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The Euclidean $\lambda\phi_4^4$-model on noncommutative geometries: a status report

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CRC 1442 GEOMETRY: DEFORMATIONS AND RIGIDITY

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- [Landau, Abrikosov, Khalatnikov 54] gave a heuristic argument that 4D quantum electrodynamics cannot exist as a renormalised QFT.

  The problem is called Landau ghost, triviality, positivity of β-function.
- This was considered as death of QFT, rescued only by the discovery of asymptotic freedom in non-Abelian Yang-Mills theory by Gross, Politzer, Wilczek in 1973.
- [Aizenman 81] and [Fröhlich 82] gave a rigorous proof of triviality for the $\lambda\Phi^4$-model in $D = 4 + \epsilon$ dimensions.
- Almost 40 years later, [Aizenman, Duminil-Copin 19] proved triviality of $\lambda\Phi^4$ in $D = 4$.

The situation of 4D QFT is discouraging

The only interacting model which seems to exist is non-Abelian Yang-Mills theory, but the proof is one of the Millenium Prize problems!

To get a little insight we relax a key requirement on space(-time) geometry.

Noncommutative geometry

... is a vast generalisation of topology and differential geometry of manifolds, based on operator algebras. Noncommutative geometry gives up the idea of a point in favour of states. Symmetries are very different.

In the 80s-90s, the question was brought up whether physics and quantum fields can reasonably be defined on noncommutative geometries. Here are some early milestones:

- [Bellissard 86] integer quantum Hall effect in terms of K-theory
- [Connes, Rieffel 87] Yang-Mills for non-commutative two-tori
- [Dubois-Violette, Kerner, Madore 90] NC differential geometry of matrix algebras
- [Connes 90] NCG on two points gives Higgs potential; details in [Connes, Lott 91]
- [Grosse, Madore 92] Schwinger model on fuzzy sphere
- [Doplicher, Fredenhagen, Roberts 95] quantum spacetime
- [Filk 96] deformed Feynman rules

Measures

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We focus here on the Euclidean approach in which candidates for Schwinger functions are moments and cumulants of a measure on a space $\mathcal{V}'$ of distributions.
As usual, we intend to deform a Gaußian measure by an interaction functional.

- By Bochner-Minlos-Schur, the Gaußian measure results from an inner product on a real nuclear vector space $\mathcal{V}$:
$$\int_{\mathcal{V}'} d\mu_0(\Phi)\, e^{\mathrm{i}\Phi(f)} := \exp\Big(-\frac{1}{2}\langle f, f\rangle\Big)$$
- The substitute for locality, in the sense of point-like interactions, is that the interaction is the trace of a polynomial P in the operator algebra. We would like to give a meaning to
$$d\mu(\Phi)\text{``:=''}\frac{1}{\mathcal{Z}} d\mu_0(\Phi)\; \exp(-\mathrm{Tr}(P(\Phi)))\;.$$
- Since the product of distributions is not defined, we meet various divergences of QFT.

We sketch how the construction succeeds for $P(\Phi)=\frac{\lambda}{4}\Phi^4$. Although it is still far away from a QFT in 4D, we learn a lot.

Approximation by matrix algebras

To make sense of a product of distributions we need to pass to finite-dimensional approximations of the problem, together with a careful limiting procedure.

Finite-dimensional approximations of noncommutative algebras are matrix algebras. A good choice is to take $\mathcal{V}$ as space of selfadjoint matrices of infinite size and rapidly decaying entries.

Setting $\Phi(f_1 + \mathrm{i} f_2) := \Phi(f_1) + \mathrm{i}\Phi(f_2)$ we can define $\Phi_{kl} := \Phi(e_{kl})$, where (e_{kl}) is the standard matrix basis. Then

$$\mathrm{Tr}_\Lambda(\Phi^4) = \mathcal{N} \sum_{k,l,m,n=0}^{\Lambda^D \mathcal{N}} \Phi_{kl}\Phi_{lm}\Phi_{mn}\Phi_{nk} \,, \qquad d\mu_\lambda^\Lambda(\Phi) := \frac{\exp(-\frac{\lambda}{4}\mathrm{Tr}_\Lambda(\Phi^4))}{\mathcal{Z}_\Lambda}\, d\mu_0(\Phi)$$

where $\mathcal{N}$ is the 'noncommutativity/Planck volume', D a dimension identified later and Λ a cut-off. The goal is to construct renormalised moments in the limit $\Lambda \to \infty$,

$$\langle f_1 .... f_n \rangle := \lim_{\Lambda\to\infty} Z^{-\frac{n}{2}}(\Lambda) \int_{\mathcal{V}'} d\mu_\lambda^\Lambda(\Phi)\, \Phi(f_1)\cdots\Phi(f_n) \,,$$

for an appropriate wavefunction renormalisation $Z(\Lambda)$.

The Gaußian

Here we decisively restrict the generality and take the measure $d\mu_0(\Phi)$ for the inner product

$$\langle f, g\rangle = \frac{1}{\mathcal{N}} \sum_{k,l=0}^{\Lambda^D \mathcal{N}} \frac{f_{kl} g_{lk}}{E_k + E_l + M^2(\Lambda)}$$

where $M^2(\Lambda)$ is a mass renormalisation and $E_k > 0$ the spectral values of a Laplacian on NCG of spectral dimension

$$d_{spec} = \inf\Big\{p \;:\; \sum_{k=0}^{\infty} (1+E_k)^{-p/2} < \infty\Big\}.$$

- ex: 2D Moyal space has $(E_k) = \frac{1}{\mathcal{N}}(0,1,2,3,4,\dots)$ and $d_{spec} = 2$.
- ex: 4D Moyal space has $(E_k) = \frac{1}{\sqrt{\mathcal{N}}}(0,1,1,2,2,2,3,3,3,3,\dots)$ and $d_{spec} = 4$.

Intermediate steps need E_k pairwise different. Renormalisation depends on $D := 2[\frac{d_{spec}}{2}]$.

Of particular interest are the two-point functions

$$G_{|ab|} = \frac{\mathcal{N}}{Z(\Lambda)} \int_{\mathcal{V}'} d\mu_\lambda^\Lambda(\Phi)\; \Phi(e_{ab})\Phi(e_{ba})\,, \qquad G_{|a|b|} = \frac{\mathcal{N}^2}{Z(\Lambda)} \int_{\mathcal{V}'} d\mu_\lambda^\Lambda(\Phi)\; \Phi(e_{aa})\Phi(e_{bb})\,.$$

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Lemma [Schürmann, W 19]

The Fourier transform $\mathcal{Z}(f) := \int_{\mathcal{V}'} d\mu_\lambda(\Phi)\, e^{\mathrm{i}\Phi(f)}$ of the measure satisfies

1. $$\frac{1}{\mathrm{i}}\frac{\partial\mathcal{Z}(f)}{\partial f_{ab}} = \frac{\mathrm{i} f_{ba}\mathcal{Z}(f)}{\mathcal{N}(E_a+E_b+M^2)} - \frac{\lambda}{\mathrm{i}^3(E_a+E_b+M^2)}\sum_{k,l=0}^{\Lambda^D\mathcal{N}}\frac{\partial^3\mathcal{Z}(f)}{\partial f_{ak}\partial f_{kl}\partial f_{lb}}$$

2. $$\frac{1}{\mathcal{N}}\frac{\partial\mathcal{Z}(f)}{\partial E_a} = \sum_{k=0}^{\Lambda^D\mathcal{N}}\frac{\partial^2\mathcal{Z}(f)}{\partial f_{ak}\partial f_{ka}} + \Big(\frac{1}{\mathcal{N}}\sum_{k=0}^{\Lambda^D\mathcal{N}} ZG_{|ak|} + \frac{1}{\mathcal{N}^2} ZG_{|a|a|}\Big)\mathcal{Z}(f)$$

Corollary (Ward-Takahashi identity [Disertori, Gurau, Magnen, Rivasseau 06])

3. $$-\mathcal{N}\sum_{k=0}^{\Lambda^D\mathcal{N}}(E_a-E_b)\frac{\partial^2\mathcal{Z}(f)}{\partial f_{ak}\partial f_{kb}} = \sum_{k=0}^{\Lambda^D\mathcal{N}}\Big(f_{ka}\frac{\partial\mathcal{Z}(f)}{\partial f_{kb}} - f_{bk}\frac{\partial\mathcal{Z}(f)}{\partial f_{ak}}\Big)$$

Dyson-Schwinger equations

The equation of motion ❶ induces Dyson-Schwinger equations between moments. Thereby an n-point function is expressed in terms of $n+2$-point functions. Using ❸ and ❷ one can avoid this. For the 2-point function one finds

$$(E_a+E_b)ZG_{|ab|} = 1+\frac{\lambda}{\mathcal{N}}\sum_{\substack{k=0\\k\neq a}}^{\Lambda^D\mathcal{N}}\frac{ZG_{|kb|}-ZG_{|ab|}}{E_k-E_a}+\frac{\lambda}{\mathcal{N}^2}\frac{ZG_{|b|b|}-ZG_{|a|b|}}{E_b-E_a}$$

$$-ZG_{|ab|}\Big(\frac{\lambda}{\mathcal{N}}\sum_{k=0}^{\Lambda^D\mathcal{N}}ZG_{|ak|}+\frac{\lambda}{\mathcal{N}^2}ZG_{|a|a|}\Big)+\frac{Z\lambda}{\mathcal{N}}\frac{\partial G_{|ab|}}{\partial E_a}$$

Complexification

- The dependence of $G_{|ab|}$ on the matrix indices a, b is of the form of an evaluation $G_{|ab|} = G(E_a, E_b)$ of a function G of two complex variables at E_a, E_b.
- $G(\zeta, \eta)$ still depends on summation variables E_k. Differentiating wrt some E_c is also an evaluation at E_c of another function $-\mathcal{N}\frac{\partial}{\partial E_c} G(\zeta,\eta) = T(E_c\|\zeta,\eta|)$.
- Similarly for $G_{|a|b|} = G(E_a|E_b)$.

Complexified Dyson-Schwinger equation

$$(\zeta+\eta+M^2)ZG(\zeta,\eta) = 1 + \frac{\lambda}{\mathcal{N}}\sum_{k=0}^{\Lambda^D\mathcal{N}} \frac{ZG(E_k,\eta)-ZG(\zeta,\eta)}{E_k-\zeta} + \frac{\lambda}{\mathcal{N}^2}\frac{ZG(\eta|\eta)-ZG(\zeta|\eta)}{\eta-\zeta}$$
$$- ZG(\zeta,\eta)\Big(\frac{\lambda}{\mathcal{N}}\sum_{k=0}^{\Lambda^D\mathcal{N}} ZG(\zeta,E_k) + \frac{\lambda}{\mathcal{N}^2}ZG(\zeta|\zeta)\Big) - \frac{\lambda}{\mathcal{N}^2}ZT(\zeta\|\zeta,\eta|)$$

The genus expansion

We approach the solution of such equations in a formal genus expansion

$$G(\zeta,\eta) = \sum_{g=0}^{\infty} \mathcal{N}^{-2g} G^{(g)}(\zeta,\eta)\,, \qquad G(\zeta|\eta) = \sum_{g=0}^{\infty} \mathcal{N}^{-2g} G^{(g)}(\zeta|\eta)\,,$$

$$T(\xi\|\zeta,\eta|) = \sum_{g=0}^{\infty} \mathcal{N}^{-2g} T^{(g)}(\xi\|\zeta,\eta|)$$

together with the convention that $\frac{1}{\mathcal{N}}$ in front of a summation is neutral.

These series have zero radius of convergence! Making sense of them via Borel resummation is a main challenge for the future. Connects to resurgence.

Theorem [Grosse, W 09]

The planar two-point function satisfies the closed non-linear equation

$$\Big(\zeta + \eta + M^2 + \frac{\lambda}{\mathcal{N}} \sum_{k=0}^{\Lambda^D \mathcal{N}} ZG^{(0)}(\zeta, E_k)\Big) ZG^{(0)}(\zeta,\eta) = 1 + \frac{\lambda}{\mathcal{N}} \sum_{k=0}^{\Lambda^D \mathcal{N}} \frac{ZG^{(0)}(E_k,\eta) - ZG^{(0)}(\zeta,\eta)}{E_k - \zeta}$$

Integral equation

We can arrange the two-point function of a large family of matrix and QFT models with quartic interaction into the integral equation

$$\Big(\zeta+\eta+M^2+\lambda\int_0^\infty dt\,\varrho_0(t) ZG^{(0)}(\zeta,t)\Big) ZG^{(0)}(\zeta,\eta) = 1+\lambda\int_0^\infty dt\,\varrho_0(t)\frac{Z(G^{(0)}(t,\eta)-G^{(0)}(\zeta,\eta))}{t-\zeta}$$

- large-θ 4D Moyal: $\varrho_0(t)=t\chi_{[0,\Lambda^2]}(t)$ and $M=M(\Lambda)$, $Z=Z(\Lambda)$
- large-θ 2D Moyal: $\varrho_0(t)=\chi_{[0,\Lambda^2]}(t)$ and $M=M(\Lambda)$, $Z=1$
- $N\times N$ matrix model $\varrho(t)=\frac{1}{N}\sum_{k=1}^d r_k\delta(t-e_k)$, $r_1+\ldots+r_d=N$, $M=0$, $Z=1$

The spectral dimension becomes $d_{spec}:=\inf(p\;:\;\int_0^\infty\frac{\varrho_0(t)}{(1+t)^{p/2}}<\infty)$

- In [Panzer, W 18] we solved the case $\varrho_0(t)=1$. Key step was to extrapolate a computer algebra evaluation of iterated integrals.
- In [Grosse, Hock, W 19a] we succeeded in solving the integral equation for any Hölder-continuous measure ϱ_0 of spectral dimension $d_{spec}<6$.

Solution

Theorem [Panzer-W 18 for $\varrho_0 = 1$, Grosse-Hock-W 19a]

1. Ansatz $G^{(0)}(\zeta,\eta) = \dfrac{e^{\mathcal{H}_\zeta[\tau_\eta(\bullet)]}\sin\tau_\eta(\zeta)}{Z\lambda\pi\varrho_0(\zeta)}$, $\quad \mathcal{H}_\zeta[f] := \frac{1}{\pi}\,⨍_0^{\Lambda^2} \frac{dp\, f(p)}{p-\zeta}$ finite Hilbert transf.
2. $\tau_\eta(\zeta) = \operatorname{Im}\log\big(\eta + I(\zeta + \mathrm{i}\epsilon)\big)$ with $I(\zeta) = -R_D(-m^2 - R_D^{-1}(\zeta))$
3. $$\boxed{R_D(z) = z - \lambda(-z)^{D/2}\int_0^\infty \frac{dt\;\varrho_\lambda(t)}{(m^2+t)^{D/2}(t+m^2+z)}} \qquad D = 2[\tfrac{\delta}{2}]$$
4. ϱ_λ is implicit solution of $\varrho_0(R_D(\zeta)) = \varrho_\lambda(\zeta)$.

- Proof: [Cauchy 1831] residue theorem, [Lagrange 1770] inversion theorem, [Bürmann 1799] formula
- $\varrho_0(t) \equiv 1$ (2D Moyal, $m = 1$) in terms of Lambert-W satisfying $W(z)e^{W(z)} = z$:
  $I(\zeta) := \lambda W_0\big(\frac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\big) - \lambda\log\big(1 - \lambda W_0\big(\frac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\big)\big)$

$D = 4$ Moyal space: $\varrho_0(t) = t$ [Grosse-Hock-W 19b]

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- $\varrho_\lambda(x) \equiv \varrho_0(R_4(x)) = R_4(x) = x - \lambda x^2 \int_0^\infty \frac{dt\; \varrho_\lambda(t)}{(m^2+t)^2(t+x)}$
- If $\varrho_\lambda(t) \sim \varrho_0(t) = t$, then $R_4(x)$ bounded above. Consequently, R_4^{-1} would not be globally defined: triviality!
- Fredholm equation perturbatively solved by iterated integrals: Hyperlogarithms and $\zeta(2n)$ which can be summed to

$$R_4(x) \equiv \varrho_\lambda(x) = x \cdot {}_2F_1\left(\begin{matrix}\alpha_\lambda, 1-\alpha_\lambda \\ 2\end{matrix}\middle| -\frac{x}{m^2}\right) \qquad \alpha_\lambda = \begin{cases} \dfrac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda| \leq \frac{1}{\pi} \\ \frac{1}{2} + \mathrm{i}\dfrac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{for } \lambda \geq \frac{1}{\pi} \end{cases}$$

Corollary

The interaction alters the spectral dimension to $4 - 2\frac{\arcsin(\lambda\pi)}{\pi}$ and thus avoids the triviality problem (in the planar sector).

Gives non-perturbative integral representation for $G^{(0)}(\xi, \eta)$.

How solid is the non-triviality result?

Recall we employed the formal power series $G_{|ab|} = \sum_{g=0}^{\infty} \mathcal{N}^{-2g} G^{(g)}_{|ab|}$.

- There are techniques in probability theory (contentration of measures) which combined with Dyson-Schwinger equations [Guionnet] should allow to prove asymptotic estimates $\big|G_{|ab|} - G^{(0)}_{|ab|}\big| < \frac{const_{a,b}}{\mathcal{N}^2}$.
- Such a non-triviality of the noncommutative QFT says nothing about our commutative world, which corresponds to $\mathcal{N} \to 0$.

Being in a mathematics environment, I am currently more interested in another direction:

- . . . to understand the higher genus contributions $G^{(g\geq 1)}_{|ab|}$ and their connection to algebraic geometry (intersection theory on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves) and enumerative geometry.
- Work in progress for finite $N \times N$-matrices. A dream would be to establish Borel summability of $\sum_{g=0}^{\infty} N^{-2g} G^{(g)}_{|ab|}$.

Some further discussion

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1. We show on the next slides that (parts of) higher genus contributions are given by localisation at critical points of $R_0(z)$. These are incompatible with a limit $N = \Lambda^D \mathcal{N} \to \infty$.

   Cf. #(zeros) of $\sum_{k=0}^{N} \frac{z^k}{k!}$ and of $\exp(z)$

2. [Glimm, Jaffe 87] formulate Osterwalder-Schrader axioms for Fourier transform $\mathcal{Z}(f) = \int_{\mathcal{V}'} d\mu(\Phi)\, e^{\mathrm{i}\Phi(f)}$. We have only access to renormalised moments; the measure must be constructed by solving a moment problem.

   Statements about axioms are not in sight.

The Kontsevich model

Consider the Gaußian measure $d\mu_0$ on $\mathcal{V}'$, for $\mathcal{V} = H_N$ Hermitean $N \times N$-matrices, induced by $\langle f, g \rangle = \frac{1}{N} \sum_{k,l=1}^{N} \frac{f_{kl} g_{lk}}{E_k + E_l}$.

- Deforming it to $d\mu_K(\Phi) := \frac{1}{\mathcal{Z}} e^{\frac{\mathrm{i}}{6} N \operatorname{Tr}(\Phi^3)} d\mu_0(\Phi)$ gives the [Kontsevich 92] model which generates intersection numbers of ψ- and κ-classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves. Proves a conjecture by [Witten 90].
- This is a key example for topological recursion [Eynard, Orantin 07]. Its spectral curve is $x(z) = z^2$, $y(z) = -z + \frac{1}{N} \sum_{k=1}^{N} \frac{1}{\hat{E}_k(\hat{E}_k - z)}$ and $\omega_{0,2}(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$ where $\hat{E}_k = \sqrt{E_k^2 + c}$ and $c = \frac{2}{N} \sum_{k=1}^{N} \frac{1}{\sqrt{E_k^2 + c}}$.
- One can turn the Kontsevich model into a QFT on Moyal space (or other NCG) [Grosse, Steinacker 05/06], [Grosse, Sato, W 16], [Grosse, Hock, W 19c]. But $d\mu_K$ is not a valid measure.

We will now deform the same $d\mu_0$ of the Kontsevich model by a quartic potential $d\mu_\lambda(\Phi) := \frac{1}{\mathcal{Z}} e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_0(\Phi)$.

- We called this the quartic Kontsevich model. There are another variations of the Kontsevich model, e.g. the family of generalised Kontsevich models which relate to r-spin intersection numbers [Belliard, Charbonnier, Eynard, Garcia-Failde 21].
- One can also take for $\mathcal{V}$ all complex $N \times N$-matrices [Langmann, Szabo, Zarembo 03]. Very recently, [Hock, Branahl 22] proved that the complex quartic model obeys topological recursion, whereas the real model (discussed below) needs blobbed TR. They clearly located the differences responsible for the blobs.

Recall that if $(e_1, ..., e_d)$ are the pairwise different values in (E_k) which arise with multiplicites r_i, then the planar 2-point function of the real quartically deformed model satisfies

$$\Big(\eta+\zeta+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(\zeta,e_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{e_k-\zeta}\Big)G^{(0)}(\zeta,\eta) = 1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{G^{(0)}(e_k,\eta)}{e_k-\zeta}$$

‘Elementary’ solution

Suppose there is a rational function R of degree $d+1$, with simple pole at ∞ of residue -1 and

$$R(z)+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(R(z),e_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{e_k-R(z)}=-R(-z)$$

Setting $\zeta=R(z)$, $\eta=R(w)$ and $G^{(0)}(\zeta,\eta)=\mathcal{G}^{(0)}(z,w)$ and chosing $\varepsilon_k\in R^{-1}(e_k)$, the non-linear equation becomes

$$(R(w)-R(-z))\mathcal{G}^{(0)}(z,w)=1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{\mathcal{G}^{(0)}(\varepsilon_k,w)}{R(\varepsilon_k)-R(z)}$$

Theorem [Schürmann, W 19]

1. $R(z)=z-\frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{\varepsilon_k+z}$ where $R(\varepsilon_k)=E_k$ and $R'(\varepsilon_k)\varrho_k=r_k$.
2. $\mathcal{G}^{(0)}(z,w)=\dfrac{P(R(z),R(w))}{(R(z)-R(-w))(R(w)-R(-z))}$ where

$$P(R(z),R(w))=\frac{\prod_{u\in R^{-1}(\{w\})}(R(z)-R(-u))}{\prod_{k=1}^{d}(R(z)-R(\varepsilon_k))}\equiv P(R(w),R(z))$$

Higher genus and auxiliary functions [Branahl, Hock, W 20]

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The DSE for 2-point functions contain $T^{(g)}(E_a\|\zeta,\eta|) := -N\frac{\partial}{\partial E_a}G^{(g)}(\zeta,\eta)$ and $T^{(g)}(E_a\|\zeta|\eta|) := -N\frac{\partial}{\partial E_a}G^{(g)}(\zeta|\eta)$. To get equations for them we differentiate again, and so on. Another combination appears:

$$\Omega^{(g)}_{q_1,\dots,q_n} := \frac{(-N)^{n-1}\partial^{n-1}\big(\frac{1}{N}\sum_{k=1}^N G^{(g)}_{|kq_1|} + G^{(g-1)}_{|q_1|q_1|}\big)}{\partial E_{q_2}\cdots\partial E_{q_n}} + \frac{\delta_{g,0}\delta_{n,2}}{(E_{q_1}-E_{q_2})^2}$$

- Complexify all these auxiliary functions and pass to preimages

$$\begin{aligned}\tilde{\Omega}_n(R(z_1),..,R(z_n)) &=: \Omega_n(z_1,\dots,z_n),\\ T(R(z_1),..,R(z_n)\|R(w_1),R(w_2)|) &=: \mathcal{T}(z_1,\dots,z_n\|w_1,w_2|),\\ T(R(z_1),..,R(z_n)\|R(w_1)|R(w_2)|) &=: \mathcal{T}(z_1,\dots,z_n\|w_1|w_2|)\end{aligned}$$

  with $\Omega^{(g)}_n(\varepsilon_{q_1},\dots,\varepsilon_{q_n}) = \Omega^{(g)}_{q_1,\dots,q_n}$ and similarly for the T.
- Pass to meromorphic differentials $\omega_{g,n}(z_1,\dots,z_n) = \lambda^{2-2g-n}\Omega^{(g)}_n(z_1,\dots,z_m)\prod_{k=1}^n dR(z_k)$.

Contact with topological recursion

A lengthy calculation gives $\omega_{0,2}(z_1,z_2) = \frac{dz_1\,dz_2}{(z_1-z_2)^2} + \frac{dz_1\,dz_2}{(z_1+z_2)^2}$ and then

$$\omega_{0,3}(u_1,u_2,z) = -\sum_{i=1}^{2d} \frac{\Big(\frac{1}{(u_1-\beta_i)^2}+\frac{1}{(u_1+\beta_i)^2}\Big)\Big(\frac{1}{(u_2-\beta_i)^2}+\frac{1}{(u_2+\beta_i)^2}\Big)du_1\,du_2\,dz}{R'(-\beta_i)R''(\beta_i)(z-\beta_i)^2}$$
$$+\Big[d_{u_1}\Big(\frac{\omega_{0,2}(u_2,u_1)}{(dR)(u_1)}\frac{dz}{R'(-u_1)(z+u_1)^2}\Big)+u_1\leftrightarrow u_2\Big]$$

$$\omega_{1,1}(z) = \sum_{i=1}^{2d} \frac{dz}{R'(-\beta_i)R''(\beta_i)}\Big\{-\frac{1}{8(z-\beta_i)^4}+\frac{\frac{1}{24}x_{1,i}}{(z-\beta_i)^3}+\frac{(x_{2,i}+y_{2,i}-x_{1,i}y_{1,i}-x_{1,i}^2-\frac{6}{\beta_i^2})}{48(z-\beta_i)^2}$$
$$-\frac{dz}{8(R'(0))^2z^3}+\frac{R''(0)dz}{16(R'(0))^3z^2}$$

where $\beta_{1,\dots,2d}$ solve $dR(\beta_i)=0$ (ramification pnts), $x_{n,i} := \frac{R^{(n+2)}(\beta_i)}{R''(\beta_i)}$, $y_{n,i} := \frac{(-1)^n R^{(n+1)}(-\beta_i)}{R'(-\beta_i)}$

Observation [Branahl, Hock, W 20]

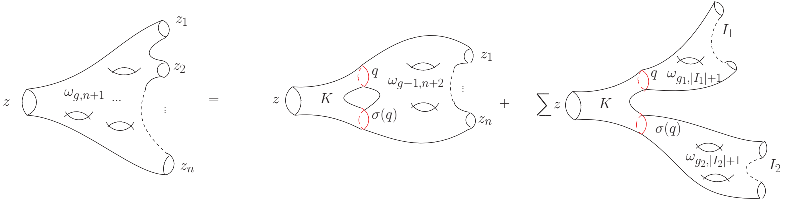
The blue terms correspond to topological recursion for $x(z)=R(z)$ & $y(z)=-R(-z)$, the magenta terms signal an extension to blobbed topological recursion [Borot-Shadrin 15].

Topological recursion [Eynard, Orantin 07]

TR recursively constructs, starting from a spectral curve consisting of

- a ramified covering $x : \Sigma \to \Sigma_0$ of Riemann surfaces,
- meromorphic differentials $\omega_{0,1}{=}ydx$ on Σ and $\omega_{0,2}$ on $\Sigma\times\Sigma$,

a family $\omega_{g,n}$ of meromorphic differentials on Σ^n, with poles at zeros of dx (ramification points), schematically by



- Surprisingly many examples in mathematical physics, algebraic and enumerative geometry follow this TR-construction scheme.
- The $\omega_{g,n}$ generate intersection numbers of ψ- and κ-classes on $\overline{\mathcal{M}}_{g,n}$ [Eynard 11]
- The free energy $\mathcal{F} = N^2\mathcal{F}_0 + \mathcal{F}_1 + \sum_{g=2}^{\infty} N^{2-2g}\omega_{g,0}$ is a τ-function for a Hirota equation

Conjecture: NC $\lambda\Phi^4$-model obeys BTR!

When trying to prove the conjecture for $g = 0$ we noticed surprising identities between $\omega_{0,m+1}(u_1, \dots, u_m, -z)$ and $\omega_{0,k+1}(u_1, \dots, u_k, z)$. They were generalised in [Hock 22] to a general approach to the x-y symmetry in TR.

Definition

Let $x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a ramified covering with ramification points $\beta_1, \dots, \beta_r$. For a global involution $\iota : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, which neither fixes nor permutes the β_i, let $y(z) := -x(\iota z)$. Then a family $\{\omega_{0,n}\}_{n\geq 2}$ of meromorphic differentials is introduced by

$$\omega_{0,2}(w,z) = \frac{1}{2}\frac{dw\,dz}{(w-z)^2} + \frac{1}{2}\frac{d(\iota w)\,d(\iota z)}{(\iota w - \iota z)^2} - \frac{1}{2}\frac{dw\,d(\iota z)}{(w-\iota z)^2} - \frac{1}{2}\frac{d(\iota w)\,dz}{(\iota w - z)^2}$$

and for $m \geq 2$ by the involution identity (here $I := \{u_1, \dots, u_m\}$)

$$\omega_{0,m+1}(I,z) + \omega_{0,m+1}(I,\iota z) = \sum_{s=2}^{m} \sum_{I_1 \uplus \dots \uplus I_s = I} \frac{1}{s} \operatorname*{Res}_{w\to z} \left(\frac{dy(z)dx(w)}{(y(z)-y(w))^s} \prod_{i=1}^{s} \frac{\omega_{0,|I_i|+1}(I_i, w)}{dx(w)} \right).$$

Recursion of the genus-0 sector

Theorem [Hock-W 21]

The involution identity has the unique solution

$$\omega_{0,m+1}(I,z) = \sum_{i=1}^{r} \operatorname*{Res}_{q\to\beta_i} K_i(z,q) \sum_{I_1\uplus I_2=I} \omega_{0,|I_1|+1}(I_1,q)\,\omega_{0,|I_2|+1}(I_2,\sigma_i(q))$$

$$-\sum_{k=1}^{m} d_{u_k}\Big[\operatorname*{Res}_{q\to\iota u_k} \sum_{I_1\uplus I_2=I} \tilde{K}(z,q,u_k)\, d_{u_k}^{-1}\big(\omega_{0,|I_1|+1}(I_1,q)\,\omega_{0,|I_2|+1}(I_2,q)\big)\Big].$$

(for $m\geq 2$). Here σ_i is the local Galois involution near β_i, i.e. $x(z)=x(\sigma_i(z))$, $\sigma_i(\beta_i)=\beta_i$, $\sigma_i\neq \mathrm{id}$. The recursion kernels are given by

$$K_i(z,q) := \frac{\frac{1}{2}\big(\frac{dz}{z-q}-\frac{dz}{z-\sigma_i(q)}\big)}{dx(\sigma_i(q))(y(q)-y(\sigma_i(q)))}\,, \qquad \tilde{K}(z,q,u) := \frac{\frac{1}{2}\big(\frac{d(\iota z)}{\iota z-\iota q}-\frac{d(\iota z)}{\iota z-u}\big)}{dx(q)(y(q)-y(\iota u))}.$$

For the choice $\iota z=-z$ and $x(z)=R(z):=z-\frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{\varepsilon_k+z}$, the solution of the involution identity coincides with the solution of the system for $(\Omega_n^{(0)},\mathcal{T}^{(0)})$ found in [BHW 20].

Linear and quadratic loop equations

The best strategy seems to prove that the Dyson-Schwinger equations imply linear and quadratic loop equations [Borot, Eynard, Orantin 13]. If $R^{-1}(R(z)) = \{\hat{z}^0 \equiv z, \hat{z}^1, \dots \hat{z}^d\}$,

$$\sum_{j=0}^{d} \omega_{g,|I|+1}(I, \hat{z}^j) = f_1(R(z); I),$$

$$\sum_{\substack{j,k=0 \\ j\neq k}}^{d} \Big(\omega_{g-1,|I|+2}(I, \hat{z}^j, \hat{z}^k) + \sum_{\substack{g_1+g_2=g \\ I_1 \uplus I_2 = I}} \omega_{g_1,|I_1|+1}(I_1, \hat{z}^j)\omega_{g_2,|I_2|+1}(I_2, \hat{z}^k)\Big) = f_2(R(z); I),$$

where f_1, f_2 are *holomorphic* at ramification points of R.

- Blobbed TR is the general solution of such loop equations [Borot, Shadrin 15]. To reduce to TR one needs more.
- In our case f_1, f_2 are of particular structure which completely fixes the poles at $z = -u_k$ and $z = 0$.
- Work in progress: extension to higher g (kernel K_0 for pole at $z = 0$ already found).

Enumerative geometry [Branahl, PhD thesis 22]

Moments/Cumulants and even the $\omega_{g,n}$ expand into (products of) ribbon graphs.

- For $d = 1$ (combinatorial limit) all graphs of given topology carry the same weight. The λ^v-coefficient just counts the number of ribbon graphs of given topology and v vertices.
- By Euler duality vertices $\leftrightarrow$ faces and edges $\leftrightarrow$ edges, this is the same as the number of maps (in the sense of geography) of particular polygonal countries and polygonal oceans on a genus-g world. Such investigations have been pioneered by [Tutte, 60s].

Types of maps

- $\lim_{d=1}[N^{-2g}]\langle\prod_{\beta=1}^{b} e_{k_1^\beta k_2^\beta} e_{k\beta_2 k_3^\beta} \cdots e_{k_{n_\beta}^\beta k_1^\beta}\rangle_c$ generates fully simple quadrangulations of a genus-g Riemann surface with b boundaries of lengths $n_1, ..., n_b$. For $g = 0$ there is an explicit formula [Bernardi, Fusy 18] which we reproduce.
- $\Omega_1^{(g)}(\varepsilon)\big|_{d=1}$ generates rooted ordinary quadrangulations with one boundary length of 2.
- The part of $\Omega_1^{(g)}(\varepsilon)\big|_{d=1}$ with poles at ramification points generates rooted bipartite ordinary quadrangulations with one boundary length of 2.

Outlook I: Intersection numbers

Eynard proved in 2011 a general formula which expresses $\omega_{g,n}$ for *any* spectral curve $(x : \Sigma \to \Sigma_0, \omega_{0,1}, \omega_{0,2})$ with simple ramification points in terms of intersection numbers of ψ- and κ-classes on several copies of $\overline{\mathcal{M}}_{g,n}$.

- Extensions to higher-order ramifications are known.
- According to [Borot, Shadrin 15], Eynard's formula survives with some modifications to blobbed TR.

Thus, expressing our $\omega_{g,n}$ in terms of intersection numbers is an achievable goal in very near future.

- This expression can be interesting (like the ELSV formula) or not.
- We hope it captures aspects of the involution $z \mapsto -z$ which plays a decisive rôle in the residue formula.

For the complex model everything fits nicely.

Outlook II: Integrability

Consider $\tau(\{t_i\}) = N^2\mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \sum_{g=2}^{\infty} N^{2-2g}\omega_{g,0}$

- In the [Kontsevich 92] model, for $t_i = -\frac{(2i-1)!!}{N}\mathrm{Tr}(E^{-2i-1})$, τ satisfies the Hirota bilinear PDE of the KdV-hierarchy.
- [Eynard, Orantin 07] describe how to obtain from any spectral curve of TR a formal Hirota equation (order by order in $1/N^2$).
- Whether or not integrability extends to blobbed TR is not known.
- We remain optimistic for our case because the recursion formula with residue kernel is very close to TR.
- Recall that the complex $\lambda(\Phi^\dagger\Phi)^2$ model obeys topological recursion exactly [Branahl, Hock 22]. Its free energy is the τ-function of a KP hierarchy.

  This should give hints in which variables to search for integrability of the real model.