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Blobbed topological recursion of quartic matrix models

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CRC 1442 GEOMETRY: DEFORMATIONS AND RIGIDITY

MM Mathematics Münster Cluster of Excellence

The (incomplete) zoo of Hermitian one-matrix models

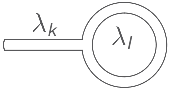
For $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_N)$ and $V(M) = \sum_{i=1}^{p} \frac{g_i}{i} M^i$, various measured on $H_N = \{\text{Hermitian } N \times N \text{ matrices}\}$ have been studied (dM is Lebesgue):

1. Original model $d\mu_{\Lambda,V}(M) = \frac{1}{\mathcal{Z}} e^{-\mathrm{Tr}(V(M))} dM$ [Brézin, Itzykson, Parisi, Zuber 78]
   - Generating function to enumerate maps
   - TR for spectral curve $x(z) = \alpha + \gamma(z + \frac{1}{z})$ and $y(z) \simeq W_1^{(0)}(x(z))$
2. Kontsevich model $d\mu_{\Lambda,V}(M) = \frac{1}{\mathcal{Z}} e^{-\mathrm{Tr}(\Lambda M^2 + \frac{\mathrm{i}}{3} M^3)} dM$ [Kontsevich 92]
   - generates intersection numbers on $\overline{\mathcal{M}}_{g,n}$, equivalent to 1 ([Witten 91] conjecture)
   - also an external field model $d\mu_{\Lambda,V}(M) = \frac{1}{\mathcal{Z}} e^{-\mathrm{i}\,\mathrm{Tr}(\Lambda^2 M + \frac{1}{3} M^3)} dM$
   - TR for spectral curve $x(z) = z^2$ and $y(z) = -z + \sum_{k=1}^{N} \frac{1}{\hat{\lambda}_k(\hat{\lambda}_k - z)}$
3. Generalised Kontsevich model $d\mu_{\Lambda,V}(M) = \frac{1}{\mathcal{Z}} e^{-\mathrm{Tr}(V(M) - V(\Lambda) - (M-\Lambda)V'(\Lambda))} dM$ [Belliard, Charbonnier, Eynard, Garcia-Failde 21]
   - generates r-spin intersection numbers where $r = p - 1$
   - TR for spectral curve $x(z) = z^r$ and $y(z) = z$,
4. External field matrix model $d\mu_{\Lambda,V}(M) = \frac{1}{\mathcal{Z}} e^{-Tr(\Lambda M - V(M))} dM$ [Eynard-Orantin 08]
   - TR for spectral curve $x(z) = Q(z)$ and $y(z) = z - \mathrm{Tr}(\frac{1}{Q'(\Lambda)(z-\Lambda)})$ with Q built from V'

Moments of the free Hermitian matrix model

Consider measure $d\mu_\Lambda(M) = \frac{1}{\mathcal{Z}} e^{-\operatorname{Tr}(\Lambda M^2)} dM$ on H_N.

- Correlators $\langle \mathcal{O}(M) \rangle_\Lambda := \int_{H_N} d\mu_\Lambda(M)\, \mathcal{O}(M)$ of polynomials in matrix entries given as sum over pairings with covariance $\langle M_{kl} M_{mn} \rangle_\Lambda = \frac{\delta_{kn}\delta_{lm}}{\lambda_k + \lambda_l}$
- $\langle \operatorname{Tr} M^{r_1} \cdots \operatorname{Tr} M^{r_v} \rangle_\Lambda$ represented as sum over closed ribbon graphs:
  - v vertices of valencies $r_1, ..., r_v$
  - n faces labelled $\lambda_{k_1}, ..\lambda_{k_n}$, at the end summation over $k_i = 1, ..., N$
  - edges given weight $\frac{1}{\lambda_k + \lambda_l}$ if shared by faces with labels λ_k and λ_l



$$\langle \operatorname{Tr} M \operatorname{Tr} M^3 \rangle_\Lambda = 3 \sum_{k,l=1}^{N} \frac{1}{(2\lambda_k)(\lambda_k + \lambda_l)} = \frac{3}{4} \Big(\sum_{k=1}^{N} \frac{1}{\lambda_k} \Big)^2$$

Observation

$\langle \operatorname{Tr} M^{r_1} \cdots \operatorname{Tr} M^{r_v} \rangle_\Lambda$ with all r_i odd evaluate to polynomials in power sum symmetric functions $p_{2n+1} = \sum_{k=1}^{N} \frac{1}{\lambda_k^{2n+1}}$. Any presence of an even $\operatorname{Tr} M^{2j}$ gives much more complicated expressions without apparent structure.

Why should we look at expectation values involving $\mathrm{Tr}\, M^{2j}$?

- No interacting QFT-model in 4 dimensions is in sight. 4D models are either too difficult (Yang-Mills, millenium prize problem), or trivial (ϕ^4_4 [Aizenman, Duminil-Copin 19]).
- Quantum field theories on noncomutative geometries provide a new class of 4D QFT-models to try. They violate symmetry axioms, but renormalisation is very similar.
- The simplest one is the Φ^4-model on noncommutative Moyal space, which is Fréchet-isomorphic to infinite matrices with rapidly decaying entries.
- In Euclidean approach, have (formal) measure

$$d\mu_\lambda(\Phi) \text{``}{:=}\text{''} \frac{1}{\mathcal{Z}} d\mu_0(\Phi)\ \exp\Big(-\frac{g}{4}\mathcal{N}\,\mathrm{Tr}(\Phi^4)\Big)\,, \qquad \mathcal{N} := \Big(\frac{\theta}{4}\Big)^{D/2}\,.$$

- $d\mu_0$ is Gaußian, defined by covariance. Simplest choice is

$$\langle \Phi_{kl}\Phi_{mn}\rangle = \int d\mu_0(\Phi)\ \Phi_{kl}\Phi_{mn} = \frac{\delta_{kn}\delta_{lm}}{\mathcal{N}(\lambda_k+\lambda_l)}\,,$$

  where $\lambda_k > 0$ are the eigenvalues of a Laplacian in D dimensions.

A better knowledge of the even case would help to understand this QFT model.

$\mathcal{M}_{g,n}$ moduli space of genus-g curves with n marked points

- Assigning positive numbers $(b_1, \ldots, b_n)$ to the marked points, we obtain the decorated moduli space $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$.
- Strebel differentials give rise to cell decomposition $\mathcal{M}_{g,n} \times \mathbb{R}_+^n \simeq \sum_{\Gamma \in Rib_{g,n}} \mathbb{R}_+^{e(\Gamma)}/Aut(\Gamma)$ with sum over genus g-ribbon graphs with n faces and e edges.
- Restricting this homeomorphism to a fixed tuple $(b_1, \ldots, b_n)$ of $b_i \in \mathbb{Z}_{>0}$ yields a space homeomorphic to $\mathcal{M}_{g,n}$ decomposed into compact convex integral polytopes.
- Lattice count polynomial $N_{g,n}(b_1, \ldots, b_n) = \sum_{\Gamma \in Rib_{g,n}} N_\Gamma(b_1, \ldots, b_n)/Aut(\Gamma)$ [Norbury 10] as weighted sum of the numbers $N_\Gamma(b_1, \ldots, b_n)$ of lattice points in such a polytope.

1. Top degree of $N_{g,n}$ is a half of [Kontsevich 92] volume polynomial and expressible in terms of intersection numbers of ψ-classes on $\overline{\mathcal{M}}_{g,n}$. Only ribbon graphs with 3-valent vertices contribute to top degree.
2. The orbifold Euler characteristic [Harer, Zagier 86] is $\chi(\mathcal{M}_{g,n}) = N_{g,n}(0, \ldots, 0)$ and needs ribbon graphs with vertices of ***any*** valency.

Ribbon graphs with even-valent vertices are relevant and deserve a better understanding.

BKP hierarchy [Date, Jimbo, Kashiwara, Miwa 82]

Deform previous Hermitian matrix measure to $d\mu_{\Lambda,V}(M) = \frac{1}{\mathcal{Z}} e^{-\operatorname{Tr}(\Lambda M^2 + V(M))} dM$, where $V = \sum_{i=3}^{p} \frac{g}{i} M^i$ is ***any*** polynomial potential. Set $\langle \mathcal{O}(M) \rangle_{\Lambda,V} := \int_{H_N} \mathcal{O}(M) d\mu_{\Lambda,V}(M)$. Then:

Theorem [Borot]

$\tau_{\Lambda,V}(\boldsymbol{t}_B) := \langle e^{\sum_{i=0}^{\infty} t_{2i+1} \operatorname{Tr}(M^{2i+1})} \rangle_{\Lambda,V}$ is a τ-function of the BKP hierarchy, i.e. it satisfies

$$\tau_{\Lambda,V}(\boldsymbol{t}_B)\tau_{\Lambda,V}(\boldsymbol{s}_B) \equiv \operatorname*{Res}_{z=0} \Big[\frac{dz}{z} e^{\sum_{i=0}^{\infty} z^{2i+1}(t_{2i+1} - s_{2i+1})} \tau_{\Lambda,V}(\boldsymbol{t}_B - 2[z^{-1}]_B) \tau_{\Lambda,V}(\boldsymbol{s}_B + 2[z^{-1}]_B) \Big]$$

identically in $\boldsymbol{t}_B = (t_1, t_3, t_5, \ldots)$ and $\boldsymbol{s}_B = (s_1, s_3, s_5, \ldots)$, where $[z^{-1}]_B = (\frac{1}{z}, \frac{1}{3z^3}, \frac{1}{5z^5}, \ldots)$.

Gives hierarchy of quadratic equations between moments.

First BKP equation for even potential $V(M) = V(-M)$

$$\begin{aligned} 0 \equiv\, & 15\langle (\operatorname{Tr} M)^4 \rangle_{\Lambda,V} \langle (\operatorname{Tr} M)^2 \rangle_{\Lambda,V} - 15\langle (\operatorname{Tr} M \operatorname{Tr} M^3 \rangle_{\Lambda,V} \langle (\operatorname{Tr} M)^2 \rangle_{\Lambda,V} \\ & + \langle (\operatorname{Tr} M)^6 \rangle_{\Lambda,V} - 5\langle (\operatorname{Tr} M)^3 \operatorname{Tr} M^3 \rangle_{\Lambda,V} - 5\langle (\operatorname{Tr} M^3)^2 \rangle_{\Lambda,V} + 9\langle (\operatorname{Tr} M \operatorname{Tr} M^5 \rangle_{\Lambda,V} . \end{aligned}$$

Statement of the challenge

We are interested in measures $d\mu_{\Lambda,V}(M) = \frac{1}{\mathcal{Z}} e^{-\operatorname{Tr}(\Lambda M^2 + V(M))} dM$ for Hermitian matrices where V contains even monomials.

- Expanding V produces expectation values without apparent pattern.
- Nevertheless, moments of $d\mu_{\Lambda,V}$ involving only odd $\operatorname{Tr} M^{2n+1}$ satisfy BKP equations.

In a first step we focus on $V(M) = \frac{1}{4} M^4$.

Programme

1. Give definition of $\omega_n^{(g)}$ and identify a spectral curve $(x, y : \mathbb{P}^1 \to \mathbb{P}^1, \omega_{0,2})$.
2. Turn Dyson-Schwinger equations of matrix model into linear & quadratic loop equations.

Without additional identities (patternless structure of correlators does not give them!), such loop equations imply blobbed topological recursion [Borot, Shadrin 15].

- First found in enumeration of stuffed maps [Borot 13]
- Also in intermediate field rep. of tensor models [Bonzom, Dartois 16; Bonzom, Dubb 20]

Blobbed topological recursion [Borot, Shadrin 15]

Set $\omega_n^{(g)}(z_1,\dots,z_n) =: W_n^{(g)}(z_1,\dots,z_n)\prod_{k=1}^n dx(z_k)$ and $I=\{z_1,\dots,z_n\}$

Linear and quadratic loop equations

$$L_{n+1}^{(g)}(x(z);I) = \sum_{\hat{z}\in x^{-1}\{x(z)\}} W_{n+1}^{(g)}(\hat{z},I)\;,$$

$$Q_{n+1}^{(g)}(x(z);I) = \sum_{\hat{z}\in x^{-1}\{x(z)\}} \Big(\sum_{\substack{g_1+g_2=g\\ I_1\uplus I_2=I}} W_{|I_1|+1}^{(g_1)}(\hat{z},I_1)W_{|I_2|+1}^{(g_2)}(\hat{z},I_3) + W_{|I|+2}^{(g-1,reg)}(\hat{z},\hat{z},I)\Big)\;,$$

where $W_1^{(0)}(z)=y(z)$ and $W_{|I|+2}^{(g,reg)}(z,w,I) = W_{|I|+2}^{(g)}(z,w,I) - \frac{\delta_{g,0}\delta_{|I|,0}}{(x(z)-w(z))^2}$.

The $\omega_n^{(g)}$ obey blobbed TR if $L_{n+1}^{(g)}(x(z);I)$, $Q_{n+1}^{(g)}(x(z);I)$ holomorphic at branch points of x.

They obey standard TR if $L_n^{(g)}$ is constant for $2g+n-2>0$.

Remark: BTR is just as good as TR and determines the $\omega_n^{(g)}$ uniquely. More precisely:

- Part of $\omega_n^{(g)}$ with poles at ramification points of x from TR recursion formula.
- Knowledge of $L_{n+1}^{(g)}$, $Q_{n+1}^{(g)}$ near their poles implies recursion formula for the whole $\omega_n^{(g)}$.

Quartic potential

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Consider the partition function $\mathcal{Z}_{\Lambda,4} := \int_{H_N} dM\, e^{-N\,\mathrm{Tr}(\Lambda M^2 + \frac{1}{4}M^4)}$, define
$\langle \mathcal{O}(M)\rangle_{\Lambda,4} = \frac{1}{\mathcal{Z}_{\Lambda,4}} \int_{H_N} dM\, \mathcal{O}(M) e^{-N\,\mathrm{Tr}(\Lambda M^2 + \frac{1}{4}M^4)}$.

Main definition [Branahl, Hock W 20]

$W^{(g)}_{a_1,\dots,a_n} := [N^{2-2g-n}] \frac{(-1)^n \partial^n \log \mathcal{Z}_{\Lambda,4}}{\partial\lambda_{a_1}\cdots\partial\lambda_{a_n}} + \frac{\delta_{g,0}\delta_{n,2}}{(\lambda_{a_1}-\lambda_{a_2})^2} + \delta_{g,0}\delta_{n,1} f(\lambda_{a_1})$ for $a_1,\dots,a_n$ pairwise different

- Procedure consists in deriving equations for the $W^{(g)}_{a_1,\dots,a_n}$ which should extend to complexified equations for $\tilde{W}^{(g)}_n(\xi_1,\dots,\xi_n)$ with $\tilde{W}^{(g)}_n(\lambda_{a_1},\dots,\lambda_{a_n}) = W^{(g)}_{a_1,\dots,a_n}$.
- Need auxiliary functions $\frac{(-1)^n \partial^n \langle M_{kl} M_{lk}\rangle_{\Lambda,4}}{\partial\lambda_{a_1}\cdots\partial\lambda_{a_n}}$ and $\frac{(-1)^n \partial^n \langle M_{kk} M_{ll}\rangle_{\Lambda,4}}{\partial\lambda_{a_1}\cdots\partial\lambda_{a_n}}$ also to complexify.
- Non-linear equation for $G^{(0)}_{kl} := [N^{-1}]\langle M_{kl} M_{lk}\rangle_{\Lambda,4} \mapsto G^{(0)}(\xi_1,\xi_2)$ can be solved and provides $\xi \equiv x(z)$ for TR.
- Get $y(z) \simeq \frac{1}{N}\sum_{k=1}^{N} G^{(0)}(\underbrace{x(z),\lambda_k}_{\dots}) + f(x(z))$ for TR.

The planar 2-point function

$(\lambda_1, \ldots, \lambda_d)$ – pairwise different eigenvalues with multiplicities $(r_1, \ldots, r_d)$.

Theorem [Grosse, W 09]

$$\Big(\zeta+\eta+\frac{1}{N}\sum_{k=1}^{d} r_d G^{(0)}(\zeta,\lambda_k)\Big)G^{(0)}(\zeta,\eta) = 1+\frac{1}{N}\sum_{k=1}^{d} r_d \frac{G^{(0)}(\lambda_k,\eta)-G^{(0)}(\zeta,\eta)}{\lambda_k-\zeta}$$

Theorem [Schürmann, W 19]

A solution can be implicitly found in the form $G^{(0)}(x(z),x(w)) =: \mathcal{G}^{(0)}(z,w)$ with $x(z) = z - \frac{1}{N}\sum_{k=1}^{N}\frac{\varrho_k}{\varepsilon_k+z}$ and $x(\varepsilon_k)=\lambda_k$ and $x'(\varepsilon_k)\varrho_k = r_k$:

$$\mathcal{G}^{(0)}(z,w) = \frac{P_1^{(0)}(x(z),x(w))}{(x(z)+y(w))(x(w)+y(z))} \quad \text{where } \boxed{y(z)=-x(-z)} \text{ and}$$

$$P_1^{(0)}(x(z),x(w)) = \frac{\prod_{u\in x^{-1}(\{x(w)\})}(x(z)+y(u))}{\prod_{k=1}^{d}(x(z)-x(\varepsilon_k))} \equiv P_1^{(0)}(x(w),x(z))$$

Contact with topological recursion

A lengthy calculation gives $\omega_2^{(0)}(z_1,z_2) = \frac{dz_1\,dz_2}{(z_1-z_2)^2} + \frac{dz_1\,dz_2}{(z_1+z_2)^2}$ and then

$$\omega_3^{(0)}(u_1,u_2,z) = -\sum_{i=1}^{2d} \frac{\Big(\frac{1}{(u_1-\beta_i)^2}+\frac{1}{(u_1+\beta_i)^2}\Big)\Big(\frac{1}{(u_2-\beta_i)^2}+\frac{1}{(u_2+\beta_i)^2}\Big)du_1\,du_2\,dz}{y'(\beta_i)x''(\beta_i)(z-\beta_i)^2}$$

$$+\Big[d_{u_1}\Big(\frac{\omega_2^{(0)}(u_2,u_1)}{(dx)(u_1)}\frac{dz}{y'(u_1)(z+u_1)^2}\Big)+u_1\leftrightarrow u_2\Big]$$

$$\omega_1^{(1)}(z) = \sum_{i=1}^{2d}\frac{dz}{y'(\beta_i)x''(\beta_i)}\Big\{-\frac{1}{8(z-\beta_i)^4}+\frac{\frac{1}{24}x_{1,i}}{(z-\beta_i)^3}+\frac{(x_{2,i}+y_{2,i}-x_{1,i}y_{1,i}-x_{1,i}^2-\frac{6}{\beta_i^2})}{48(z-\beta_i)^2}$$

$$-\frac{dz}{8(x'(0))^2z^3}+\frac{x''(0)dz}{16(x'(0))^3z^2}$$

where $\beta_{1,\dots,2d}$ solve $dx(\beta_i)=0$ (ramification points), $x_{n,i} := \frac{x^{(n+2)}(\beta_i)}{x''(\beta_i)}$, $y_{n,i} := \frac{y^{(n+1)}(\beta_i)}{y'(\beta_i)}$

Observation [Branahl, Hock, W 20]

The blue terms correspond to topological recursion for $y(z)=-x(-z)$, the magenta terms signal an extension to blobbed topological recursion.

Details: Dyson-Schwinger equations

We are guided by [Chekhov, Eynard, Orantin 06] who discovered for the 2-matrix model

$$(y(w)-y(z))H^{(g),TR}_{n+1}(y(w);z;I)+P^{(g),TR}_{n+1}(y(w);x(z);I)$$
$$=-\sum W^{(g_2),TR}_{|I_1|+1}(z,I_1)H^{(g_2),TR}_{|I_2|+1}(y(w);z;I_2)-H^{(g-1),TR}_{n+2}(y(w);z;z',I)\big|_{z'=z}$$

From this and $H^{(g),TR}_{n+1}(y(w);z;I)=\frac{1}{y(w)}W^{(g),TR}_{n+1}(z,I)+\mathcal{O}(y(w)^{-2})$ they proved TR.

Our situation is more complicated: We need extended loop equations for 7 families of functions. Matrix model gives coupled system ①, ②, ③ of DSE's between $(U^{(g)}_n, V^{(g)}_n, W^{(g)}_n)$.

① $$W^{(g)}_{|I|+1}(z,I)=\frac{\delta_{|I|,1}\delta_{g,0}}{(x(z)-x(u_1))^2}+\delta_{|I|,0}\delta_{g,0}\Big(x(z)+\frac{1}{N}\sum_{k=1}^{d}\frac{r_k}{x(\varepsilon_k)-x(z)}\Big)$$
$$+\frac{1}{N}\sum_{l=1}^{d}r_l U^{(g)}_{|I|+1}(z,\varepsilon_l;I)-\sum_{j=1}^{|I|}\frac{\partial}{\partial x(u_j)}U^{(g)}_{|I|}(z,u_j;I\backslash u_j)+V^{(g-1)}_{|I|+1}(z,z;I)\,.$$

- Here and below, $I=\{u_1,...,u_n\}$ collects spectator variables $u_j\in\mathbb{P}^1$.
- There are $d+1$ preimages $x^{-1}(x(z))=\{z=\hat{z}^0,\hat{z}^1,....,\hat{z}^d\}$ of $x(z)$.

Auxiliary functions partially symmetric in preimages

Introducing

$$H^{(g)}_{|I|+1}(x(v);z;I) := \delta_{g,0}\delta_{I,\emptyset} - \frac{1}{N}\sum_{l=1}^{d}\frac{r_l U^{(g)}_{|I|+1}(z,\varepsilon_l;I)}{x(v)-x(\varepsilon_l)} + \sum_{j=1}^{|I|}\frac{\partial}{\partial x(u_j)}\frac{U^{(g)}_{|I|}(z,u_j;I\backslash u_j)}{x(v)-x(u_j)} - \frac{V^{(g-1)}_{|I|+1}(z,z;I)}{x(v)-x(z)}\,,$$

$$M^{(g)}_{|I|+1}(x(v);z;I) := -\frac{1}{N}\sum_{l=1}^{d}\frac{r_l V^{(g)}_{|I|+1}(z,\varepsilon_l;I)}{x(v)-x(\varepsilon_l)} + \sum_{j=1}^{|I|}\frac{\partial}{\partial x(u_j)}\frac{V^{(g)}_{|I|}(z,u_j;I\backslash u_j)}{x(v)-x(u_j)} - \frac{U^{(g)}_{|I|+1}(z,z;I)}{x(v)-x(z)}\,,$$

then (sums below are $\sum' = \sum_{I_1\uplus I_2=I,\; g_1+g_2=g,\; (g_1,I_1)\neq(0,\emptyset)}$)

2.
$$H^{(g)}_{|I|+1}(x(v);z;I) = (x(z)+y(v))U^{(g)}_{|I|+1}(v,z;I) + \sum{}' W^{(g_1)}_{|I_1|+1}(v,I_1)U^{(g_2)}_{|I_2|+1}(v,z;I_2) + U^{(g-1)}_{|I|+2}(v,z;I\cup s)\Big|_{s=v} + \frac{V^{(g-1)}_{|I|+1}(v,z;I)}{x(z)-x(v)}\,,$$

3.
$$M^{(g)}_{|I|+1}(x(v);z;I) = (x(v)+y(v))V^{(g)}_{|I|+1}(v,z;I) + \sum{}' W^{(g_1)}_{|I_1|+1}(v,I_1)V^{(g_2)}_{|I_2|+1}(v,z;I_2) + V^{(g-1)}_{|I|+2}(v,z;I\cup s)\Big|_{s=v} + \frac{U^{(g)}_{|I|+1}(v,z;I)}{x(z)-x(v)}\,.$$

Auxiliary functions completely symmetric in preimages

Set

$$P^{(g)}_{|I|+1}(x(v),x(z);I) := \delta_{|I|,0}\delta_{g,0}x(z) - \frac{1}{N}\sum_{k=1}^{d}\frac{r_k H^{(g)}_{|I|+1}(x(v);\varepsilon_k;I)}{x(z)-x(\varepsilon_k)} + \sum_{j=1}^{|I|}\frac{\partial}{\partial x(u_j)}\frac{H^{(g)}_{|I|}(x(v);u_j;I\setminus u_j)}{x(z)-x(u_j)}$$
$$+\frac{\delta_{|I|,1}\delta_{g,0}}{x(v)-x(u_1)} + \delta_{|I|,0}\delta_{g,0}\Big(x(v) - \frac{1}{N}\sum_{k=1}^{d}\frac{r_k}{x(v)-x(\varepsilon_k)}\Big),$$

$$Q^{(g)}_{|I|+1}(x(v),x(z);I) := -\frac{1}{N}\sum_{k=1}^{d} r_k\frac{M^{(g)}_{|I|+1}(x(v);\varepsilon_k;I)}{x(z)-x(\varepsilon_k)} + \sum_{j=1}^{|I|}\frac{\partial}{\partial x(u_j)}\frac{M^{(g)}_{|I|+1}(x(v);u_j;I\backslash u_j)}{x(z)-x(u_j)},$$

then

$$P^{(g)}_{|I|+1}(x(v),x(z);I) = (x(v)+y(z))H^{(g)}_{|I|+1}(x(v);z;I) + {\sum}' W^{(g_1)}_{|I_1|+1}(z,I_1)H^{(g)}_{|I_2|+1}(x(v);z;I_2)$$
$$+ H^{(g-1)}_{|I|+2}(x(v);z;I\cup s)\Big|_{s=z} + \frac{M^{(g-1)}_{|I|+1}(x(v);z;I)}{x(v)-x(z)},$$

$$Q^{(g)}_{|I|+1}(x(v),x(z);I) = (x(z)+y(z))M^{(g)}_{|I|+1}(x(v);z;I) + {\sum}' W^{(g_1)}_{|I_1|+1}(z,I_1)M^{(g_2)}_{|I_2|+1}(x(v);z;I_2)$$
$$+ M^{(g-1)}_{|I|+2}(x(v);z;I\cup s)\Big|_{s=z} + \frac{H^{(g)}_{|I|+1}(x(v);z;I)}{x(v)-x(z)}.$$

Solution for $g = 0$

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- Loop insertion operators D_u and $D_{\{u_1,\dots,u_n\}} = D_{u_1}\cdots D_{u_n}$ as convenient abbreviation

$$D_u(x(z)) = 0, \quad D_u y(z) = W_2^{(0)}(z,u), \quad D_u W^{(g)}_{|I|+1}(z,I) = W^{(g)}_{|I|+2}(z,u,I),$$

and $D_u \hat{F}^{(g)}_{|I|+1}(v,z;I) = \hat{F}^{(g)}_{|I|+2}(v,z;I\cup u)$ for $F\in\{U,H,P,V,M,Q\}$.

- Equations for $g = 0$ turn into

$$D_I \log P_1^{(0)}(x(v),x(z)) = D_I \log(x(v)+y(z)) + D_I \log H_1^{(0)}(x(v);z)$$

Proposition [Hock, W 23]

$$D_I \log H_1^{(0)}(x(v);z) = \sum_{k=1}^{d} D_I \log(x(v)+y(\hat{z}^k)) + F^{(0)}_{|I|+1}(x(v);x(z);I)\ ,$$

$$D_I \log P_1^{(0)}(x(v),x(z)) = \sum_{k=0}^{d} D_I \log(x(v)+y(\hat{z}^k)) + F^{(0)}_{|I|+1}(x(v);x(z);I)$$

with $F^{(0)}_{|I|+1}(x(v);x(z);I) = \sum_{j=1}^{|I|} \frac{\partial}{\partial x(u_j)} D_{I\setminus u_j} \frac{1}{(x(v)-x(u_j))(x(z)+y(u_j))}$ (from eq. for $H^{(0)}$)

Linear and quadratic loop equations for $g = 0$

Comparing the obvious expansion of $P^{(0)}_{|I|+1}(x(v), x(z); I)$ at $x(v) = \infty$ with the one resulting from the defining equation and $H^{(g)}_{|I|+1}(x(v); z; I) = -\frac{1}{x(v)} W^{(g)}_{|I|+1}(z, I) + \mathcal{O}((x(v))^{-2})$ gives:

Proposition

The functions $W^{(0)}_{|I|+1}$ satisfy for $I \neq \emptyset$ the global linear loop equations

$$\sum_{k=0}^{d} W^{(0)}_{|I|+1}(\hat{z}^k, I) = \frac{\delta_{|I|,1}}{(x(z) - x(u_1))^2} - \sum_{j=1}^{|I|} \frac{\partial}{\partial x(u_j)} D_{I \setminus u_j}\Big(\frac{1}{x(z) + y(u_j)}\Big)$$

and the global quadratic loop equations

$$\frac{1}{2} \sum_{I_1 \uplus I_2 = I} \sum_{k=0}^{d} W^{(0)}_{|I_1|+1}(\hat{z}^k, I_1) W^{(0)}_{|I_2|+1}(\hat{z}^k, I_2)$$

$$= \sum_{j=1}^{|I|} \frac{\partial}{\partial x(u_j)} D_{I \setminus u_j}\Big(\frac{x(u_j)}{x(z)+y(u_j)}\Big) - \frac{1}{N} \sum_{k=1}^{d} \frac{r_k W^{(0)}_{|I|+1}(\varepsilon_k, I)}{x(z) - x(\varepsilon_k)} + \sum_{j=1}^{|I|} \frac{\partial}{\partial x(u_j)} \frac{W^{(0)}_{|I|}(I)}{x(z) - x(u_j)} .$$

- With Alex Hock and Maciej Dołęga we obtained these equations before in a combinatorial approach which is a version of the *x*-*y* symmetry in TR. Alex Hock generalised it later.
- The loop equations determine the $\omega_n^{(g)}$ uniquely if they are known not only near branch points of x (where they are holomorphic), but also near all of their poles.
- In applications, this extended knowledge of

$$\sum_{\hat{z}\in x^{-1}(\{x(z)\})} W^{(g)}_{|I|+1}(\hat{z},I) \quad \text{and} \quad \sum_{\hat{z}\in x^{-1}(\{x(z)\})} \Big(W^{(g-1)reg}_{|I_2|+1}(\hat{z},\hat{z},I) + \sum_{\substack{I_1\uplus I_1=I\\ g_1+g_2=g}} W^{(g_1)}_{|I_1|+1}(\hat{z},I_1) W^{(g_2)}_{|I_2|+1}(\hat{z},I_2)\Big)$$

should in principle be achievable whenever they are derived from some generating series. This is the case for the considered quartic model.

$$\hat{Q}^{(0)}_{|I|+1}(x(v),x(z);I) = Q^{(0)}_{|I|+1}(x(v),x(z);I) - \frac{P^{(0)}_{|I|+1}(x(v),x(z);I)}{(x(v)-x(z))^2}$$

GEOMETRY: DEFORMATIONS AND RIGIDITY CRC 1442 MM

By a similar method applied to

$$\hat{Q}^{(0)}_1(x(v),x(z)) = -\frac{(x(v)+x(z)-2x(0))}{2(x(v)-x(z))^2}\sqrt{\frac{P^{(0)}_1(x(v),x(v))P^{(0)}_1(x(z),x(z))}{(x(v)-x(0))(x(z)-x(0))}}$$

(found in [Schürmann, W 19], here $y(z)=-x(-z)$ is essential) one finds:

Proposition

$$\begin{aligned}
&D_I \log \hat{Q}^{(0)}_1(x(v),x(z)) \\
&= \frac{1}{2}D_I \log P^{(0)}_1(x(v),x(v)) + \frac{1}{2}D_I \log P^{(0)}_1(x(z),x(z)) \\
&- \frac{1}{2}\sum_{l=1}^{|I|}\frac{(-1)^{l-1}}{l}\Big(\frac{2^{l+1}}{(x(v)+x(z)-2x(0))^l} - \frac{1}{(x(v)-x(0))^l} - \frac{1}{(x(z)-x(0))^l}\Big)\sum_{\substack{I_1\uplus\ldots\uplus I_l=I\\ I_1,\ldots,I_l\neq\emptyset}}\prod_{i=1}^{l} D^0_{I_i}x(0)\,,
\end{aligned}$$

where $D^0_{I_i}x(0)$ is uniquely determined by holomorpicity of $D_I \log \hat{Q}^{(0)}_1(x(v),x(z))$ at $v=0$.

Theorem [Hock, W 23]: $P^{(1)}_{|I|+1}(x(v),x(z);I)$

$$D_I\Big(\frac{P_1^{(1)}(x(v),x(z))}{P_1^{(0)}(x(v),x(z))}+\frac{Q_1^{(0)}(x(v),x(z))}{(x(v)-x(z))^2P_1^{(0)}(x(v),x(z))}\Big)$$
$$=\sum_{k=0}^{d}D_I\frac{W_1^{(1)}(\hat z^k)}{x(v)+y(\hat z^k)}+\frac{1}{2}\sum_{j,k=0,\ j\neq k}^{d}D_I\frac{W_2^{(0)}(\hat z^j,\hat z^k)}{(x(v)+y(\hat z^j))(x(v)+y(\hat z^k))}$$
$$-\sum_{j=1}^{|I|}\frac{\partial}{\partial x(u_j)}\Big[\frac{1}{(x(v)-x(u_j))}\sum_{k=0}^{d}D_{I\setminus u_j}\frac{W_2^{(0)}(\hat z^k,u_j)}{(x(v)+y(\hat z^k))(x(z)+y(u_j))^2}\Big]$$
$$-\frac{1}{(x(v)-x(z))^3}\frac{\partial}{\partial x(w)}\Big(D_I\log P_1^{(0)}(x(v),x(w))-D_I\log P_1^{(0)}(x(z),x(w))\Big)\Big|_{w=z}$$
$$+\frac{1}{(x(v)-x(z))^2}\Big(\frac{1}{2}\frac{\partial^2(D_I\log P_1^{(0)}(x(w),x(z)))}{\partial x(w)\partial x(z)}\Big|_{w=z}-D_I^0\frac{1}{8(x(z)-x(0))^2}\Big)+\frac{1}{x(v)-x(z)}D_I^0\frac{1}{8(x(z)-x(0))^3}$$
$$+\sum_{j=1}^{|I|}\frac{\partial}{\partial x(u_j)}\Big[\frac{1}{(x(v)-x(u_j))}D_{I\setminus u_j}\Big\{\frac{W_2^{(0)reg}(u_j,u_j)}{(x(z)+y(u_j))^3}-\frac{W_1^{(1)}(u_j)}{(x(z)+y(u_j))^2}+\frac{1}{2(x(z)+y(u_j))^2}\frac{\partial^2}{\partial(x(u_j))^2}\frac{1}{(x(z)+y(u_j))}\Big\}\Big]$$
$$+\sum_{i,j=1,\ \ i<j}^{|I|}\frac{\partial^2}{\partial x(u_i)\partial x(u_j)}\Big[D_{I\setminus\{u_i,u_j\}}\frac{W_2^{(0)}(u_i,u_j)}{(x(v)-x(u_j))(x(v)-x(u_i))(x(z)+y(u_i))^2(x(z)+y(u_j))^2}\Big]$$

$$W_2^{(0)reg}(u,u)=\lim_{u'\to u}\Big(W_2^{(0)}(u,u')-\frac{1}{(x(u)-x(u'))^2}\Big)$$

Linear and quadratic loop equations for $g = 1$

Proposition [Hock, W 23]

The genus-1 meromorphic functions $W^{(1)}_{|I|+1}(z, I)$ satisfy the linear loop equation

$$\sum_{k=0}^{d} W^{(1)}_{|I|+1}(\hat{z}^k, I) = -D^0_I \frac{1}{8(x(z)-x(0))^3}$$
$$-\sum_{j=1}^{|I|} \frac{\partial}{\partial x(u_j)} D_{I\setminus u_j} \Big\{ \frac{W^{(0)reg}_2(u_j, u_j)}{(x(z)+y(u_j))^3} - \frac{W^{(1)}_1(u_j)}{(x(z)+y(u_j))^2}$$
$$- \frac{1}{2(x(z)+y(u_j))^2} \frac{\partial^2}{\partial(x(u_j))^2} \frac{1}{(x(z)+y(u_j))} \Big\}$$

and . . .

Linear and quadratic loop equations for $g = 1$

Proposition [Hock, W 23]

...the quadratic loop equation

$$\frac{1}{2}\sum_{\substack{g_1+g_2=1\\ I_1\uplus I_2=I}}\sum_{k=0}^{d} W^{(g_1)}_{|I|+1}(\hat{z}^k,I_1)W^{(g_2)}_{|I|+1}(\hat{z}^k,I_2)+\frac{1}{2}\sum_{k=0}^{d} W^{(0)reg}_{2}(\hat{z}^k,\hat{z}^k,I)$$

$$=\frac{1}{6}\sum_{j=1}^{|I|}\frac{\partial^2}{\partial(x(u_j))^2}\Big(D_{I\setminus u_j}\frac{1}{(x(z)+y(u_j))^3}\Big)-D^0_I\frac{1}{8(x(z)-x(0))^2}+x(z)D^0_I\frac{1}{8(x(z)-x(0))^3}$$

$$+\sum_{j=1}^{|I|}\frac{\partial}{\partial x(u_j)}\Big[x(u_j)D_{I\setminus u_j}\Big\{\frac{W^{(0)reg}_2(u_j,u_j)}{(x(z)+y(u_j))^3}-\frac{W^{(1)}_1(u_j)}{(x(z)+y(u_j))^2}-\frac{1}{2(x(z)+y(u_j))^2}\frac{\partial^2}{\partial(x(u_j))^2}\frac{1}{(x(z)+y(u_j))}\Big\}\Big]$$

$$-\frac{1}{N}\sum_{l=1}^{d}\frac{W^{(1)}_{|I|+1}(\varepsilon_l,I)}{x(z)-x(\varepsilon_l)}+\sum_{j=1}^{|I|}\frac{\partial}{\partial x(u_j)}\frac{W^{(1)}_{|I|}(I)}{x(z)-x(u_j)}\,.$$

The linear and quadratic loop equations are equivalent to

$$
\begin{aligned}
\omega^{(g)}_{|I|+1}(z,I) = &-\sum_{\beta_i} \operatorname*{Res}_{q\to\beta_i} \frac{\frac{1}{2}\big(\frac{dz}{z-q}-\frac{dz}{z-\sigma_i(q)}\big)}{(y(q)-y(\sigma_i(q)))dx(q)} \Big\{ {\sum}' \omega^{(g_1)}_{|I_1|+1}(q,I_1)\omega^{(g_2)}_{|I_2|+1}(q,I_2) + \omega^{(g-1)}_{|I|+2}(q,q,I)\Big\} \\
&- \sum_{j=1}^{|I|} d_{u_j}\Big[\operatorname*{Res}_{q\to -u_j} \frac{\frac{1}{2}\big(\frac{dz}{z-q}-\frac{dz}{z+u_j}\big)}{(y(q)-y(-u_j))dx(q)} \Big\{ {\sum}' d^{-1}_{u_j}(\omega^{(g_1)}_{|I_1|+1}(q,I_1)\omega^{(g_2)}_{|I_2|+1}(q,I_2)) \\
&\qquad + d^{-1}_{u_j}\omega^{(g-1)}_{|I|+2}(q,q,I) + \frac{(dx(q))^2}{6}\frac{\partial^2}{\partial (x(q))^2}\Big(\frac{\omega^{(g-1)}_{|I|+1}(q,I))}{dx(q)dx(u_j)}\Big)\Big\}\Big] \\
&- \operatorname*{Res}_{q\to 0} \frac{\frac{1}{2}\big(\frac{dz}{z-q}-\frac{dz}{z}\big)}{(y(q)-y(-q))dx(q)} \Big\{ {\sum}' \omega^{(g_1)}_{|I_1|+1}(q,I_1)\omega^{(g_2)}_{|I_2|+1}(q,I_2) + \omega^{(g-1)}_{|I|+2}(q,q,I) \\
&\qquad + \frac{(dx(q))^2}{2}\frac{\partial}{\partial x(q)}\Big(\frac{d^{-1}_{q'}\omega^{(g-1)}_{|I|+2}(q,q',I)}{dx(q)}\Big|_{q'=q}\Big)\Big\}\,,
\end{aligned}
$$

where β_i we denote the ramification points of x, σ_i is the Galois involution near β_i and $\omega_2^{(0)}(q,q) \mapsto \lim_{q'\to q}(\omega_2^{(0)}(q,q') - \frac{dx(q)\,dx(q')}{(x(q)-x(q'))^2})$ is understood. $\sum'$ means $(g_i, I_i) \neq (0,\emptyset)$.

Outlook

There is no principal problem which prevents pushing these structures to any genus g.

- The terms which after expansion at $x(v)=\infty$ give $\sum_{k=0}^{d} W^{(g)}_{|I|+1}(\hat{z}^k;I)$ and $\sum_{k=0}^{d}\big(W^{(g-1)reg}_{|I_2|+1}(\hat{z}^k,\hat{z}^k;I)+\sum W^{(g_1)}_{|I_1|+1}(\hat{z}^k;I_1)W^{(g_2)}_{|I_2|+1}(\hat{z}^k;I_2)\big)$ are always there. There cannot be any contribution with poles at β_i.
- Hence, the linear and quadratic loop equation terms are locally, in neighbourhood of ramification points, always holomorphic.
- Thus, the quartic matrix model satisfies blobbed topological recursion in its original sense.
- The stronger question, to establish the linear and quadratic loop equation for all g globally, is a difficult combinatorial problem. The number of terms increases quickly with g, we need Taylor expansions up to order $4g-3$.
- Maybe it is realistic to reach $g=2$ to check whether the recursion kernel representation remains stable or needs additional $g-2$ contributions.