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Matrix models and topological recursion

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CRC 1442 GEOMETRY: DEFORMATIONS AND RIGIDITY

MM Mathematics Münster Cluster of Excellence

Overview

Matrix models are a common topic e.g. in

- enumerative geometry and combinatorics,
- quantum gravity in two dimensions,
- complex algebraic geometry,
- quantum fields on noncommutative geometry.

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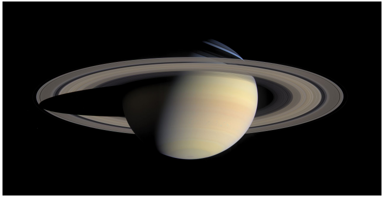
In form of random matrix theory, they are important in

- stochastics,
- free probability.

They are examples for a universal structure called topological recursion.

Enumerative geometry

Consider a planet of genus g on which all countries are (possibly degenerate) polygons neighbouring each other.



We are interested in world maps of

- n_3 triangle countries, n_4 quadrangle countries, etc.

We admit a fixed number of oceans:

- l_3 of them triangles, l_4 of them quadrangles, etc.

How many different world maps are there?

William Tutte (1963) counted these numbers in the case of a spherical planet (genus 0) with one ocean (rooted planar maps).

- For special case that all countries and the ocean form $n = n_4 + 1$ quadrangles, there are $\dfrac{2 \cdot 3^n}{(n+2)} C_n$ different world maps, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

Hermitian 1-matrix model

(Formal) matrix integral [Brézin, Itzykson, Parisi, Zuber 78] as generating function of maps:

$$\mathcal{Z}(t_3,\dots,t_d;t) := \int_{H_N} dM \exp\left(-N\,\mathrm{Tr}\left(\frac{M^2}{2t}\right)\right)\exp\left(\frac{N}{t}\,\mathrm{Tr}\left(\frac{t_3}{3}M^3+\dots+\frac{t_d}{d}M^d\right)\right)$$

$$= \sum_{\Sigma\in\substack{\text{disconn. maps}\\ \text{no ocean}}} \left(\frac{N}{t}\right)^{\chi(\Sigma)} t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \cdots t_d^{n_d(\Sigma)} \cdot \frac{t^{v(\Sigma)}}{\#\mathrm{Aut}(\Sigma)}$$

where

- integral is over self-adjoint $N\times N$-matrices M, with dM normalised Lebesgue measure,
- $\chi(\Sigma)$ is the Euler characteristic of Σ,
- Σ has $n_3(\Sigma)$ triangles, ..., $n_d(\Sigma)$ d-gons and in total $v(\Sigma)$ vertices.

Quantum gravity

- Challenge: make sense of $\sum_{\text{topologies}} \int_{\text{metrics}} dg\ e^{-\int_{M_g} \frac{\kappa}{2}(\mathrm{scal}(g)-2\Lambda)d\mathrm{vol}(g)}$
- In $D=2$ dimensions, Gauß-Bonnet reduces this to Euler characteristic and volume.
- Substitute for g-integral is sum over world maps where each country has unit weight.

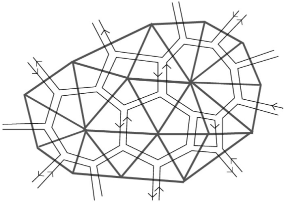
Remarks

$$\exp\left(\frac{N}{t}\mathrm{Tr}\Big(\sum_{i=3}^{d}\frac{t_i}{i}M^i\Big)\right) = \sum_{n_3,\dots,n_d=0}^{\infty}\frac{1}{n_3!\cdots n_d!}\left(\frac{N}{t}\right)^{n_3+\dots+n_d}\left(\frac{t_3}{3}\mathrm{Tr}(M^3)\right)^{n_3}\cdots\left(\frac{t_d}{d}\mathrm{Tr}(M^d)\right)^{n_d}$$

- Gaußian integral

$$\int_{H_N} dM\ \exp\left(-N\,\mathrm{Tr}\Big(\frac{M^2}{2t}\Big)\right)\prod_{i=1}^{v}M_{k_il_i} = \begin{cases}\displaystyle\sum_{\text{pairings}}\prod_{\text{pairs (i,j)}}\frac{t}{N}\delta_{k_il_j}\delta_{l_ik_j}\\ 0 \text{ if } v \text{ is odd}\end{cases}$$

  gives sum over closed ribbon graphs with n_3 trivalent vertices, ..., n_d d-valent vertices.
- Factorials and $(\frac{1}{k})^{n_k}$ combine to $\frac{1}{\#\mathrm{Aut}(\Sigma)}$
- Each such ribbon graph Γ comes with prefactor $(\frac{N}{t})^{\chi(\Gamma)}$, where $\chi(\Gamma) = v(\Gamma) - e(\Gamma) + f(\Gamma)$ is Euler characteristic.
- Ribbon graphs Γ and maps Σ are dual to each other:



Including oceans

Oceans which are $(l_1, \dots, l_s)$-gons are generated by

$$\frac{\displaystyle\int_{H_N} dM\, \mathrm{Tr}(M^{l_1}) \cdots \mathrm{Tr}(M^{l_s}) e^{-N\,\mathrm{Tr}(\frac{M^2}{2t})} e^{\frac{N}{t}\mathrm{Tr}(\frac{t_3}{3}M^3 + \frac{t_4}{4}M^4 + \dots + \frac{t_d}{d}M^d)}}{\displaystyle\int_{H_N} dM\; e^{-N\,\mathrm{Tr}(\frac{M^2}{2t})} e^{\frac{N}{t}\mathrm{Tr}(\frac{t_3}{3}M^3 + \frac{t_4}{4}M^4 + \dots + \frac{t_d}{d}M^d)}}$$

They can formally be collected into resolvents $W(x) = \mathrm{Tr}((x-M)^{-1})$, for $x \notin \mathbb{R}$.

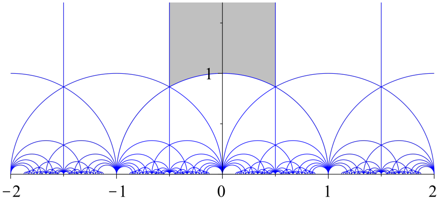
- Integration by parts gives identities between derivatives of $\mathcal{Z}(t_3, \dots, t_d)$:

$$0 = \Big(\sum_{j=1}^{\infty}(k+j)t_j \frac{\partial}{\partial t_{k+j}} + \frac{t^2}{N^2}\sum_{l=1}^{k-1} l(k-l)\frac{\partial^2}{\partial t_l \partial t_{k-l}} + 2t_k \frac{\partial}{\partial t_k}\Big)\mathcal{Z}$$

- Up to conjugation, these differential operators become generators L_k of the Witt/Virasoro algebra, $[L_k, L_l] = (k-l)L_{k+l}$.
- Identifies KdV integrable hierarchy in Hermitian 1-matrix model.

The moduli space of complex curves

- Any two tori (of genus 1) are homotopic, but not necessarily (complex-) diffeomorphic.
- The equivalence classes of tori with marked point 0 are parametrised by points in the fundamental domain



- Compactified by adding the unique sphere with three marked points, two of them glued to a pinched torus.

In general, the moduli space $\mathcal{M}_{g,n}$ of genus-g curves with n marked points is a space of complex dimension $3g+n-3$. It is an orbifold, similar to a manifold, but with corners.

Deligne-Mumford compactification to moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves.

Intersection numbers

Consider on $\overline{\mathcal{M}}_{g,n}$ a family $\{\mathcal{L}_1, \dots, \mathcal{L}_n\}$ of line bundles:

- Fibre of $\mathcal{L}_i$ over $x \in \overline{\mathcal{M}}_{g,n}$, which is a (nodal) curve $x = \mathcal{C}$, is the cotangent space of $\mathcal{C}$ at the i-th marked point.
- These bundles are classified by their first Chern class $\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.
- Intersection numbers $\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \in \mathbb{Q}$, non-zero iff $d_1 + \ldots + d_n = 3g - 3 + n$

Collect them to generating function $\mathcal{F}_g(t_0, t_1, \dots) = \sum_{\substack{n=1 \\ 2g+n\geq 3}}^{\infty} \frac{1}{n!} \sum_{d_1,\dots,d_n=0}^{\infty} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \prod_{i=1}^{n} t_{d_i}$

Conjecture [Witten 91]

$\tau(t_0, t_1, ..) := \exp(\sum_{g=0}^{\infty} N^{2-2g} \mathcal{F}_g(t_0, t_1, \dots))$ is a KdV τ-function, thus equivalent to partition function of the 1-matrix model

The Kontsevich matrix model

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Theorem [Kontsevich 92]

The generating function of intersection numbers is the $1/N$-expansion of a matrix integral

$$\sum_{g=0}^{\infty} N^{2-2g}\mathcal{F}_g(t_0,t_1,\dots) = \log\left(\frac{\displaystyle\int_{H_N} dM\, e^{-\frac{N}{2}\mathrm{Tr}(\Lambda M^2)+\frac{\mathrm{i}N}{6}\mathrm{Tr}(M^3)}}{\displaystyle\int_{H_N} dM\, e^{-\frac{N}{2}\mathrm{Tr}(\Lambda M^2)}}\right)$$

where $t_i := -(2i-1)!!\mathrm{Tr}(\Lambda^{-2i-1})$. In particular, $\exp(\sum_{g=0}^{\infty} N^{2-2g}\mathcal{F}_g)$ is a KdV τ-function.

- The Kontsevich matrix model was understood later [Eynard, Orantin 07] as the simplest example of topological recursion (TR).
- Historically, TR was discovered in approach [Chekhov, Eynard, Orantin 06] to the 2-matrix model. Shortly later it was also found in the 1-matrix model.

We sketch the main ideas of TR for the Kontsevich model.

Cumulants of the Kontsevich matrix model

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Similar to the generating functions including oceans, consider

$$\langle M_{a_1a_1}\cdots M_{a_na_n}\rangle_c := \log\left(\frac{\mathrm{i}^n\int_{H_N} dM\, M_{a_1a_1}\cdots M_{a_na_n}\, e^{-\frac{N}{2}\mathrm{Tr}(\Lambda M^2)+\frac{\mathrm{i}N}{6}\mathrm{Tr}(M^3)}}{\int_{H_N} dM\, e^{-\frac{N}{2}\mathrm{Tr}(\Lambda M^2)+\frac{\mathrm{i}N}{6}\mathrm{Tr}(M^3)}}\right)$$

- Expanding $\exp(\frac{\mathrm{i}N}{6}\mathrm{Tr}(M^3))$ produces ribbon graphs with n marked faces each containing one 1-valent vertex and any number of unmarked faces. All other vertices are 3-valent.
- Forgetting the marking gives in algebraic geometry rise to κ-classes:

$$[N^{2-2g-2n}]\langle M_{a_1a_1}\cdots M_{a_na_n}\rangle_c$$
$$=\sum_{k=0}^{\infty}\frac{1}{(1-t_0)^{2g+n-2}k!}\sum_{\substack{d_1+\dots+d_n+\\+l_1+\ldots+l_k=3g-3+n}}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_{l_1}\cdots\kappa_{l_k}\prod_{i=1}^{n}\frac{(2d_i+1)!!}{\lambda_{a_i}^{2d_i+3}}\prod_{j=1}^{k}s_{l_j}$$

where $2g+n\geq 3$, $d_j\geq 0$, $l_j\geq 1$ and for $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)$, renormalised to $\sum_{k=1}^{N}\frac{1}{\lambda_k}=0$, $s_l=-[x^l]\log\left(1-\sum_{m=0}^{\infty}t_mx^m\right)$ if $t_m=-(2m-1)!!\frac{1}{N}\sum_{k=1}^{N}\frac{1}{\lambda_k^{2m+3}}$

Integration by parts

...gives relations between cumulants: loop equations, Dyson-Schwinger equations (DSE)

Expanding $\langle M_{a_1a_1}\cdots M_{a_na_n}\rangle_c - \lambda_{a_1}\delta_{n,1} =: \sum_{g=0}^\infty (N/2)^{2-2g-2n} W^{(g)}_{a_1,\dots,a_n}$, these equations read

Dyson-Schwinger equations of Kontsevich model

$$\sum_{\substack{I_1\uplus I_2=\{1,\dots,n\}\\ g_1+g_2=g}} W^{(g_1)}_{a,I_1} W^{(g_2)}_{a,I_2} = \lambda_a^2\delta_{n,0}\delta_{g,0} - W^{(g-1)}_{a,a,a_1,\dots,a_n} - \frac{2}{N}\sum_{k=1}^N \frac{W^{(g)}_{k,a_1,\dots,a_n} - W^{(g)}_{a,a_1,\dots,a_n}}{\lambda_k^2-\lambda_a^2}$$

$$-\sum_{j=1}^n \frac{\partial}{\partial\lambda_{a_j}^2} \frac{W^{(g)}_{a_1,\dots,a_n} - W^{(g)}_{a_1,\dots,a_{j-1},a,a_{j+1},\dots,a_n}}{\lambda_{a_j}^2-\lambda_a^2}$$

- Non-linear equation for $W_a^{(0)}$ if $g=n=0$; solved by [Makeenko, Semenoff 91] $W_a^{(0)} = -\sqrt{\lambda_a^2+c} + \frac{1}{N}\sum_{l=1}^N \frac{1}{\sqrt{\lambda_l^2+c}\left(\sqrt{\lambda_a^2+c}+\sqrt{\lambda_l^2+c}\right)}$ where $c = \frac{2}{N}\sum_{k=1}^N \frac{1}{\sqrt{\lambda_k^2+c}}$.
- otherwise affine with known inhomogeneity

Complexify DSE to system of equations

$$\sum_{\substack{I_1 \uplus I_2 = \{z_1,\dots,z_n\} \\ g_1+g_2=g}} \hat{W}^{(g_1)}_{|I_1|+1}(z, I_1)\hat{W}^{(g_2)}_{|I_2|+1}(z, I_2) + \hat{W}^{(g-1)}_{n+2}(z, z, z_1, \dots, z_n)$$

$$= (z^2 - c)\delta_{n,0}\delta_{g,0} - \frac{2}{N}\sum_{k=1}^{N} \frac{\hat{W}^{(g)}_{n+1}(\hat{\lambda}_k, z_1, \dots, z_n) - \hat{W}^{(g)}_{n+1}(z, z_1, \dots, z_n)}{\hat{\lambda}_k^2 - z^2}$$

$$- \sum_{j=1}^{n} \frac{\partial}{\partial z_j^2} \frac{\hat{W}^{(g)}_{n}(z_1, \dots, z_n) - \hat{W}^{(g)}_{n}(z_1, \dots, z_{j-1}, z, z_{j+1}, \dots, z_n)}{z_j^2 - z^2}$$

for complex functions $\hat{W}^{(g)}_n$ satisfying $W^{(g)}_{a_1,\dots,a_n} \equiv \hat{W}^{(g)}_n(\hat{\lambda}_{a_1}, \dots, \hat{\lambda}_{a_n})$, where $\hat{\lambda}_k := \sqrt{\lambda_k^2 + c}$

- $\hat{W}^{(0)}_2(z, z_1) = \dfrac{1}{4zz_1(z + z_1)^2}$
- $\hat{W}^{(0)}_3(z_1, z_2, z_3) = \dfrac{1}{16(1 - \hat{t}_3)z_1^3 z_2^3 z_3^3}$ where $\hat{t}_3 = -\frac{1}{N}\sum_{k=1}^{N} \frac{1}{\hat{\lambda}_k^3}$

Linear and quadratic loop equations

The complexified DSE imply inductively for $2g+n\geq 3$:

- $W_n^{(g)}(z_1,\dots,z_n)$ has poles only at $z_i=0$
- **Linear loop equation $(2g+n\geq 3)$**

  $$W_n^{(g)}(z,z_2,\dots,z_n)+W_n^{(g)}(-z,z_2,\dots,z_n)=0$$

Use this and splitting of $\hat{W}_1^{(0)}$ and $\hat{W}_2^{(0)}$ into parts with $\pm z$ to rearrange DSE into

Quadratic loop equation $(2g+n\geq 3)$

$$\sum_{\substack{I_1\uplus I_2=\{z_2,\dots,z_n\}\\ g_1+g_2=g}} W^{(g_1)}_{|I_1|+1}(z,I_1)W^{(g_2)}_{|I_2|+1}(-z,I_2)+W^{(g-1)}_{n+1}(z,-z,z_2,\dots,z_n)$$
$$=\frac{1}{N}\sum_{k=1}^{N}\frac{W_n^{(g)}(\hat{\lambda}_k,z_2,\dots,z_n)}{\hat{\lambda}_k^2-z^2}+\sum_{j=2}^{n}\frac{\partial}{\partial z_j^2}\Big(\frac{W_{n-1}^{(g)}(z_2,..,z_n)}{z_j^2-z^2}\Big)$$

where $W_1^{(0)}\equiv y(z):=-z+\frac{1}{N}\sum_{k=1}^{N}\frac{1}{\hat{\lambda}_k(\hat{\lambda}_k-z)}$, $\quad W_2^{(0)}(z_1,z_2)=\frac{1}{4z_1z_2(z_1-z_2)^2}$

and $W_n^{(g)}=\hat{W}_n^{(g)}$ for $2g+n\geq 3$

Topological recursion

- [Eynard, Orantin 07] noticed that many matrix models admit such recursive structures for meromorphic functions $W_n^{(g)}$.
- Only the choice of initial data, called the spectral curve, was specific to the model, the recursion itself was universal.

Spectral curve

- Complex curve/Riemann surface Σ and two ramified coverings $x, y : \Sigma \to \mathbb{P}^1$
- Bergman kernel B: symmetric bidifferential on $\Sigma \times \Sigma$, with double pole on diagonal, no other pole, normalised

Soon later many important examples other than matrix models were identified:

- Weil-Peterssen volumes of moduli spaces of bordered hyperbolic surfaces [Mirzakhani 07]
- ELSV formula, expresses simple Hurwitz numbers as integral of ψ- and λ-classes over $\overline{\mathcal{M}}_{g,n}$ [Bouchard, Mariño 07; Eynard, Mulase, Safnuk 09]
- semisimple cohomological field theories [Dunin-Barkowski, Orantin, Shadrin, Spitz 14]

Recursion formula

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The setting

Given a spectral curve $(x, y : \Sigma \to \mathbb{P}^1, B)$ where

- x has simple ramification points $\beta_1, \ldots, \beta_r$ (which solve $dx(\beta_i) = 0$),
- y is holomorphic and non-zero at β_i,
- there is a family $W_n^{(g)}$ of meromorphic functions, extending $W_1^{(0)}(z) = y(z)$ and $B(z, w) =: W_2^{(0)}(z, w)dx(z)dx(w)$.

These data specify:

- the local Galois involution σ_i near β_i with fixed point β_i and $x(\sigma_i(z)) = x(z)$,
- the list $\{\hat{z}^0 = z, \hat{z}^1, \ldots, \hat{z}^d\} = x^{-1}(x(z))$ of preimages of $x(z)$ under x,
- meromorphic differentials $\omega_n^{(g)}(z_1, \ldots, z_n) = W_n^{(g)}(z_1, \ldots, z_n) \prod_{i=1}^n dx(z_i)$

Kontsevich: $\Sigma = \mathbb{C}$, $x(z) = z^2$, $\beta_1 = 0$, $\sigma_1(z) = -z$, $\{\hat{z}^0 = z, \hat{z}^1 = -z\}$, $B(z, w) = \frac{dzdw}{(z-w)^2}$

Recursion formula

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Theorem [Borot-Eynard-Orantin 15]

The previous setting satisfies topological recursion if for $2g+n\geq 3$ the functions

$$L(x(z);z_2,\dots,z_n):=\sum_{j=0}^{d} W_n^{(g)}(\hat{z}^j,z_2,\dots,z_n)$$

$$Q(x(z);z_2,\dots,z_n)=\sum_{j=0}^{d}\Bigg(\sum_{\substack{I_1\uplus I_2=\{z_2,\dots,z_n\}\\ g_1+g_2=g}} W_{|I_1|+1}^{(g_1)}(\hat{z}^j,I_1)W_{|I_2|+1}^{(g_2)}(\hat{z}^j,I_2)+W_{n+1,reg}^{(g-1)}(\hat{z}^j,\hat{z}^j,z_2,\dots,z_n)\Bigg)$$

are holomorphic at any branch point $x(z)=x(\beta_i)$. If in addition a projection property

$$\omega_{g,n}(z,z_2,\dots,z_n)=\sum_{i=1}^{r}\operatorname{Res}_{q=\beta_i}\int_{\beta_i}^{q}\omega_{0,2}(z,.)\omega_{g,n}(q,z_2,\dots,z_n)$$

holds, then the $\omega_n^{(g)}$ are recursively evaluated by

$$\omega_n^{(g)}(z,z_2,\dots,z_n)$$
$$=\sum_{i=1}^{r}\operatorname{Res}_{q=\beta_i}\frac{\frac{1}{2}\int_{\sigma_i(q)}^{q}B(z,.)}{\omega_1^{(0)}(q)-\omega_1^{(0)}(\sigma_i(q))}\Bigg(\sum_{\substack{I_1\uplus I_2=\{z_2,\dots,z_n\}\\ g_1+g_2=g,(g_i,I_i)\neq(0,\emptyset)}}\omega_{|I_1|+1}^{(g_1)}(q,I_1)\omega_{|I_2|+1}^{(g_2)}(\sigma_i(q),I_2)+\omega_{n+1}^{(g-1)}(q,\sigma_i(q),z_2,\dots,z_n)\Bigg)$$

Further remarks

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- Laurent expansion of $\omega_n^{(g)}(z_1, \dots, z_n)$ near an n-tupel of ramification points can be expressed in terms of intersection numbers of ψ- and κ-classes on $\overline{\mathcal{M}}_{g,n}$ [Eynard 11] (generalised by [Dunin-Barkowski, Orantin, Shadrin, Spitz 14] to CohFT).
- Absence of projection property gives blobbed topological recursion [Borot, Shadrin 15].
- Deformations of spectral curve express formal Baker-Akhiezer kernel in terms of $\omega_n^{(g)}$. Gives rise to formal KP τ-function [Eynard, Orantin 07; Borot, Eynard 12].
- Symplectic invariance of $dy \wedge dx$: previously open x-y swap understood in [Hock 22; Bychkov, Dunin-Barkowski, Kazarian, Shadrin 22].
- Application to higher-order free cumulants in free probability [Borot, Charbonnier, Garcia-Failde, Leid, Shadrin 21].

QFT on noncommutative geometries

- No interacting QFT-model in 4 dimensions is in sight. 4D models are either too difficult (Yang-Mills, millenium prize problem), or trivial (ϕ^4_4 [Aizenman, Duminil-Copin 19]).
- Quantum field theories on noncomutative geometries provide a new class of 4D QFT-models to try. They violate symmetry axioms, but renormalisation is very similar.
- The simplest one is the Φ^4-model on noncommutative Moyal space, which is Fréchet-isomorphic to infinite matrices with rapidly decaying entries.
- In Euclidean approach, have (formal) measure
$$d\mu_\lambda(\Phi) \text{“:=”} \frac{1}{\mathcal{Z}} d\mu_0(\Phi)\ \exp\Big(-\frac{g}{4}\mathcal{N}\,\mathrm{Tr}(\Phi^4)\Big)\,, \qquad \mathcal{N} := \Big(\frac{\theta}{4}\Big)^{D/2}\,.$$
- $d\mu_0$ is Gaußian, defined by covariance. Simplest choice is
$$\langle \Phi_{kl}\Phi_{mn}\rangle = \int d\mu_0(\Phi)\ \Phi_{kl}\Phi_{mn} = \frac{\delta_{kn}\delta_{lm}}{\mathcal{N}(\lambda_k+\lambda_l)}\,,$$
where $\lambda_k > 0$ are the eigenvalues of a Laplacian in D dimensions.

Meromorphic functions for quartic Kontsevich model

Consider the partition function $\mathcal{Z}_{\Lambda,4} := \int_{H_N} dM\, e^{-N\,\mathrm{Tr}(\Lambda M^2 + \frac{1}{4}M^4)}$, where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$. Define $\langle \mathcal{O}(M) \rangle_{\Lambda,4} = \frac{1}{\mathcal{Z}_{\Lambda,4}} \int_{H_N} dM\, \mathcal{O}(M) e^{-N\,\mathrm{Tr}(\Lambda M^2 + \frac{1}{4}M^4)}$.

Main definition [Branahl, Hock W 20]

$W^{(g)}_{a_1,\dots,a_n} := [N^{2-2g-n}] \frac{(-1)^n \partial^n \log \mathcal{Z}_{\Lambda,4}}{\partial\lambda_{a_1} \cdots \partial\lambda_{a_n}} + \frac{\delta_{g,0}\delta_{n,2}}{(\lambda_{a_1} - \lambda_{a_2})^2} + \delta_{g,0}\delta_{n,1} f(\lambda_{a_1})$ for $a_1, \dots, a_n$ pairwise different

- Procedure consists in deriving equations for the $W^{(g)}_{a_1,\dots,a_n}$ which should extend to complexified equations for $\tilde{W}^{(g)}_n(\xi_1, \dots, \xi_n)$ with $\tilde{W}^{(g)}_n(\lambda_{a_1}, \dots, \lambda_{a_n}) = W^{(g)}_{a_1,\dots,a_n}$.
- Need auxiliary functions $\frac{(-1)^n \partial^n \langle M_{kl} M_{lk} \rangle_{\Lambda,4}}{\partial\lambda_{a_1} \cdots \partial\lambda_{a_n}}$ and $\frac{(-1)^n \partial^n \langle M_{kk} M_{ll} \rangle_{\Lambda,4}}{\partial\lambda_{a_1} \cdots \partial\lambda_{a_n}}$ also to complexify.
- Non-linear equation for $G^{(0)}_{kl} := [N^{-1}] \langle M_{kl} M_{lk} \rangle_{\Lambda,4} \mapsto G^{(0)}(\xi_1, \xi_2)$ can be solved and provides $\xi \equiv x(z)$ and $y(z)$ for TR.

The planar 2-point function

$(\lambda_1, \ldots, \lambda_d)$ – pairwise different eigenvalues with multiplicities $(r_1, \ldots, r_d)$.

Theorem [Grosse, W 09]

$$\Big(\zeta + \eta + \frac{1}{N}\sum_{k=1}^{d} r_d G^{(0)}(\zeta, \lambda_k)\Big) G^{(0)}(\zeta, \eta) = 1 + \frac{1}{N}\sum_{k=1}^{d} r_d \frac{G^{(0)}(\lambda_k, \eta) - G^{(0)}(\zeta, \eta)}{\lambda_k - \zeta}$$

Theorem [Schürmann, W 19]

A solution can be implicitly found in the form $G^{(0)}(x(z), x(w)) =: \mathcal{G}^{(0)}(z, w)$ with $x(z) = z - \frac{1}{N}\sum_{k=1}^{N} \frac{\varrho_k}{\varepsilon_k + z}$ and $x(\varepsilon_k) = \lambda_k$ and $x'(\varepsilon_k)\varrho_k = r_k$:

$$\mathcal{G}^{(0)}(z, w) = \frac{P_1^{(0)}(x(z), x(w))}{(x(z) + y(w))(x(w) + y(z))}$$ where $\boxed{y(z) = -x(-z)}$ and

$$P_1^{(0)}(x(z), x(w)) = \frac{\prod_{u \in x^{-1}(\{x(w)\})}(x(z) + y(u))}{\prod_{k=1}^{d}(x(z) - x(\varepsilon_k))} \equiv P_1^{(0)}(x(w), x(z))$$

Linear and quadratic loop equations for $g = 0$

Extract from DSE (which relate $W_n^{(g)}$ to auxiliary functions) the lin./quad. loop equations:

Proposition [Hock, W 21; Hock, W 23]

The functions $W^{(0)}_{|I|+1}$ satisfy for $\emptyset \neq I = \{u_1, \dots, u_n\}$ the global linear loop equations

$$\sum_{k=0}^{d} W^{(0)}_{|I|+1}(\hat{z}^k, I) = \frac{\delta_{|I|,1}}{(x(z)-x(u_1))^2} - \sum_{j=1}^{|I|} \frac{\partial}{\partial x(u_j)} D_{I\setminus u_j}\Big(\frac{1}{x(z)+y(u_j)}\Big)$$

and the global quadratic loop equations

$$\frac{1}{2} \sum_{I_1 \uplus I_2 = I} \sum_{k=0}^{d} W^{(0)}_{|I_1|+1}(\hat{z}^k, I_1) W^{(0)}_{|I_2|+1}(\hat{z}^k, I_2)$$

$$= \sum_{j=1}^{|I|} \frac{\partial}{\partial x(u_j)} D_{I\setminus u_j}\Big(\frac{x(u_j)}{x(z)+y(u_j)}\Big) - \frac{1}{N} \sum_{k=1}^{d} \frac{r_k W^{(0)}_{|I|+1}(\varepsilon_k, I)}{x(z)-x(\varepsilon_k)} + \sum_{j=1}^{|I|} \frac{\partial}{\partial x(u_j)} \frac{W^{(0)}_{|I|}(I)}{x(z)-x(u_j)} ,$$

where $D_{\{u_1,\dots,u_n\}} = \prod_{j=1}^{n} D_{u_j}$ for derivations $D_u W_m^{(g)}(z_1, \dots, z_m) = W_m^{(g)}(z_1, \dots, z_m, u)$, $D_u x(z) = 0$

Projection property does not hold: blobbed topological recursion

Linear and quadratic loop equations for $g = 1$

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Proposition [Hock, W 23]

The genus-1 meromorphic functions $W^{(1)}_{|I|+1}(z, I)$ satisfy the linear loop equation

$$\sum_{k=0}^{d} W^{(1)}_{|I|+1}(\hat{z}^k, I) = -D^0_I \frac{1}{8(x(z)-x(0))^3} - \sum_{j=1}^{|I|} \frac{\partial}{\partial x(u_j)} D_{I\setminus u_j}\Big\{ \frac{W_2^{(0)reg}(u_j, u_j)}{(x(z)+y(u_j))^3} - \frac{W_1^{(1)}(u_j)}{(x(z)+y(u_j))^2} - \frac{1}{2(x(z)+y(u_j))^2} \frac{\partial^2}{\partial(x(u_j))^2} \frac{1}{(x(z)+y(u_j))} \Big\}$$

and . . .

Linear and quadratic loop equations for $g = 1$

Proposition [Hock, W 23]

…the quadratic loop equation

$$\frac{1}{2}\sum_{\substack{g_1+g_2=1\\ I_1\uplus I_2=I}}\sum_{k=0}^{d} W^{(g_1)}_{|I|+1}(\hat{z}^k, I_1)W^{(g_2)}_{|I|+1}(\hat{z}^k, I_2) + \frac{1}{2}\sum_{k=0}^{d} W_2^{(0)reg}(\hat{z}^k,\hat{z}^k, I)$$

$$= \frac{1}{6}\sum_{j=1}^{|I|}\frac{\partial^2}{\partial(x(u_j))^2}\Big(D_{I\setminus u_j}\frac{1}{(x(z)+y(u_j))^3}\Big) - D_I^0\frac{1}{8(x(z)-x(0))^2} + x(z)D_I^0\frac{1}{8(x(z)-x(0))^3}$$

$$+\sum_{j=1}^{|I|}\frac{\partial}{\partial x(u_j)}\Big[x(u_j)D_{I\setminus u_j}\Big\{\frac{W_2^{(0)reg}(u_j,u_j)}{(x(z)+y(u_j))^3} - \frac{W_1^{(1)}(u_j)}{(x(z)+y(u_j))^2} - \frac{1}{2(x(z)+y(u_j))^2}\frac{\partial^2}{\partial(x(u_j))^2}\frac{1}{(x(z)+y(u_j))}\Big\}\Big]$$

$$-\frac{1}{N}\sum_{l=1}^{d}\frac{W^{(1)}_{|I|+1}(\varepsilon_l, I)}{x(z)-x(\varepsilon_l)} + \sum_{j=1}^{|I|}\frac{\partial}{\partial x(u_j)}\frac{W^{(1)}_{|I|}(I)}{x(z)-x(u_j)}\,.$$

Final remarks

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- The global linear and quadratic loop equations give explicit recursion formulae for $\omega_n^{(g)}$ (so far for $g \leq 1$).
- Original blobbed TR [Borot, Shadrin 15] defined for local curves; this leaves large freedom (called 'blobs') in $\omega_n^{(g)}$. Validity of local loop equations is clear.
- $\mathcal{Z}(\mathbf{t}) = \int_{H_N} d\mu_\Lambda(M)\ \exp\big(\mathrm{Tr}\big(-\frac{1}{4}M^4 + \sum_{k=0}^\infty t_{2k+1}M^{2k+1}\big)\big)$ is a BKP τ-function (in fact for any potential; with $d\mu_\Lambda(M) = \frac{1}{Z}\exp\big(-\frac{1}{2}\mathrm{Tr}(\Lambda M^2)\big)dM$) [Borot, W 23]
- $\lambda\Phi^4$ on 4D noncommutative Moyal space leads to

$$x(z) = z \cdot {}_2F_1\Big(\begin{matrix}\alpha_\lambda, 1-\alpha_\lambda \\ 2\end{matrix}\Big| -\frac{z}{m^2}\Big) \qquad \alpha_\lambda = \begin{cases} \dfrac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda| \leq \frac{1}{\pi} \\ \frac{1}{2} + \mathrm{i}\dfrac{\mathrm{arcosh}(\lambda\pi)}{\pi} & \text{for} \lambda \geq \frac{1}{\pi} \end{cases}$$

  The effective spectral dimension of $x(z)$ is $D_\lambda = 4 - \frac{2}{\pi}\arcsin(\lambda\pi)$ [Grosse, Hock, W 19]. This dimension drop avoids the triviality problem of the usual $\lambda\phi^4$-model.