x(u_j))} \sum_{k=0}^{d} D_{I\setminus u_j} \frac{\lambda^3 \Omega^{(0)}_2(\hat{z}^k,u_j)}{(x(v)+y(\hat{z}^k))(x(z)+y(u_j))^2} \\
&- \frac{\lambda^2}{(x(v)-x(z))^3} \frac{\partial}{\partial x(w)} \Big(D_I \log P^{(0)}_1(x(v),x(w)) - D_I \log P^{(0)}_1(x(z),x(w)) \Big)_{w=z} \\
&+ \frac{\lambda^2}{(x(v)-x(z))^2} \Big(-D^0_I \frac{1}{8(x(z)-x(0))^2} + \frac{1}{2} \frac{\partial^2 (D_I \log P^{(0)}_1(x(w),x(z)))}{\partial x(w) \partial x(z)} \Big|_{w=z} \Big) \\
&+ \frac{1}{x(v)-x(z)} D^0_I \frac{\lambda^2}{8(x(z)-x(0))^3} \\
&+ \sum_{j=1}^{|I|} \frac{1}{(x(v)-x(u_j))} D_{I\setminus u_j} \Big\{ \frac{\lambda^3 \Omega^{(0)reg}_2(u_j,u_j)}{(x(z)+y(u_j))^3} - \frac{\lambda^2 W^{(1)}_1(u_j)}{(x(z)+y(u_j))^2} \\
&\qquad + \frac{\lambda^3}{2(x(z)+y(u_j))^2} \frac{\partial^2}{\partial (x(u_j))^2} \frac{1}{(x(z)+y(u_j))} \Big\} \\
&+ \sum_{i,j=1,\ i<j}^{|I|} D_{I\setminus\{u_i,u_j\}} \frac{\lambda^4 \Omega^{(0)}_2(u_i,u_j)}{(x(v)-x(u_j))(x(v)-x(u_i))(x(z)+y(u_i))^2(x(z)+y(u_j))^2} .
\end{aligned}
$$

$$
\Omega^{(0)reg}_2(u,u) = \lim_{u'\to u} \Big(\Omega^{(0)}_2(u,u') - \frac{1}{(x(u)-x(u'))^2} \Big)
$$

Linear and quadratic loop equations for $g = 1$

Proposition [Hock, W 23]

The genus-1 meromorphic functions $W^{(1)}_{|I|+1}(z; I)$ satisfy the linear loop equation

$$\sum_{k=0}^{d} W^{(1)}_{|I|+1}(\hat{z}^k; I) = -D^0_I \frac{\lambda}{8(x(z)-x(0))^3}$$
$$- \sum_{j=1}^{|I|} D_{I\setminus u_j} \Big\{ \frac{\lambda^2 \Omega^{(0)reg}_2(u_j,u_j)}{(x(z)+y(u_j))^3} - \frac{\lambda W^{(1)}_1(u_j)}{(x(z)+y(u_j))^2}$$
$$- \frac{\lambda^2}{2(x(z)+y(u_j))^2} \frac{\partial^2}{\partial(x(u_j))^2} \frac{1}{(x(z)+y(u_j))} \Big\}$$

and ...

Linear and quadratic loop equations for $g = 1$

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Proposition [Hock, W 23]

...the quadratic loop equation

$$\sum_{k=0}^{d} y(\hat{z}^k) W^{(1)}_{|I|+1}(\hat{z}^k; I) + \lambda \sum_{\substack{I_1 \uplus I_2 = I \\ I_2 \neq \emptyset}} \sum_{k=0}^{d} W^{(1)}_{|I|+1}(\hat{z}^k; I_1) W^{(0)}_{|I|+1}(\hat{z}^k; I_2) + \frac{\lambda}{2} \sum_{k=0}^{d} D_I \Omega_2^{(0)reg}(\hat{z}^k, \hat{z}^k)$$

$$= \frac{\lambda^2}{6} \sum_{j=1}^{|I|} \frac{\partial}{\partial x(u_j)} \Big(D_{I \setminus u_j} \frac{1}{(x(z) + y(u_j))^3} \Big) - D_I^0 \frac{\lambda}{8(x(z) - x(0))^2} + x(z) D_I^0 \frac{\lambda}{8(x(z) - x(0))^3}$$

$$+ \sum_{j=1}^{|I|} x(u_j) D_{I \setminus u_j} \Big\{ \frac{\lambda^2 \Omega_2^{(0)reg}(u_j, u_j)}{(x(z) + y(u_j))^3} - \frac{\lambda W_1^{(1)}(u_j)}{(x(z) + y(u_j))^2} - \frac{\lambda^2}{2(x(z) + y(u_j))^2} \frac{\partial^2}{\partial (x(u_j))^2} \frac{1}{(x(z) + y(u_j))} \Big\}$$

$$- \frac{\lambda}{N} \sum_{l=1}^{d} \frac{W^{(1)}_{|I|+1}(\varepsilon_l; I)}{x(z) - x(\varepsilon_l)} + \lambda \sum_{j=1}^{|I|} \frac{W^{(1)}_{|I|}(u_j; I \setminus u_j)}{x(z) - x(u_j)} .$$

Main theorem [Hock, W 23]: recursion formula for $g \leq 1$

The linear and quadratic loop equations are equivalent to

$$\begin{aligned}
\omega^{(g)}_{|I|+1}(z,I) = &-\sum_{\beta_i} \operatorname*{Res}_{q\to\beta_i} \frac{\frac{1}{2}\big(\frac{dz}{z-q}-\frac{dz}{z-\sigma_i(q)}\big)}{(y(q)-y(\sigma_i(q)))dx(q)} \Big\{ {\sum}' \omega^{(g_1)}_{|I_1|+1}(q,I_1)\omega^{(g_2)}_{|I_2|+1}(q,I_2) + \omega^{(g-1)}_{|I|+2}(q,q,I)\Big\} \\
&- \sum_{j=1}^{|I|} d_{u_j}\Big[\operatorname*{Res}_{q\to -u_j} \frac{\frac{1}{2}\big(\frac{dz}{z-q}-\frac{dz}{z+u_j}\big)}{(y(q)-y(-u_j))dx(q)} \Big\{ {\sum}' d^{-1}_{u_j}(\omega^{(g_1)}_{|I_1|+1}(q,I_1)\omega^{(g_2)}_{|I_2|+1}(q,I_2)) \\
&\qquad + d^{-1}_{u_j}\omega^{(g-1)}_{|I|+2}(q,q,I) + \frac{(dx(q))^2}{6}\frac{\partial^2}{\partial(x(q))^2}\Big(\frac{\omega^{(g-1)}_{|I|+1}(q,I))}{dx(q)dx(u_j)}\Big)\Big\}\Big] \\
&- \operatorname*{Res}_{q\to 0} \frac{\frac{1}{2}\big(\frac{dz}{z-q}-\frac{dz}{z}\big)}{(y(q)-y(-q))dx(q)} \Big\{ {\sum}' \omega^{(g_1)}_{|I_1|+1}(q,I_1)\omega^{(g_2)}_{|I_2|+1}(q,I_2) + \omega^{(g-1)}_{|I|+2}(q,q,I) \\
&\qquad + \frac{(dx(q))^2}{2}\frac{\partial}{\partial x(q)}\Big(\frac{d^{-1}_{q'}\omega^{(g-1)}_{|I|+2}(q,q',I)}{dx(q)}\Big|_{q'=q}\Big)\Big\},
\end{aligned}$$

where $\omega^{(g)}_{n+1}(z,u_1,...u_n) = \lambda^{2g+n-1} d_{u_1}\cdots d_{u_n} W^{(g)}_{n+1}(z;u_1,...,u_n)dx(z)$, the sums are ${\sum}' = \sum_{I_1 \uplus I_2 = I,\ g_1+g_2=g,\ (g_i,I_i)\neq(0,\emptyset)}$. By β_i we denote the ramification points of x, σ_i is the Galois involution near β_i and $\omega^{(0)}_2(q,q) \mapsto \lim_{q'\to q}(\omega^{(0)}_2(q,q') - \frac{dx(q)\,dx(q')}{(x(q)-x(q'))^2})$ is understood.

Outlook

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There is no principal problem which prevents pushing these structures to any genus g.

- The terms which after expansion at $x(v)=\infty$ give $\sum_{k=0}^{d} W^{(g)}_{|I|+1}(\hat{z}^k;I)$ and $\sum_{k=0}^{d}\big(W^{(g-1)reg}_{|I_2|+1}(\hat{z}^k,\hat{z}^k;I)+\sum W^{(g_1)}_{|I_1|+1}(\hat{z}^k;I_1)W^{(g_2)}_{|I_2|+1}(\hat{z}^k;I_2)\big)$ are always there. There cannot be any contribution with poles at β_i.
- Hence, the linear and quadratic loop equation terms are locally, in neighbourhood of ramification points, always holomorphic.
- Thus, the $\lambda\Phi^4$-matrix model satisfies blobbed topological recursion in its original sense.
- The stronger question, to establish the linear and quadratic loop equation for all g globally, is a difficult combinatorial problem. The number of terms increases quickly with g, we need Taylor expansions up to order $4g-3$.
- Maybe it is realistic to reach $g=2$ to check whether the recursion kernel representation remains stable or needs additional $g-2$ contributions.