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How topological recursion organises quantum fields on noncommutative geometries

Raimar Wulkenhaar

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Wightman axioms

- [von Neumann 32] axioms for quantum mechanics
- [Wightman, Gårding 50s] unique extension to quantum fields
  = unbounded operator valued distributions $f \mapsto \Phi(f) : \mathcal{D} \to \mathcal{D} \subset \mathcal{H}$

Wightman axioms (sketched, see [Streater, Wightman 64])

- **Relativistic invariance.** Poincaré group represented by unitary operators on $\mathcal{H}$ under which field operators transform as $U(a,L)\Phi(f)U(a,L)^* = S(a,L)\Phi(f^{a,L})$
- **Spectrum condition.** Joint spectrum of generators P_i of translations contained in forward lightcone, i.e. $P_0^2 - \sum_{i=1}^{D-1} P_i^2 > 0$ and $P^0 > 0$
- **Vacuum.** There is a cyclic and Poincaré-invariant vector $\Omega \in \mathcal{D}$.
- **Locality.** $\Phi(f)\Phi(g) = \pm\Phi(g)\Phi(f)$ if f, g have causally disjoint support.

Reformulated by [Haag, Kastler 64] in terms of C^* and von Neumann algebras (today the preferred approach)

Wightman and Schwinger functions

Consider Wightman distributions

$$(f_1, \ldots, f_n) \mapsto W_n(f_1, \ldots, f_n) := \langle \Omega, \Phi(f_1) \cdots \Phi(f_n) \Omega \rangle$$

Axioms for $\Phi(f)$ induce axioms for W_n, and conversely. Moreover:

Theorem [Hall, Wightman 57], Bargmann, [Bros, Epstein, Glaser 67]

Wightman functions $\langle \Omega, \Phi(x_1) \cdots \Phi(x_n) \Omega \rangle$ are boundary values of holomorphic functions.

- Their restriction to real subspace of Euclidean points (minus diagonals) defines Schwinger functions S_n.
- Schwinger functions inherit real analyticity, Euclidean invariance, complete symmetry and reflection positivity.

Osterwalder-Schrader axioms

Reflection positivity

Let $(x^0, \vec{x})^\theta := (-x^0, \vec{x})$ be the reflection in $\mathbb{R}^D$ at the plane $x^0 = 0$. Let f_n be a test function on $(\mathbb{R}^D)^n \ni (x_1, ..., x_n)$ whose support is contained in $0 < x_1^0 < x_2^0 < \cdots < x_n^0$. Then for any tuple $f_0, f_1, \cdots, f_N$ of such test functions with ordered time support one has

$$\sum_{m,n=0}^{N} \int dx_1...dx_m\, dy_1..dy_n S_{m+n}(x_1, \ldots, x_m, y_1, \ldots, y_n)\overline{f_m(x_1^\theta, \ldots, x_m^\theta)} f_n(y_1, \ldots, y_n) \geq 0$$

- [Nelson 73] formulated axioms for random distributions where a Markov property was essential.
- [Osterwalder, Schrader 74] gave a first proof that the natural properties of Schwinger functions are also sufficient to imply the Wightman axioms.
- [Glaser 74] noticed a gap in the proof. [Osterwalder, Schrader 75] found a growth condition which together with their other axioms implies the Wightman axioms.
- Reflection positivity is related to the Hausdorff-Bernstein-Widder theorem.

Probability measures on distributions

Some axioms are automatic if $S_n(f_1, \dots, f_n) = \int_{\mathcal{V}'} d\mu(\Phi)\, \Phi(f_1) \cdots \Phi(f_n)$ are moments of a measure $d\mu$ on distributions, where $\mathcal{V} = C_0^\infty(\mathbb{R}^D)$ or $\mathcal{V} = \mathcal{S}(\mathbb{R}^D)$.

Consider the Fourier transform $\mathcal{Z}(f) := \int_{\mathcal{V}'} d\mu(\Phi)\, e^{\mathrm{i}\Phi(f)}$

[Glimm, Jaffe 87] axioms for a probability measure of Euclidean quantum fields

- OS0 **Analyticity.** For any $f_1, \dots, f_N \in \mathcal{V}$, $\mathbb{C}^N \ni (z_1, \dots, z_N) \mapsto \mathcal{Z}(z_1 f_1 + \cdots + z_N f_N)$ is holomorphic.
- OS1 **Regularity.** $|\mathcal{Z}(f)| \leq \exp(c_1\|f\|_1 + c_p\|f\|_p^p)$ for some $1 < p \leq 2$.
- OS2 **Euclidean invariance.** $\mathcal{Z}(f) = \mathcal{Z}(f^{R,a})$ where $f^{R,a}(x) = f(R^{-1}(x-a))$.
- OS3 **Reflection positivity.** For any real $f_1, \dots, f_N \in \mathcal{V}$, the $N \times N$ matrix $M_{ij} = \mathcal{Z}(f_i - f_j^\theta)$ is positive semidefinite, where $f^\theta(x) := f(x^\theta)$.
- OS4 **Ergodicity.** The time translation subgroup acts ergodically on $(\mathcal{V}', d\mu)$.

Gaußian measure

Theorem (Bochner, Minlos, Schur)

Any continuous inner product $\langle\ ,\ \rangle$ on a real nuclear vector space $\mathcal{V}$ defines a unique probability measure $d\mu_0$ on $\mathcal{V}'$ with $\exp(-\frac{1}{2}\langle f, f\rangle) = \int_{\mathcal{V}'} d\mu_0(\Phi) e^{i\Phi(f)}$.

A measure of this type defines a Gaußian field $\Phi \in \mathcal{V}'$.

Example

$\mathcal{V} = \mathcal{S}(\mathbb{R}^D)$ and $\langle f, g\rangle = \int_{\mathbb{R}^{2D}} dx\, dy\, f(x)(-\Delta + m^2)^{-1}(x,y)\, g(y)$

- One would like to define interacting fields by a deformation $d\mu(\Phi)\,\text{"}{:=}\text{"}\ \frac{1}{\mathcal{Z}} e^{-P(\Phi)} d\mu_0(\Phi)$ of the Gaußian measure $d\mu_0$, for P a polynomial of degree > 2 bounded from below.
- This is not straightforward! At a technical level, a product of distributions in $P(\Phi)$ is not defined.
- More concretely, various types of divergences occur whose treatment (renormalisation) requires a sophisticated analysis.

Selected success stories

The construction of interacting measures succeded in only a few cases, all in dimension < 4.

- A few exactly solvable 2D models, such as the [Thirring 58] model (fermions with quartic self-interaction) and the [Schwinger 62] model (QED in 2D).
- The $\lambda\Phi^4_2$-model in 2D, $P(\Phi) = \frac{\lambda}{4!}\int_{\mathbb{R}^2} dx\,(\Phi(x))^4$, first by hard work in relativistic formulation [Glimm, Jaffe 68–72].
- The $P[\Phi]_2$ model, i.e. any polynomial interaction in 2D. First spectacular success of the Euclidean method [Simon 74].
- Many conformal field theories [Belavin, Polyakov, Zamolodchikov 84] in 2D.
- Gross-Neveu model in 2D by fermionic summation techniques [Gawędzki-Kupiainen 85, Feldman-Magnen-Rivasseau-Sénéor 86].
- The $\lambda\Phi^4_3$-model in 3D [Feldman, Osterwalder 76].
  Recently the target of spectacular developments in probability theory [Hairer 14; Gubinelli, Imkeller, Perkowski 15; Mourrat, Weber 17].

Triviality

- [Landau, Abrikosov, Khalatnikov 54] gave a heuristic argument that quantum electrodynamics cannot exist as a renormalised QFT.

  The problem is called Landau ghost, triviality, positivity of β-function.

- This was considered as death of QFT, rescued only by the discovery of asymptotic freedom in non-Abelian Yang-Mills theory by Gross, Politzer, Wilczek in 1973.
- [Aizenman 81] and [Fröhlich 82] gave a rigorous proof of triviality for the $\lambda\Phi^4$-model in $D = 4 + \epsilon$ dimensions.
- Almost 40 years later, [Aizenman, Duminil-Copin 19] proved triviality of $\lambda\Phi^4$ in $D = 4$.

The situation of 4D QFT is disappointing

The only interacting model which seems to exist is non-Abelian Yang-Mills theory, but the proof is one of the Millenium Prize problems!

This motivates us to look at QFT on noncommutative geometries.

Noncommutative geometry

[Murray, von Neumann 36] introduced operator algebras and studied their properties. They defined the types of von Neumann factors.

- [Gelfand, Naimark 43] proved that commutative C^*-algebras are isometrically isomorphic to $C(X)$ for a locally-compact Hausdorff space X.

Noncommutative geometry relaxes the commutativity assumption and studies general operator algebras.

- [Takesaki 70] worked out Tomita's theory of modular automorphisms. These allowed [Connes 73] to finish the classification of amenable von Neumann algebras.

K-theory and various (co)homology theories were extended to the noncommutative world.

- [Kasparov 80] introduced KK-theory as bivariant functor on $C*$-algebras.
- [Connes 81] developed cyclic cohomology to achieve an index pairing with K-theory.

Non-commutative differential geometry as established research area [Connes 85].

First appearance of NCG in physics

- [Bellissard 86] understood the integer quantum Hall effect in terms of K-theory
- [Connes, Rieffel 87] defined Yang-Mills for non-commutative two-tori
- [Dubois-Violette, Kerner, Madore 90] worked out noncommutative differential geometry of matrix algebras
- [Connes 90] showed that noncommutative geometry on two points gives the Higgs potential; details worked out by [Connes, Lott 91]
- [Madore 91] proposed the fuzzy sphere, [Grosse, Madore 92] formulated a noncommutative version of the Schwinger model
- [Doplicher, Fredenhagen, Roberts 95] discussion (later more)
- [Filk 96] introduced Feynman rules for noncommutative quantum fields and showed that planar diagrams have the same divergences

First appearance of NCG in physics

- [Connes 96] introduced spectral triples and showed later that commutative spectral triples are manifolds [Connes 08]. Contributions by [Figueroa, Gracia-Bondía, Varilly 00] and [Rennie, Varilly 06].
- [Connes, Chamseddine 96] proposed the spectral action principle as unification of gravity and standard model of particle physics. Improvements by [Connes, Chamseddine, Marcolli 07] and [Connes, Chamseddine, van Suijlekom 13]
- [Kreimer 97] understood that the combinatorics of renormalisation in quantum field theory is encoded in the antipode of a Hopf algebra. It is closely related to a Hopf algebra which governs the local index formula [Connes, Moscovici 98] for foliations.

Summary so far (and outlook)

In the first lecture we sketched the path to the Euclidean formulation of quantum field theory, described by a measure on a space of distributions. We also gave a few historical notes about noncommutative geometry.

We will now bring both fields together. It is relatively straightforward to write down formal measures for Euclidean quantum fields on noncommutative spaces.

- A free field measure only requires an inner product on test functions; no product.
- Interaction is encoded in a product in a noncommutative algebra, possibly with differential forms. This gives up locality, which we argue with Doplicher, Fredenhagen & Roberts as desired.
- In contrast to first expectation [Grosse, Klimčík, Prešnajder 95], the non-locality does not rule out divergences in Feynman graps [Filk 96]. These can be treated by usual QFT methods, but another problem was discovered from the string theory side.

With Harald Grosse we understood that this problem just signals an incomplete model, which can be consistently completed. Surprisingly, this completion improves the behaviour concerning triviality.

Limits to localisability

[Doplicher, Fredenhagen, Roberts 95] combined an argument of [Wheeler 55] with noncommutative geometry (in particular the fuzzy sphere [Madore 91]).

- To resolve a structure of size ℓ, a scattering experiment with wavelength $\lambda < \ell$ is necessary. These waves carry a quantum-mechanical energy $E = \frac{\hbar c}{\lambda} > \frac{\hbar c}{\ell}$.
- Energy is a source of gravitational fields; the simplest type is the Schwarzschild solution with characteristic radius $r_S = \frac{2GE}{c^4} > \frac{2G\hbar}{c^3\ell}$.
- To avoid trapped surfaces (in the sense of Penrose), the structure to resolve must stay outside the Schwarzschild horizon, $\ell > 2r_s > \frac{4G\hbar}{c^3\ell}$.

Hence, only structures larger than the Planck length $\ell_P = \sqrt{\frac{G\hbar}{c^3}} = 10^{-33}\,\text{cm}$ can be meaningful if quantum physics and general relativity are combined.

The detailed anlysis of Doplicher, Fredenhagen, Roberts established unvertainty relations

$$\Delta x^0(\Delta x^1 + \Delta x^2 + \Delta x^3) \geq \ell_P^2 \text{ and } \Delta x^1\Delta x^2 + \Delta x^2\Delta x^3 + \Delta x^3\Delta x^1 \geq \ell_P^2.$$

They are induced by noncommutative coordinate operators $\hat{x}^\mu = (\hat{x}^\mu)^*$ with

$$[[\hat{x}^\mu, \hat{x}^\nu], \hat{x}^\rho] = 0\,, \qquad [\hat{x}_\mu, \hat{x}_\nu][\hat{x}^\mu, \hat{x}^\nu] = 0\,, \qquad \big(\tfrac{1}{8}[\hat{x}^\mu, \hat{x}^\nu][\hat{x}^\rho, \hat{x}^\sigma]\epsilon_{\mu\nu\rho\sigma}\big)^2 = \ell_P^8\,.$$

Input from String theory

- [Witten 86] discussed Non-commutative geometry and string field theory.
- [Banks, Fischler, Shenker, Susskind 97] and [Ishibashi, Kawai, Kitazawa, Tsuchiya 97] introduced two formulations of M-theory as matrix models.
- [Connes, Douglas, Schwarz 97] studied compactifications of M-theory on noncommutative tori.
- Yang-Mills on noncommutative 4-torus is one-loop renormalisable [Krajewski-W 99]. Similar results from Filk's rules [Martín, Sánchez-Ruiz 99].
- [Schomerus 99] discovered that *D*-branes in flat background with constant magnetic field have operator product expansion given by noncommutative Moyal product.
- [Seiberg, Witten 99] vastly extended Schomerus' ideas and found a transformation between objects on noncommutative and commutative space.
- [Minwalla, van Raamsdonk, Seiberg 99] discovered UV/IR-mixing. Thoroughly investigated by [Chepelev, Roiban 99].

Remarks on relativistic formulation

A direct relativistic (with time) construction of true quantum fields on noncommutative spaces is much harder.

- [Doplicher, Fredenhagen, Roberts 95] outline a definition of quantum fields directly on non-local space-time.
- Further developed in [Bahns, Doplicher, Fredenhagen, Piacitelli 02], with emphasis on unitarity.
- *Free* Wightman functions on quantum space remain boundary values of holomorphic functions [Bahns 09], but these differ from the usual Euclidean approach (except for commutative time [Grosse, Lechner, Ludwig, Verch 11]).
- New achievements in perturbative algebraic quantum field theory [Doplicher, Morsella, Pinamonti 20].

There are thoughts that time should be introduced differently: Noncommutative algebras come with a canonical time evolution (Tomita-Takesaki theory). See [Connes, Rovelli 94]

Approximation by matrix algebras

To define interacting quantum fields one needs to make sense of a product of distributions.

If correctly implemented, the various divergences of the naïve product (of individual Feynman graphs [Filk], UV/IR-mixing [Minwalla, van Raamsdonk, Seiberg] and triviality [Landau; Aizenman]) should all disappear.

The only known strategy consists in finite-dimensional approximations of the problem, together with a careful limiting procedure.

Finite-dimensional approximations of noncommutative algebras are matrix algebras. E.g.

- nuclear C^*-algebras in which the identity map, as a completely positive map, approximately factors through matrix algebras;
- AF-algebras, which are inductive limit of sequences of matrix algebras.

These two classes of C^*-algebras are not appropriate for us; we need (smooth) test functions, equipped which a Fréchet topology. There are enough examples where Fréchet algebras have a matrix approximation.

The Moyal algebra

Let $\Theta = -\Theta^t$ be a skew-symmetric real $D \times D$-matrix. Then by

$$(f \star g)(x) := \int_{\mathbb{R}^D \times \mathbb{R}^D} \frac{dydk}{(2\pi)^D} f(x + \tfrac{1}{2}\Theta k) g(x+y)\, e^{\mathrm{i}\langle k,y \rangle}$$

an associative non-commutative product on Schwartz functions $f, g \in \mathcal{S}(\mathbb{R}^D)$ is defined.

- Prototype example for strict deformation quantisation by $\mathbb{R}^D$-actions [Rieffel 93].
- Defining $(\hat{x}^\mu f)(x) = (x^\mu \star f)(x)$ by obvious extension, then $[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\Theta^{\mu\nu}$ is central and a simple-minded implementation of [Doplicher, Fredenhagen, Roberts 95].
- $\overline{f \star g} = \overline{g} \star \overline{f}$, $\int_{\mathbb{R}^D} dx\, (f \star g)(x) = \int_{\mathbb{R}^D} dx\, f(x) g(x)$.

The inner product $\langle f, g \rangle = \int_{\mathbb{R}^{2D}} dx\, dy\, f(x)(-\Delta + M^2)^{-1}(x,y)\, g(y)$ gives, as usual, rise to a Gauß measure $d\mu_0$ on $(\mathcal{S}(\mathbb{R}^D))'$ which now can be noncommutatively deformed to $d\mu_\lambda(\Phi) := \frac{1}{\mathcal{Z}} e^{-\frac{\lambda}{4}\int_{\mathbb{R}^D} dx\, (\Phi \star \Phi \star \Phi \star \Phi)(x)} d\mu_0(\Phi)$. The resulting QFT suffers from UV/IR-mixing.

The matrix basis

Let $D = 2$ and $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$. We recommend to check:

Exercise

1. The Gaußian $b_{00}(x) := 2e^{-\frac{\|x\|^2}{\theta}}$ is a projector, $b_{00} \star b_{00} = b_{00}$.
2. Let $a := \frac{1}{\sqrt{2}}(x_1 + \mathrm{i}x_2)$, $\bar{a} := \frac{1}{\sqrt{2}}(x_1 - \mathrm{i}x_2)$. Then $a \star b_{00} = 0 = b_{00} \star \bar{a}$ and $[\bar{a}, a]_\star = \theta$.
3. Consequently, $b_{mn} := \frac{1}{\sqrt{m!n!\theta^{m+n}}} \bar{a}^{\star m} \star b_{00} \star a^{\star n}$ satisfies $b_{kl} \star b_{mn} = \delta_{lm} b_{kn}$. Moreover, $\int_{\mathbb{R}^2} dx\ b_{kl}(x) = 2\pi\theta\delta_{kl}$.
4. $b_{mn}(x) = 2(-1)^m \sqrt{\frac{m!}{n!}} \big(\sqrt{\frac{2}{\theta}}(x_1 + \mathrm{i}x_2)\big)^{n-m} L_m^{n-m}\big(\frac{2}{\theta}\|x\|^2\big) e^{-\frac{\|x\|^2}{\theta}}$ where L_m^α are associate Laguerre polynomials.

Let $\mathcal{A}_\theta = \{f = (f_{kl})_{k,l\in\mathbb{N}}\}$ be the vector space of infinite matrices, completed in the Fréchet topology induced by the family $\langle f, g\rangle_m := \sum_{k,l\in\mathbb{N}} \theta^{2m}(k + \frac{1}{2})^m (l + \frac{1}{2})^m \overline{f_{kl}} g_{kl}$.

Then $\mathcal{A}_\theta \ni f \mapsto \sum_{k,l\in\mathbb{N}} f_{kl} b_{kl}$ is an isomorphism of Fréchet algebras between $(\mathcal{A}_\theta, \cdot)$ and $(\mathcal{S}(\mathbb{R}^2), \star)$ [Gracia-Bondía, Varilly 88].

The $\lambda\Phi^{\star 4}$-model in the matrix basis

- The isomorphism extends to $\mathcal{S}(\mathbb{R}^D)$, for D even, via $b_{k_1 l_1}(x_1, x_2) \cdots b_{k_{D/2} l_{D/2}}(x_{D-1}, x_D)$. Use bijection $\pi : \mathbb{N}^{D/2} \to \mathbb{N}$, e.g. $\pi(k_2, k_2) = \frac{1}{2}(k_1+k_1)(k_1+k_1+1)+k_2$ [Cantor].
- Setting $\Phi_{kl} := \Phi(b_{k_1 l_1} \otimes .. \otimes b_{k_{D/2} l_{D/2}})\big|_{k=\pi(k_1,\dots k_{D/2}), l=\pi(l_1,..,l_{D/2})}$, we turn the measure deformation into

$$d\mu_\lambda(\Phi) \mapsto \frac{1}{\mathcal{Z}} e^{-\frac{\lambda}{4}\sqrt{\det(2\pi\Theta)}\sum_{k,l,m,n\in\mathbb{N}} \Phi_{kl}\Phi_{lm}\Phi_{mn}\Phi_{nk}} d\mu_0(\Phi) \;.$$

We should also express the Gauß measure $d\mu_0(\Phi)$ in the matrix basis. One first notices

$$\int_{\mathbb{R}^2} dx\, f(x)((-\Delta + M^2)g)(x) = \sum_{k,l,m,n\in\mathbb{N}} \Delta_{kl;mn} f_{kl} g_{mn} \qquad \text{where}$$

$$\Delta_{kl;mn} = (M^2 + \frac{2}{\theta}(m+n+2))\delta_{nk}\delta_{ml} - \frac{2}{\theta}\sqrt{kl}\delta_{n+1,k}\delta_{m+1,l} - \frac{2}{\theta}\sqrt{mn}\delta_{n-1,k}\delta_{m-1,l}$$

- The inner product to define $d\mu_0$ needs the kernel of the inverse $(-\Delta + M^2)^{-1}$. Can be found by diagonalisation via Meixner polynomials [Grosse, W 04].
- Then UV/IR-mixing traced back to off-diagonal terms too large.

The harmonic oscillator potential [Grosse, W 04]

Scaling down the off-diagonal term to cure the UV/IR-mixing problem amounts to introduce a harmonic oscillator potential. We thus arrive at the inner product (in $D=4$)

$$\langle f,g\rangle := \int_{\mathbb{R}^4\times\mathbb{R}^4} dxdy\; f(y)\big(-\Delta+\tfrac{4\Omega^2}{\theta^2}|x|^2+M^2\big)^{-1}(y,x)g(x) = \sum_{\substack{k_1,k_2,\dots\\ n_1,n_2=0}}^{\infty} f_{\substack{k_1l_1\\k_2l_2}}\, g_{\substack{m_1n_1\\m_2n_2}}\, C_{\substack{k_1l_1\\k_2l_2};\substack{m_1n_1\\m_2n_2}}$$

$$C_{\substack{k_1l_1\\k_2l_2};\substack{m_1n_1\\m_2n_2}} = \frac{\theta}{2(1+\Omega)^2}\delta_{m_1+k_1,n_1+l_1}\delta_{m_2+k_2,n_2+l_2}$$

$$\times \sum_{v_1=\frac{|m_1-l_1|}{2}}^{\frac{m_1+l_1}{2}} \sum_{v_2=\frac{|m_2-l_2|}{2}}^{\frac{m_2+l_2}{2}} B\Big(1+\tfrac{M^2\theta}{8\Omega}+\tfrac{1}{2}(m_1+k_1+m_2+k_2)-v_1-v_2,\,1+2v_1+2v_2\Big)$$

$$\times\, {}_2F_1\left(\begin{matrix}1+2v_1+2v_2\,,\; \frac{M^2\theta}{8\Omega}-\frac{1}{2}(m_1+k_1+m_2+k_2)+v_1+v_2\\ 2+\frac{M^2\theta}{8\Omega}+\frac{1}{2}(m_1+k_1+m_2+k_2)+v_1+v_2\end{matrix}\,\middle|\,\frac{(1-\Omega)^2}{(1+\Omega)^2}\right)$$

$$\times \prod_{i=1}^{2}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{2v_i}\sqrt{\binom{n_i}{v_i+\frac{n_i-k_i}{2}}\binom{k_i}{v_i+\frac{k_i-n_i}{2}}\binom{m_i}{v_i+\frac{m_i-l_i}{2}}\binom{l_i}{v_i+\frac{l_i-m_i}{2}}}\,.$$

Perturbative renormalisation and β-function

Aim is to make sense of $\frac{1}{\mathcal{Z}}\sum_{p=0}^{\infty}\frac{1}{p!}\Big(-\frac{\lambda}{4}\sqrt{\det(2\pi\Theta)}\sum_{k,l,m,n=0}^{\infty}\Phi_{kl}\Phi_{lm}\Phi_{mn}\Phi_{nk}\Big)^p d\mu_0(\Phi)$ for $d\mu_0$ defined with inner product with kernel C.

- Restrict everything to finite $N\times N$-matrices and set $\Phi_{mn}\mapsto\sqrt{Z}(N)\Phi_{mn}$, $\lambda\mapsto\lambda(N)$ $M\mapsto M(N)$, $\Omega\mapsto\Omega(N)$ (regularisation).
- One proves [Grosse, W 05] that $\sqrt{Z}(N),\mu(N),\Omega(N),\lambda(N)$ can be found as formal power series in a (new) parameter λ such that the measure exists as formal power series in λ.
- Moreover, at lowest order in λ one finds $\frac{\lambda(N)}{\Omega^2(N)}=\mathit{const}$ and $\lim_{N\to\infty}\Omega(N)=1$ [Grosse W 04]. This implies existence of $\lim_{N\to\infty}\lambda(N)$, i.e. absense of the Landau ghost.
- [Disertori, Rivasseau 06] proposed to look at $\Omega\equiv 1$ independent of N. The inner product simplifies enormously, and they could check existence of $\lim_{N\to\infty}\lambda(N)$ up to third order.
- [Disertori, Gurau, Magnen, Rivasseau 06] extended this result to any order. Their method is (together with topological recursion) the key to a complete solution.

Simplifications for $\Omega = 1$

Set $\mathcal{N} := (\frac{\theta}{4})^{D/2}$. We will soon arrive at an $1/\mathcal{N}$-expansion. Note that $\mathcal{N}$ is not the size of matrices, it is the scale of noncommutativity!

The inner product on $\mathcal{V} = \mathcal{A}_\theta$ specifies for $\Omega = 1$ to

$$\langle f, g\rangle = \frac{1}{\mathcal{N}} \sum_{k,l=0}^{\infty} \frac{f_{kl} g_{lk}}{E_k + E_l}\,, \qquad E_k = \frac{(k_1 + \ldots + k_{D/2} + D/4)}{\mathcal{N}^{2/D}} + \frac{M^2}{2}\Big|_{(k_1,\ldots,k_{D/2})=\pi^{-1}(k)}$$

It gives rise to a unique measure $d\tilde{\mu}_0$ on $\mathcal{V}'$ with $\int_{\mathcal{V}'} d\tilde{\mu}_0(\Phi) e^{\mathrm{i}\Phi(f)} = \exp(-\frac{1}{2}\langle f, f\rangle)$ which we (formally) deform to

$$d\tilde{\mu}_\lambda(\Phi) = \frac{1}{\mathcal{Z}} e^{-\frac{\lambda\mathcal{N}}{4}\mathrm{Tr}(\Phi^4)} d\tilde{\mu}_0(\Phi)$$

where $\mathrm{Tr}(\Phi^4) := \sum_{k,l,m,n=0}^{\infty} \Phi(e_{kl})\Phi(e_{lm})\Phi(e_{mn})\Phi(e_{nk})$.

For technical reasons we temporarily assume that all E_k are pairwise different.

Moments and cumulants

$$\langle e_{k_1l_1}\dots e_{k_nl_n}\rangle := \int_{\mathcal{V}'} d\mu_\lambda(\Phi)\prod_{i=1}^n \Phi(e_{k_il_i}) = \frac{1}{\mathrm{i}^n}\frac{\partial^n\mathcal{Z}(M)}{\partial f_{k_1l_1}\cdots\partial f_{k_nl_n}}\Big|_{f=0}, \quad \mathcal{Z}(f) := \int_{\mathcal{V}'} d\mu_\lambda(\Phi)\, e^{\mathrm{i}\Phi(f)}$$

if $f=\sum_{k,l=0}^\infty f_{kl}e_{kl}$ with respect to standard matrix basis (e_{kl}). These moments decompose into cumulants

$$\Big\langle\prod_{i=1}^n e_{k_il_i}\Big\rangle = \sum_{\substack{\text{partitions}\\ \pi \text{ of } \{1,\dots,n\}}}\ \prod_{\text{blocks } \beta\in\pi}\Big\langle\prod_{i\in\beta} e_{k_il_i}\Big\rangle_c .$$

They are only non-zero if n is even and every block β is of even length.

Take all k_i pairwise different. Then $\langle e_{k_1l_1}\dots e_{k_nl_n}\rangle_c$ is only non-zero if $(l_1,\dots,l_n)$ is a permutation of $(k_1,\dots,k_n)$, and in this case the cumulant only depends on the *cycle type*:

$$N^{n_1+\dots+n_b}\big\langle (e_{k_1^1k_2^1}e_{k_2^1k_3^1}\cdots e_{k_{n_1}^1k_1^1})\cdots(e_{k_1^bk_2^b}e_{k_2^bk_3^b}\cdots e_{k_{n_b}^bk_1^b})\big\rangle_c =: N^{2-b}G_{|k_1^1\dots k_{n_1}^1|\dots|k_1^b\dots k_{n_b}^b|} .$$

Equations of motion

Lemma [Schürmann, W 19]

The Fourier transform $\mathcal{Z}(f) := \int_{\mathcal{V}'} d\mu_\lambda(\Phi)\, e^{\mathrm{i}\Phi(f)}$ of the measure satisfies

1. $$\frac{1}{\mathrm{i}}\frac{\partial \mathcal{Z}(f)}{\partial f_{ab}} = \frac{\mathrm{i} f_{ba}\mathcal{Z}(f)}{\mathcal{N}(E_a+E_b)} - \frac{\lambda}{\mathrm{i}^3(E_a+E_b)}\sum_{k,l=0}^{\infty}\frac{\partial^3 \mathcal{Z}(f)}{\partial f_{ak}\partial f_{kl}\partial f_{lb}}$$

2. $$\frac{1}{\mathcal{N}}\frac{\partial \mathcal{Z}(f)}{\partial E_a} = \sum_{k=0}^{\infty}\frac{\partial^2 \mathcal{Z}(f)}{\partial f_{ak}\partial f_{ka}} + \Big(\frac{1}{\mathcal{N}}\sum_{k=0}^{\infty} G_{|ak|} + \frac{1}{\mathcal{N}^2}G_{|a|a|}\Big)\mathcal{Z}(f)$$

Corollary (Ward-Takahashi identity)

3. $$-\mathcal{N}\sum_{k=0}^{\infty}(E_a-E_b)\frac{\partial^2 \mathcal{Z}(f)}{\partial f_{ak}\partial f_{kb}} = \sum_{k=0}^{\infty}\Big(f_{ka}\frac{\partial \mathcal{Z}(f)}{\partial f_{kb}} - f_{bk}\frac{\partial \mathcal{Z}(f)}{\partial f_{ak}}\Big)$$

3 was discovered in [Disertori, Gurau, Magnen, Rivasseau 06]. Alex Hock will prove it (for finite matrices) in his tutorial.

Dyson-Schwinger equations

The equation of motion ❶ induces Dyson-Schwinger equations between moments. For $n = 2$:

$$\frac{1}{\mathcal{N}} G_{|ab|} := \langle e_{ab} e_{ba} \rangle \equiv -\frac{\partial^2 \mathcal{Z}(f)}{\partial f_{ba} \partial f_{ab}}\Big|_{f=0} = \frac{1}{\mathcal{N}(E_a + E_b)} - \frac{\lambda}{(E_a + E_b)} \sum_{k,l=0}^{\infty} \frac{\partial^4 \mathcal{Z}(f)}{\partial f_{lb} \partial f_{ba} \partial f_{ak} \partial f_{kl}}\Big|_{f=0}$$

It seems that the rhs will produce 4-point functions (in general, n-point function expressed in terms of $n+2$-point functions). Using ❸ and ❷ one can avoid this:

$$\begin{aligned}(E_a + E_b) G_{|ab|} &= 1 + \lambda \sum_{\substack{l=0\\ l\neq a}}^{\infty} \frac{\partial^2}{\partial f_{lb} \partial f_{ba}} \Big[\sum_{k=0}^{\infty} \frac{1}{E_a - E_l} \Big(f_{ka} \frac{\partial \mathcal{Z}(f)}{\partial f_{kl}} - f_{lk} \frac{\partial \mathcal{Z}(f)}{\partial f_{ak}} \Big) \Big]_{f=0} \\ &\quad - \lambda \mathcal{N} \frac{\partial^2}{\partial f_{ab} \partial f_{ba}} \Big[\frac{1}{\mathcal{N}} \frac{\partial \mathcal{Z}(f)}{\partial E_a} - \Big(\frac{1}{\mathcal{N}} \sum_{k=0}^{\infty} G_{|ak|} + \frac{1}{\mathcal{N}^2} G_{|a|a|} \Big) \mathcal{Z}(f) \Big]_{f=0} \\ &= 1 + \frac{\lambda}{\mathcal{N}} \sum_{\substack{k=0\\ k\neq a}}^{\infty} \frac{G_{|kb|} - G_{|ab|}}{E_k - E_a} + \frac{\lambda}{\mathcal{N}^2} \frac{G_{|b|b|} - G_{|a|b|}}{E_b - E_a} - G_{|ab|} \Big(\frac{\lambda}{\mathcal{N}} \sum_{k=0}^{\infty} G_{|ak|} + \frac{\lambda}{\mathcal{N}^2} G_{|a|a|} \Big) + \frac{\lambda}{\mathcal{N}} \frac{\partial G_{|ab|}}{\partial E_a}\end{aligned}$$

Complexification

- The dependence of $G_{|ab|}$ on the matrix indices a, b is of the form of an evaluation $G_{|ab|} = G(E_a, E_b)$ of a function G of two complex variables at E_a, E_b.
- $G(\zeta, \eta)$ still depends on summation variables E_k. Differentiating wrt some E_c is also an evaluation at E_c of another function $-\mathcal{N}\frac{\partial}{\partial E_c} G(\zeta, \eta) = T(E_c \| \zeta, \eta|)$.
- Similarly for $G_{|a|b|} = G(E_a|E_b)$.

Complexified Dyson-Schwinger equation

$$(\zeta + \eta) G(\zeta, \eta) = 1 + \frac{\lambda}{\mathcal{N}} \sum_{k \in \mathbb{N}} \frac{G(E_k, \eta) - G(\zeta, \eta)}{E_k - \zeta} + \frac{\lambda}{\mathcal{N}^2} \frac{G(\eta|\eta) - G(\zeta|\eta)}{\eta - \zeta}$$
$$- G(\zeta, \eta) \Big(\frac{\lambda}{\mathcal{N}} \sum_{k \in \mathbb{N}} G(\zeta, E_k) + \frac{\lambda}{\mathcal{N}^2} G(\zeta|\zeta) \Big) - \frac{\lambda}{\mathcal{N}^2} T(\zeta \| \zeta, \eta|)$$

Alternatively, sum only over pairwise different e_k and include their multiplicities r_k:

$$\sum_{k \in \mathbb{N}} \frac{G(E_k, \eta) - G(\zeta, \eta)}{E_k - \zeta} = \sum_{k=0}^{\infty} r_k \frac{G(e_k, \eta) - G(\zeta, \eta)}{e_k - \zeta}, \qquad \sum_{k \in \mathbb{N}} G(\zeta, E_k) = \sum_{k=0}^{\infty} r_r G(\zeta, e_k)$$

The genus expansion

We approach the solution of such equations in a formal genus expansion (where $\mathcal{N} := (\frac{\theta}{4})^{D/2}$)

$$G(\zeta,\eta) = \sum_{g=0}^{\infty} \mathcal{N}^{-2g} G^{(g)}(\zeta,\eta)\,, \qquad G(\zeta|\eta) = \sum_{g=0}^{\infty} \mathcal{N}^{-2g} G^{(g)}(\zeta|\eta)\,,$$

$$T(\xi\|\zeta,\eta|) = \sum_{g=0}^{\infty} \mathcal{N}^{-2g} T^{(g)}(\xi\|\zeta,\eta|)$$

together with the convention that $\frac{1}{\mathcal{N}}$ in front of a summation is neutral.

Note that these series have zero radius of convergence! Making sense of them via Borel resummation is a main challenge for the future. Connects to resurgence.

Theorem [Grosse, W 09]

The planar two-point function satisfies the closed non-linear equation

$$\Big(\zeta+\eta+\frac{\lambda}{\mathcal{N}}\sum_{k\in\mathbb{N}} r_k G^{(0)}(\zeta,e_k)\Big)G^{(0)}(\zeta,\eta) = 1+\frac{\lambda}{\mathcal{N}}\sum_{k\in\mathbb{N}} r_k \frac{G^{(0)}(e_k,\eta)-G^{(0)}(\zeta,\eta)}{e_k-\zeta}$$

Regularisation

The Dyson-Schwinger equation needs a reinterpretation in view of quantum field theory.

- Recall $e_k = \frac{k}{\mathcal{N}^{2/D}} + \frac{\tilde{M}^2}{2}$ and $r_k = 1$ for $D = 2$ and $r_k = k+1$ for $D = 4$.
- We restrict the sums to $\sum_{k=0}^{\Lambda^2 \mathcal{N}^{2/D}}$. The limit $\Lambda^2 \to \infty$ can only exist if $\tilde{M}(\Lambda)$ is a carefully adapted function of Λ and if $G \mapsto Z(\Lambda)G$ is carefully rescaled.
- We better write $e_k = \tilde{e}_k + \frac{\tilde{M}^2}{2}$, $\zeta = \tilde{\zeta} + \frac{\tilde{M}^2}{2}$, $\eta = \tilde{\eta} + \frac{\tilde{M}^2}{2}$ and $G(\zeta,\eta) = \tilde{G}(\tilde{\zeta},\tilde{\eta})$

The result is for $D = 4$ (omitting the tilde)

$$\Big(\zeta + \eta + M^2(\Lambda) + \frac{\lambda}{\sqrt{\mathcal{N}}} \sum_{k=0}^{\Lambda^2\sqrt{\mathcal{N}}} \frac{k+1}{\sqrt{\mathcal{N}}} Z(\Lambda) G^{(0)}(\zeta, \tfrac{k}{\sqrt{\mathcal{N}}})\Big) Z(\Lambda) G^{(0)}(\zeta,\eta)$$

$$= 1 + \frac{\lambda}{\sqrt{\mathcal{N}}} \sum_{k=0}^{\Lambda^2\sqrt{\mathcal{N}}} \frac{k+1}{\sqrt{\mathcal{N}}} \frac{Z(\Lambda)\big(G^{(0)}(\frac{k}{\sqrt{\mathcal{N}}},\eta) - G^{(0)}(\zeta,\eta)\big)}{\frac{k}{\sqrt{\mathcal{N}}} - \zeta}$$

For $\sqrt{\mathcal{N}} = \frac{\theta}{4}$ large enough, this is arbitrarily close to integral equation

$$\Big(\zeta + \eta + M^2 + \lambda \int_0^{\Lambda^2} dt\, t Z G^{(0)}(\zeta,t)\Big) Z G^{(0)}(\zeta,\eta) = 1 + \lambda \int_0^{\Lambda^2} dt\, t \frac{Z\big(G^{(0)}(t,\eta) - G^{(0)}(\zeta,\eta)\big)}{t-\zeta}$$

Integral equation

We can arrange the two-point function of a large family of matrix and QFT models with quartic interaction into the integral equation

$$\Big(\zeta+\eta+M^2+\lambda\int_0^\infty dt\,\varrho_0(t)ZG^{(0)}(\zeta,t)\Big)ZG^{(0)}(\zeta,\eta)=1+\lambda\int_0^\infty dt\,\varrho_0(t)\frac{Z(G^{(0)}(t,\eta)-G^{(0)}(\zeta,\eta))}{t-\zeta}$$

- large-θ 4D Moyal: $\varrho_0(t)=t\chi_{[0,\Lambda^2]}$ and $M=M(\Lambda)$, $Z=Z(\Lambda)$
- large-θ 2D Moyal: $\varrho_0(t)=\chi_{[0,\Lambda^2]}$ and $M=M(\Lambda)$, $Z=1$
- $N\times N$ matrix model $\varrho(t)=\frac{1}{N}\sum_{k=1}^d r_k\delta(t-e_k)$, $r_1+\ldots+r_d=N$, $M=0$, $Z=1$

The spectral measure encodes a spectral dimension $\delta:=\inf\big(p\;:\;\int_0^\infty\frac{\varrho_0(t)}{(1+t)^{p/2}}<\infty\big)$

- In [Panzer, W 18] we solved the case $\varrho_0(t)=1$. Key step was to extrapolate a computer algebra evaluation of iterated integrals.
- In [Grosse, Hock, W 19a] we succeeded in solving the integral equation for any Hölder-continuous measure ϱ_0 of spectral dimension $\delta<6$.

Theorem [Panzer-W 18 for $\varrho_0 = 1$, Grosse-Hock-W 19a]

1. Ansatz $G^{(0)}(\zeta, \eta) = \dfrac{e^{\mathcal{H}_\zeta[\tau_\eta(\bullet)]} \sin \tau_\eta(\zeta)}{Z\lambda\pi\varrho_0(\zeta)}$, $\qquad \mathcal{H}_\zeta[f] := \frac{1}{\pi} \fint_0^{\Lambda^2} \frac{dp\, f(p)}{p-\zeta}$ finite Hilbert transf.
2. $\tau_\eta(\zeta) = \operatorname{Im} \log\big(\eta + I(\zeta + \mathrm{i}\epsilon)\big)$ with $I(\zeta) = -R_D(-m^2 - R_D^{-1}(\zeta))$
3. $$\boxed{R_D(z) = z - \lambda(-z)^{D/2} \int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(m^2+t)^{D/2}(t+m^2+z)}} \qquad D = 2[\tfrac{\delta}{2}]$$
4. ϱ_λ is implicit solution of $\varrho_0(R_D(\zeta)) = \varrho_\lambda(\zeta)$.

- Proof: [Cauchy 1831] residue theorem,
  [Lagrange 1770] inversion theorem, [Bürmann 1799] formula
- $\varrho_0(t) \equiv 1$ (2D Moyal, $m = 1$) in terms of Lambert-W satisfying $W(z)e^{W(z)} = z$:
  $I(\zeta) := \lambda W_0\big(\frac{1}{\lambda} e^{\frac{1+\zeta}{\lambda}}\big) - \lambda \log\big(1 - \lambda W_0\big(\frac{1}{\lambda} e^{\frac{1+\zeta}{\lambda}}\big)\big)$

$D = 4$ Moyal space: $\varrho_0(t) = t$ [Grosse-Hock-W 19b]

- $\varrho_\lambda(x) \equiv \varrho_0(R_4(x)) = R_4(x) = x - \lambda x^2 \int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(m^2+t)^2(t+x)}$
- If $\varrho_\lambda(t) \sim \varrho_0(t) = t$, then $R_4(x)$ bounded above. Consequently, R_4^{-1} would not be globally defined: triviality!
- Fredholm equation perturbatively solved by iterated integrals: Hyperlogarithms and $\zeta(2n)$ which can be summed to

$$R_4(x) \equiv \varrho_\lambda(x) = x \cdot {}_2F_1\left(\begin{matrix}\alpha_\lambda, 1-\alpha_\lambda \\ 2\end{matrix}\,\middle|\, -\frac{x}{m^2}\right) \qquad \alpha_\lambda = \begin{cases} \frac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda| \leq \frac{1}{\pi} \\ \frac{1}{2} + \mathrm{i}\frac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{for } \lambda \geq \frac{1}{\pi} \end{cases}$$

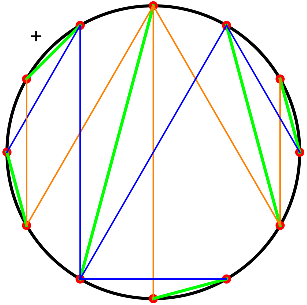
Corollary

The interaction alters the spectral dimension to $4 - 2\frac{\arcsin(\lambda\pi)}{\pi}$ and thus avoids the triviality problem (in the planar sector).

Gives non-perturbative integral representation for $G^{(0)}(\xi, \eta)$.

All planar correlation functions

- A Catalan tuple is a tuple $\tilde{p} = (p_0, p_1, ..., p_k)$ with $p_i \geq 0$, $\sum_{j=0}^{l} p_j > l$ if $l < k$ and $\sum_{j=0}^{k} p_j = k$. We let $|\tilde{p}| = k$.
- We call a collection $\mathcal{T} = \langle \tilde{p}_0, \tilde{p}_1, ..., \tilde{p}_{n+1} \rangle$ of Catalan tuples a nested Catalan table (of length n) if its length tuple $(|\tilde{p}_0| + 1, |\tilde{p}_1|, ..., |\tilde{p}_{n+1}|)$ is itself a Catalan tuple.
- There are $\frac{1}{n+1}\binom{3n+1}{n}$ nested Catalan tables of length n.



Theorem [de Jong, Hock W 19]

The planar n-point function is a sum of terms of the form $\frac{\pm G^{(0)}_{b_p b_q} \cdots G^{(0)}_{b_r b_s}}{(E_{b_t} - E_{b_u}) \cdots (E_{b_v} - E_{b_w})}$ which are in bijection with nested Catalan tables of length $\frac{n-2}{2}$.

- A green chord between a, b encodes a factor $G_{|ab|}$.
- An edge of blue or orange rooted plane tree from k to l encodes a factor $\frac{1}{E_k - E_l}$.
- There is a natural bijection between blue and orange rooted plane trees.

The Kontsevich model

Consider the Gaußian measure $d\mu_0$ on $\mathcal{V}'$, for $\mathcal{V} = H_N$ Hermitean $N \times N$-matrices, induced by $\langle f, g \rangle = \frac{1}{N} \sum_{k,l=1}^{N} \frac{f_{kl} g_{lk}}{E_k + E_l}$.

- Deforming it to $d\mu_K(\Phi) := \frac{1}{\mathcal{Z}} e^{\frac{\mathrm{i}}{6} N \,\mathrm{Tr}(\Phi^3)} d\mu_0(\Phi)$ gives the [Kontsevich 92] model which generates intersection numbers of ψ- and κ-classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves. Proves a conjecture by [Witten 90].
- This is a key example for topological recursion [Eynard, Orantin 07]. Its spectral curve is $x(z) = z^2$, $y(z) = -z + \frac{1}{N} \sum_{k=1}^{N} \frac{1}{\hat{E}_k(\hat{E}_k - z)}$ and $\omega_{0,2}(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$ where $\hat{E}_k = \sqrt{E_k^2 + c}$ and $c = \frac{2}{N} \sum_{k=1}^{N} \frac{1}{\sqrt{E_k^2 + c}}$.
- One can turn the Kontsevich model into a QFT on Moyal space (or other NCG) [Grosse, Steinacker 05/06], [Grosse, Sato, W 16], [Grosse, Hock, W 19c]. But $d\mu_K$ is not a valid measure.

The quartic Kontsevich model

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We will now deform the same $d\mu_0$ of the Kontsevich model by a quartic potential $d\mu_\lambda(\Phi) := \frac{1}{\mathcal{Z}} e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_0(\Phi)$.

- We called this the quartic Kontsevich model. There are another variations of the Kontsevich model, e.g. the family of generalised Kontsevich models which relate to r-spin intersection numbers [Belliard, Charbonnier, Eynard, Garcia-Failde 21].
- One can also take for $\mathcal{V}$ all complex $N \times N$-matrices [Langmann, Szabo, Zarembo 03]. Very recently, [Hock, Branahl 22] proved that the complex quartic model obeys topological recursion, whereas the real model (discussed below) needs blobbed TR. They clearly located the differences responsible for the blobs.

Recall that the planar 2-point function of the real quartically deformed model satisfies

$$\Big(\eta+\zeta+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(\zeta,e_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{e_k-\zeta}\Big)G^{(0)}(\zeta,\eta) = 1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k \frac{G^{(0)}(e_k,\eta)}{e_k-\zeta}$$

'Elementary' solution

Suppose there is a rational function R of degree $d+1$, with simple pole at ∞ of residue -1 and

$$R(z)+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(R(z),e_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{e_k-R(z)}=-R(-z)$$

Setting $\zeta=R(z)$, $\eta=R(w)$ and $G^{(0)}(\zeta,\eta)=\mathcal{G}^{(0)}(z,w)$ and chosing $\varepsilon_k\in R^{-1}(e_k)$, the non-linear equation becomes

$$(R(w)-R(-z))\mathcal{G}^{(0)}(z,w)=1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{\mathcal{G}^{(0)}(\varepsilon_k,w)}{R(\varepsilon_k)-R(z)}$$

Theorem [Schürmann, W 19]

1. $R(z)=z-\frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{\varepsilon_k+z}$ where $R(\varepsilon_k)=E_k$ and $R'(\varepsilon_k)\varrho_k=r_k$.
2. $\mathcal{G}^{(0)}(z,w)=\dfrac{P(R(z),R(w))}{(R(z)-R(-w))(R(w)-R(-z))}$ where

$$P(R(z),R(w))=\frac{\prod_{u\in R^{-1}(\{w\})}(R(z)-R(-u))}{\prod_{k=1}^{d}(R(z)-R(\varepsilon_k))}\equiv P(R(w),R(z))$$

Auxiliary functions [Branahl, Hock, W 20]

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Recall the complexified Dyson-Schwinger equation

$$(\zeta+\eta)G(\zeta,\eta) = 1 + \frac{\lambda}{N}\sum_{k=1}^{N}\frac{G(E_k,\eta)-G(\zeta,\eta)}{E_k-\zeta} + \frac{\lambda}{\mathcal{N}^2}\frac{G(\eta|\eta)-G(\zeta|\eta)}{\eta-\zeta}$$

$$- G(\zeta,\eta)\Big(\frac{\lambda}{N}\sum_{k=1}^{N}G(\zeta,E_k) + \frac{\lambda}{N^2}G(\zeta|\zeta)\Big) - \frac{\lambda}{N^2}T(\zeta\|\zeta,\eta|)$$

where $T(E_a\|\zeta,\eta|) := -N\frac{\partial}{\partial E_a}G(\zeta,\eta)$. To get an equation for $T(\xi\|\zeta,\eta|)$ we differentiate again, and so on. Let $I=(\zeta_1,..,\zeta_n)$ and $T(\emptyset\|\xi,\eta|)=G(\xi,\eta)$, $T(\emptyset\|\xi|\eta|)=G(\xi|\eta)$.

Definition

- $T(\zeta,I\|\xi,\eta|)$ is the complexification of $T(E_a,I\|\xi,\eta) := -N\frac{\partial}{\partial E_a}T(I\|\xi,\eta)$
- $T(\zeta,I\|\xi|\eta|)$ is the complexification of $T(E_a,I\|\xi|\eta|) := -N\frac{\partial}{\partial E_a}T(I\|\xi|\eta)$
- $\tilde{\Omega}_1(\zeta) := \frac{\lambda}{N}\sum_{k=1}^{N}G(\zeta,E_k) + \frac{\lambda}{N^2}G(\zeta|\zeta)$
- $\tilde{\Omega}_{n+1}(\zeta,I)$ is the complexification of $\tilde{\Omega}_{n+1}(E_a,I) := -N\frac{\partial}{\partial E_a}\tilde{\Omega}_n(I) + \frac{\delta_{n,1}}{(E_a-\zeta_1)^2}$

System of Dyson-Schwinger equations I

Apply the change of variables via R encoded in the 2-point function:

$$\tilde{\Omega}_n(R(z_1), .., R(z_n)) =: \Omega_n(z_1, ..., z_n),$$
$$T(R(z_1), .., R(z_n)\|R(w_1), R(w_2)|) =: \mathcal{T}(z_1, ..., z_n\|w_1, w_2|),$$
$$T(R(z_1), .., R(z_n)\|R(w_1)|R(w_2)|) =: \mathcal{T}(z_1, ..., z_n\|w_1|w_2|)$$

and the formal genus expansion $\Omega_n(I) = \sum_{g=0}^{\infty} N^{-2g}\Omega_n^{(g)}(I)$ etc.

Equation (I)

$$(R(w) - R(-z))\mathcal{T}^{(g)}(I\|z, w|) - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{r_k \mathcal{T}^{(g)}(I\|\varepsilon_k, w|)}{R(\varepsilon_k) - R(z)}$$
$$= \delta_{0,m}\delta_{g,0} - \lambda\Big\{ \sum_{\substack{I_1 \uplus I_2 = I,\; g_1+g_2=g \\ (g_1, I_1) \neq (0,\emptyset)}} \Omega^{(g_1)}_{|I_1|+1}(I_1, z)\mathcal{T}^{(g_2)}(I_2\|z, w|) + \mathcal{T}^{(g-1)}(I, z\|z, w|)$$
$$+ \sum_{i=1}^{m} \frac{\partial}{\partial R(u_i)} \frac{\mathcal{T}^{(g)}(I \setminus u_i\|u_i, w|)}{R(u_i) - R(z)} + \frac{\mathcal{T}^{(g-1)}(I\|z|w|) - \mathcal{T}^{(g-1)}(I\|w|w|)}{R(w) - R(z)}\Big\}$$

System of Dyson-Schwinger equations II

Equation (II)

$$
\begin{aligned}
&(R(z)-R(-z))\mathcal{T}^{(g)}(I\|z|w|) - \frac{\lambda}{N}\sum_{k=1}^{d} r_k \frac{\mathcal{T}^{(g)}(I\|\varepsilon_k|w|)}{R(\varepsilon_k)-R(z)} \\
&= -\lambda\Big\{ \sum_{\substack{I_1\uplus I_2=I,\ g_1+g_2=g \\ (I_1,g_1)\neq(\emptyset,0)}} \Omega^{(g_1)}_{|I_1|+1}(I_1,z)\mathcal{T}^{(g_2)}(I_2\|z|w|) + \mathcal{T}^{(g-1)}(I,z\|z|w|) \\
&\quad + \sum_{i=1}^{m} \frac{\partial}{\partial R(u_i)} \frac{\mathcal{T}^{(g)}(I\setminus u_i\|u_i|w|)}{R(u_i)-R(z)} + \frac{\mathcal{T}^{(g)}(I\|z,w|) - \mathcal{T}^{(g)}(I\|w,w|)}{R(w)-R(z)} \Big\}
\end{aligned}
$$

System of Dyson-Schwinger equations III

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Equation (III)

$$R'(z)\mathfrak{G}_0(z)\Omega^{(g)}_{|I|+1}(I,z) - \frac{\lambda}{N^2}\sum_{n,k=1}^{d} r_n r_k \frac{\mathcal{T}^{(g)}(I\|\varepsilon_k,\varepsilon_n|)}{(R(\varepsilon_k)-R(z))(R(\varepsilon_n)-R(-z))}$$

$$= \frac{\delta_{g,0}\delta_{|I|,1}}{(R(z)-R(u_1))^2} - \sum_{\substack{I_1\uplus I_2=I,\ g_1+g_2=g\\ (I_1,g_1)\neq(\emptyset,0)\neq(I_2,g_2)}} \Omega^{(g_1)}_{|I_1|+1}(I_1,z)\frac{\lambda}{N}\sum_{n=1}^{d} r_n \frac{\mathcal{T}^{(g_2)}(I_2\|z,\varepsilon_n|)}{R(\varepsilon_n)-R(-z)}$$

$$-\sum_{j=1}^{m}\frac{\partial}{\partial R(u_j)}\frac{\frac{\lambda}{N}\sum_{n=1}^{d} r_n \frac{\mathcal{T}^{(g)}(I\backslash u_j\|u_j,\varepsilon_n|)}{R(\varepsilon_n)-R(-z)}}{R(u_j)-R(z)} - \frac{\lambda}{N}\sum_{n=1}^{d} r_n \frac{\mathcal{T}^{(g-1)}(I,z\|z,\varepsilon_n|)}{R(\varepsilon_n)-R(-z)} + \mathcal{T}^{(g-1)}(I\|z|z|)$$

$$-\frac{\lambda}{N}\sum_{n=1}^{d} r_n \frac{\mathcal{T}^{(g-1)}(I\|z|\varepsilon_n|)-\mathcal{T}^{(g-1)}(I\|\varepsilon_n|\varepsilon_n|)}{(R(\varepsilon_n)-R(z))(R(\varepsilon_n)-R(-z))} - \sum_{j=1}^{m}\frac{\partial}{\partial R(u_j)}\mathcal{T}^{(g)}(I\setminus u_j\|u_j,z|)\,,$$

where $\mathfrak{G}_0(z) := \mathrm{Res}_{v\to -z}\,\mathcal{G}^{(0)}(z,v)dv$.

Dyson-Schwinger equation for $\Omega_2^{(0)}(u,z)$

$$\Omega_2^{(0)}(u,z)R'(z)\mathfrak{G}_0(z) - \frac{\lambda}{N^2}\sum_{n,k=1}^{d} \frac{r_k r_n \mathcal{T}^{(0)}(u\|\varepsilon_k,\varepsilon_n|)}{(R(\varepsilon_k)-R(z))(R(\varepsilon_n)-R(-z))}$$
$$= -\frac{\partial}{\partial R(u)}\big(\mathcal{G}^{(0)}(u,z)+\mathcal{G}^{(0)}(u,-z)\big)$$

- Seems to need $\mathcal{T}^{(0)}(u\|\varepsilon_k,\varepsilon_n|)$ which itself needs $\Omega_2^{(0)}$.
- But poles separate by partial fraction decomposition

$$\mathcal{G}^{(0)}(z,u) = \frac{\mathfrak{G}_0(z)}{u+z} + \frac{\lambda^2}{N^2}\sum_{k,l,m,n=1}^{d}\frac{C_{k,l}^{m,n}}{(z+\widehat{\varepsilon}_l{}^n)(z-\widehat{\varepsilon}_k{}^m)(u-\widehat{\varepsilon}_l{}^n)}$$

Proposition

$$\Omega_2^{(0)}(u,z) = \frac{1}{R'(u)R'(z)}\Big(\frac{1}{(u-z)^2}+\frac{1}{(u+z)^2}\Big)$$

One recognises the Bergman kernel of topological recursion!

Contact with topological recursion

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Set $\omega_{g,m}(z_1,\dots,z_m) = \lambda^{2-2g-m}\Omega^{(g)}_m(z_1,\dots,z_m)\prod_{k=1}^m dR(z_i)$. A lengthy calculation gives:

$$\omega_{0,3}(u_1,u_2,z) = -\sum_{i=1}^{2d} \frac{\Big(\frac{1}{(u_1-\beta_i)^2}+\frac{1}{(u_1+\beta_i)^2}\Big)\Big(\frac{1}{(u_2-\beta_i)^2}+\frac{1}{(u_2+\beta_i)^2}\Big)du_1\,du_2\,dz}{R'(-\beta_i)R''(\beta_i)(z-\beta_i)^2}$$
$$+\Big[d_{u_1}\Big(\frac{\omega_{0,2}(u_2,u_1)}{(dR)(u_1)}\frac{dz}{R'(-u_1)(z+u_1)^2}\Big)+u_1\leftrightarrow u_2\Big]$$

$$\omega_{1,1}(z) = \sum_{i=1}^{2d}\frac{dz}{R'(-\beta_i)R''(\beta_i)}\Big\{-\frac{1}{8(z-\beta_i)^4}+\frac{\frac{1}{24}x_{1,i}}{(z-\beta_i)^3}+\frac{(x_{2,i}+y_{2,i}-x_{1,i}y_{1,i}-x_{1,i}^2-\frac{6}{\beta_i^2})}{48(z-\beta_i)^2}$$
$$-\frac{dz}{8(R'(0))^2z^3}+\frac{R''(0)dz}{16(R'(0))^3z^2}$$

where $\beta_{1,\dots,2d}$ solve $dR(\beta_i)=0$ (ramification pnts), $x_{n,i} := \frac{R^{(n+2)}(\beta_i)}{R''(\beta_i)}$, $y_{n,i} := \frac{(-1)^nR^{(n+1)}(-\beta_i)}{R'(-\beta_i)}$

Observation

The blue terms correspond to topological recursion for $x(z)=R(z)$ & $y(z)=-R(-z)$, the magenta terms signal an extension to blobbed topological recursion [Borot-Shadrin 15].

Conjecture: NC $\lambda\Phi^4$-model obeys BTR!

When trying to prove the conjecture for $g = 0$ we noticed surprising identities between $\omega_{0,m+1}(u_1, \ldots, u_m, -z)$ and $\omega_{0,k+1}(u_1, \ldots, u_k, z)$. They were generalised in [Hock 22] to a general approach to the x-y symmetry in TR.

Definition

Let $x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a ramified covering with ramification points $\beta_1, \ldots, \beta_r$. For a global involution $\iota : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, which neither fixes nor permutes the β_i, let $y(z) := -x(\iota z)$. Then a family $\{\omega_{0,n}\}_{n\geq 2}$ of meromorphic differentials is introduced by

$$\omega_{0,2}(w,z) = \frac{1}{2}\frac{dw\,dz}{(w-z)^2} + \frac{1}{2}\frac{d(\iota w)\,d(\iota z)}{(\iota w - \iota z)^2} - \frac{1}{2}\frac{dw\,d(\iota z)}{(w-\iota z)^2} - \frac{1}{2}\frac{d(\iota w)\,dz}{(\iota w - z)^2}$$

and for $m \geq 2$ by the involution identity (here $I := \{u_1, \ldots, u_m\}$)

$$\omega_{0,m+1}(I,z) + \omega_{0,m+1}(I,\iota z) = \sum_{s=2}^{m} \sum_{I_1 \uplus \ldots \uplus I_s = I} \frac{1}{s} \operatorname*{Res}_{w\to z} \left(\frac{dy(z)dx(w)}{(y(z)-y(w))^s} \prod_{i=1}^{s} \frac{\omega_{0,|I_i|+1}(I_i, w)}{dx(w)} \right).$$

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Theorem [Hock-W 21]

The involution identity has the unique solution

$$\omega_{0,m+1}(I,z)=\sum_{i=1}^{r}\operatorname*{Res}_{q\to\beta_i}K_i(z,q)\sum_{I_1\uplus I_2=I}\omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,\sigma_i(q))$$
$$-\sum_{k=1}^{m}d_{u_k}\Big[\operatorname*{Res}_{q\to\iota u_k}\sum_{I_1\uplus I_2=I}\tilde{K}(z,q,u_k)d_{u_k}^{-1}\big(\omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,q)\big)\Big].$$

(for $m\geq 2$). Here σ_i is the local Galois involution near β_i, i.e. $x(z)=x(\sigma_i(z))$, $\sigma_i(\beta_i)=\beta_i$, $\sigma_i\neq\mathrm{id}$. The recursion kernels are given by

$$K_i(z,q):=\frac{\frac{1}{2}\big(\frac{dz}{z-q}-\frac{dz}{z-\sigma_i(q)}\big)}{dx(\sigma_i(q))(y(q)-y(\sigma_i(q)))}\,,\qquad \tilde{K}(z,q,u):=\frac{\frac{1}{2}\big(\frac{d(\iota z)}{\iota z-\iota q}-\frac{d(\iota z)}{\iota z-u}\big)}{dx(q)(y(q)-y(\iota u))}.$$

For the choice $\iota z=-z$ and $x(z)=R(z):=z-\frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{\varepsilon_k+z}$, the solution of the involution identity coincides with the solution of the system for $(\Omega_n^{(0)},\mathcal{T}^{(0)})$ found in [BHW 20].

Linear and quadratic loop equations

The best strategy seems to prove that the Dyson-Schwinger equations imply linear and quadratic loop equations [Borot, Eynard, Orantin 13]

$$\sum_{j=0}^{d} \omega_{g,|I|+1}(I, \hat{z}^j) = f_1(R(z); I)$$

$$\sum_{\substack{j,k=0 \\ j \neq k}}^{d} \Big(\omega_{g-1,|I|+2}(I, \hat{z}^j, \hat{z}^k) + \sum_{\substack{g_1+g_2=g \\ I_1 \uplus I_2 = I}} \omega_{g_1,|I_1|+1}(I_1, \hat{z}^j)\omega_{g_2,|I_2|+1}(I_2, \hat{z}^k) \Big) = f_2(R(z); I)$$

where f_1, f_2 are *holomorphic* at ramification points of R and $R^{-1}(\{R(z)\}) = \{\hat{z}^0 \equiv z, \hat{z}^1, ..., \hat{z}^d\}$.

- Blobbed TR is the general solution of such loop equations [Borot, Shadrin 15]. To reduce to TR one needs more.
- In our case f_1, f_2 are of particular structure which completely fixes the poles at $z = -u_k$ and $z = 0$.
- Work in progress: extension to higher g (kernel K_0 for pole at $z = 0$ already found).

Outlook I: Intersection numbers

Eynard proved in 2011 a general formula which expresses $\omega_{g,n}$ for *any* spectral curve $(x : \Sigma \to \Sigma_0, \omega_{0,1}, \omega_{0,2})$ with simple ramification points in terms of intersection numbers of ψ- and κ-classes on several copies of $\overline{\mathcal{M}}_{g,n}$.

- Extensions to higher-order ramifications are known.
- According to [Borot, Shadrin 15], Eynard's formula survives with some modifications to blobbed TR.

Thus, expressing our $\omega_{g,n}$ in terms of intersection numbers is an achievable goal in very near future.

- This expression can be interesting (like the ELSV formula) or not.
- We hope it captures aspects of the involution $z \mapsto -z$ which plays a decisive rôle in the residue formula.

Outlook II: Integrability

Consider $\tau(\{t_i\}) = N^2\mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \sum_{g=2}^{\infty} N^{2-2g}\omega_{g,0}$

- In the [Kontsevich 92] model, for $t_i = -\frac{(2i-1)!!}{N}\mathrm{Tr}(E^{-2i-1})$, τ satisfies the Hirota bilinear PDE of the KdV-hierarchy.
- [Eynard, Orantin 07] describe how to obtain from any spectral curve of TR a formal Hirota equation (order by order in $1/N^2$).
- Whether or not integrability extends to blobbed TR is not known.
- We remain optimistic for our case because the recursion formula with residue kernel is very close to TR.
- $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(1)}$ [Branahl, Hock 21] have been found.