From scalar fields on noncommutative geometries to blobbed topological recursion

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Introduction

This project started in 1998 as an attempt to understand quantum field theories on noncommutative geometries.

- Quantum field theory in 4 dimensions is a challenge.
- Their was some hope that QFT on noncommutative spaces is simpler. Renormalisation and improvement in β-function were established.

Since 2009, Harald Grosse and I accumulated hints that something special is behind our computations, but we were unable to locate it.

Topological recursion

... is this special structure. It governs a remarkable variety of research lines in mathematics and physics and establishes beautiful connections between different fields.

Euclidean scalar fields

- By theorems of Minlos (1963) and Schur (1911), every continuous inner product $\langle\ ,\ \rangle$ on a real nuclear vector space X defines a unique measure $d\mu_0$ on the dual X' s.t.
$$\exp\Big(-\frac{1}{2}\langle M, M\rangle\Big) =: \int_{X'} d\mu_0(\Phi)\, \mathrm{e}^{\mathrm{i}\Phi(M)} .$$
- A measure $d\mu_0$ of such type defines a free Euclidean scalar field.

We would like to deform $d\mu_0$ by a quartic potential:

$$d\mu_\lambda(\Phi) \text{ ``:=''} \frac{1}{\mathcal{Z}} d\mu_0(\Phi) \mathrm{e}^{-\lambda \mathrm{Tr}(\Phi^4)} , \qquad \mathcal{Z} \text{ ``:=''} \int_{X'} d\mu_0(\Phi) \mathrm{e}^{-\lambda \mathrm{Tr}(\Phi^4)}$$

This is NOT straightforward! For interesting free field measures $d\mu_0$ we will encounter typical aspects of a QFT: renormalisation and problem of convergence of the power series expansion in λ.

Noncommutative geometries and matrix models

We let X be the space of selfadjoint elements of some noncommutative Fréchet algebra.

- We view renormalisation as regularisation to finite-dimensional problems plus limit procedure.
- Finite-dimensional noncommutative algebras are matrix algebras. Approximation by matrix algebras is an important topic in operator algebras.
- The regularised Euclidean field theory on X is essentially a matrix model on H_N (selfadjoint $N \times N$-matrices).

There is a rich zoo of matrix models. They play an important rôle in enumerative geometry and provide a meaningful formulation of 2D quantum gravity.

Kontsevich-type matrix models

We focus on free field measures $d\mu_0$ on H'_N induced by

$$\langle M, M\rangle := \frac{1}{N}\sum_{k,l=1}^{N}\frac{M_{kl}M_{lk}}{E_k+E_l}\,, \qquad E_1,\dots,E_N>0$$

Theorem [Kontsevich 1992]

$\mathcal{Z}(E)=\int_{H'_N} d\mu_0(\Phi)\exp\big(\frac{\mathrm{i}}{6}N\,\mathrm{Tr}(\Phi^3)\big)$ is generating function of intersection numbers of ψ-classes on moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves.

Moreover, expanding Schwartz functions ϕ into eigenfunctions of 2D harmonic oscillator, then covariance of the action

$$S_2(\phi) := \int_{\mathbb{R}^2}\frac{dx}{8\pi}\Big(\tfrac{1}{2}\phi(-\Delta+\tfrac{4}{\theta^2}\|x\|^2+m^2)\phi\Big)(x)$$

coincides with covariance of $d\mu_0$ for $E_k=\frac{m^2}{2}+\frac{4k-2}{\theta}$ and $N=\frac{\theta}{4}$ (but $k\in\mathbb{N}$). Compatible with Moyal product. Generalises to 4D.

Two-point functions

Consider $\int_{H_N'} d\mu_0(\Phi)\, e^{i\Phi(M)} = \exp\big(- \frac{1}{2N}\sum_{k,l=1}^{N} \frac{M_{kl}M_{lk}}{E_k+E_k}\big)$ and $d\mu_\lambda(\Phi) = \frac{1}{\mathcal{Z}} d\mu_0(\Phi) \exp\big(-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)\big)$. Let $\Phi_{kl} := \Phi(e_{kl})$.

- Build for $a \neq b$ the two-point functions

$$G_{|ab|} = N\int_{H_N'} \Phi_{ab}\Phi_{ba} d\mu_\lambda(\Phi)$$

$$G_{|a|b|} = N^2\int_{H_N'} \Phi_{aa}\Phi_{bb} d\mu_\lambda(\Phi)$$

  The limit to $a = b$ can be discussed.

- Fact: matrix models have an asymptotic $1/N$-expansion

$$G_{|ab|} = \sum_{g=0}^{\infty} N^{-2g} G^{(g)}_{|ab|} \quad \text{and} \quad G_{|a|b|} = \sum_{g=0}^{\infty} N^{-2g} G^{(g)}_{|a|b|}$$

These $G^{(g)}_{|ab|}, G^{(g)}_{|a|b|}$ are functions of $\{E_k\}$ and λ and give rise to

$$\Omega^{(g)}_{q_1,\dots,q_n} := \frac{(-N)^{n-1}\partial^{n-1}\big(\frac{1}{N}\sum_{k=1}^{N} G^{(g)}_{|kq_1|} + G^{(g-1)}_{|q_1|q_1|}\big)}{\partial E_{q_2}\cdots\partial E_{q_n}} + \frac{\delta_{g,0}\delta_{n,2}}{(E_{q_1}-E_{q_2})^2}$$

Miracles

Dyson-Schwinger equations (which decouple in the $1/N$-expansion) and complex analysis allow to prove:

1. We provide the planar functions $G^{(0)}_{|kl|}$ and $G^{(0)}_{|k|l|}$ as exact functions of $E_1, .., E_N > 0$ and λ (taken in neighbourhood of $\mathbb{R}_+$), even for $N \to \infty$ and with explicit renormalisation parameters [Grosse-Hock-W 19, Schürmann-W 19].

2. Although $G^{(g>0)}_{|ab|}, G^{(g>0)}_{|a|b|}$ are not directly accessible, we can recursively compute all $\Omega^{(g)}_{q_1,...,q_n}$ without knowledge of the $G^{(g>0)}_{|ab|}, G^{(g>0)}_{|a|b|}$ [Branahl-Hock-W 20/21; work in progress].

   The $\Omega^{(g)}_{q_1,...,q_n}$ are surprisingly simple (in the right variables). They are universal in a precise sense and as such relate to structures in algebraic geometry.
   From the Ω's it is fairly easy to produce any moment of $d\mu_\lambda$.

A non-linear equation for the planar 2-point function

Theorem [Grosse-W 09]

There is a function $G^{(0)}$ of complex variables which interpolates $G^{(0)}(E_a, E_b) = G^{(0)}_{|ab|}$ and satisfies the non-linear closed eq.

$$\Big(\zeta + \eta + \mu^2_{bare} + \lambda \int_0^\infty dt\ \varrho_0(t)\, ZG^{(0)}(\zeta, t)\Big) ZG^{(0)}(\zeta, \eta)$$
$$= 1 + \lambda \int_0^\infty dt\ \varrho_0(t)\, \frac{ZG^{(0)}(t, \eta) - ZG^{(0)}(\zeta, \eta)}{t - \zeta}\ .$$

Here $\varrho_0(t) = \sum_{k=1}^{d} \frac{r_k}{N}\delta(t - e_k)$ if $\{e_k\}$ are the different spectral values in $\{E_l\}$ which occur with multiplicities $\{r_k\}$.

- We succeeded to exactly solve this equation in 2018/19.
- All other $1/N$-expanded correlation functions satisfy affine equations. They are always solvable, but possibly not exactly.
- In 2020/21 we found the structure which governs also their exact solution: blobbed topological recursion.

Solution

Theorem [Panzer-W 18 for $\varrho_0 = 1$, Grosse-Hock-W 19]

1. Ansatz $G^{(0)}(x,y) = \dfrac{e^{\mathcal{H}_x[\tau_y(\bullet)]} \sin \tau_y(x)}{Z\lambda\pi\varrho_0(x)}$ Z=renormalisation, $\mathcal{H}_x[f] = \frac{1}{\pi} \fint \frac{dp\, f(p)}{p-x}$

2. $\tau_y(x) = \mathrm{Im} \log\big(y + I(x+\mathrm{i}\epsilon)\big)$ with $I(\zeta) = -R(-\mu^2 - R^{-1}(\zeta))$

3. $$R(z) = z - \lambda(-z)^{D/2} \int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)^{D/2}(t+\mu^2+z)}$$
   $D = 2[\frac{\delta}{2}]$ at spectral dimension $\delta = \inf\Big(p : \int \frac{dt \varrho_0(t)}{(1+t)^{p/2}} < \infty\Big)$

4. ϱ_λ is implicit solution of $\varrho_0(R(x)) = \varrho_\lambda(x)$.

- Proof: [Cauchy 1831] residue theorem, [Lagrange 1770] inversion theorem, [Bürmann 1799] formula
- $\varrho_0(t) \equiv 1$ (2D Moyal) in terms of Lambert-W, $W(z)e^{W(z)} = z$:
  $I(\zeta) := \lambda W_0\big(\frac{1}{\lambda} e^{\frac{1+\zeta}{\lambda}}\big) - \lambda \log\Big(1 - \lambda W_0\big(\frac{1}{\lambda} e^{\frac{1+\zeta}{\lambda}}\big)\Big)$

$D=4$ Moyal space: $\varrho_0(t)=t$ [Grosse-Hock-W 19]

- $\varrho_\lambda(x) \equiv \varrho_0(R(x)) = R(x) = x - \lambda x^2 \int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)^2(t+x)}$
- If $\varrho_\lambda(t) \sim \varrho_0(t) = t$, then $R(x)$ bounded above. Consequently, R^{-1} would not be globally defined: triviality!
- Fredholm equation perturbatively solved by iterated integrals: Hyperlogarithms and $\zeta(2n)$ which can be summed to

$$\varrho_\lambda(x) = x \cdot {}_2F_1\left(\begin{matrix}\alpha_\lambda, 1-\alpha_\lambda \\ 2\end{matrix}\,\middle|\, -\frac{x}{\mu^2}\right)$$

$$\alpha_\lambda = \begin{cases} \frac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda| \leq \frac{1}{\pi} \\ \frac{1}{2} + \mathrm{i}\frac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{for } \lambda \geq \frac{1}{\pi} \end{cases}$$

Corollary

The interaction alters the spectral dimension to $4 - 2\frac{\arcsin(\lambda\pi)}{\pi}$ and thus avoids the triviality problem.

Gives non-perturbative integral representation for $G^{(0)}(\xi,\eta)$.

Direct solution for finite N

Theorem ([Schürmann-W 19], inspired by [Hock-Grosse-W 19])

Let $(\varepsilon_k, \varrho_k)$ be implicitly defined by $e_k = R(\varepsilon_k)$, $r_k = R'(\varepsilon_k)\varrho_k$

for $R(z) = z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{z+\varepsilon_k}$.

Then $G^{(0)}(\zeta,\eta) = \mathcal{G}^{(0)}(z,w)$ for $R(z) = \zeta$, $R(w) = \eta$ and

$$\mathcal{G}^{(0)}(z,w) = \frac{P_{0,1}(R(z),R(w))}{(R(w)-R(-z))(R(z)-R(-w))}$$

$$P_{0,1}(R(z),R(w)) := \frac{\prod_{j=0}^{d}(R(z)-R(-\hat{w}^j))}{\prod_{k=1}^{d}(R(z)-R(\varepsilon_k))}$$

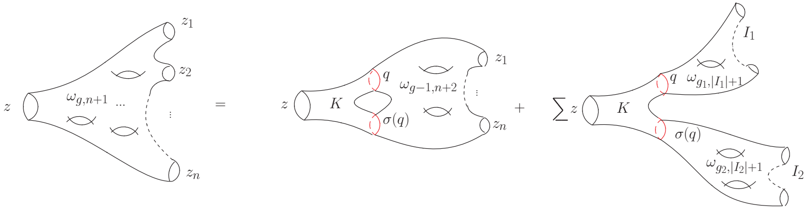
where $u \in \{z = \hat{z}^0, \hat{z}^1, \dots, \hat{z}^d\}$ are all solutions of $R(u) = R(z)$.
(The symmetry $\mathcal{G}^{(0)}(z,w) = \mathcal{G}^{(0)}(w,z)$ is automatic)

Thus, planar 2-point function solved by the composition of a rational function $\mathcal{G}^{(0)}$ with inverse of another rational function R.

Topological recursion [Eynard-Orantin 07]

- The exact solution by complex geometry suggests a link to topological recursion.
- TR recursively constructs, starting from a spectral curve consisting of
  - a ramified covering $x : \Sigma \to \Sigma_0$ of Riemann surfaces,
  - meromorphic differentials $\omega_{0,1} = y dx$ on Σ and $\omega_{0,2}$ on $\Sigma \times \Sigma$,

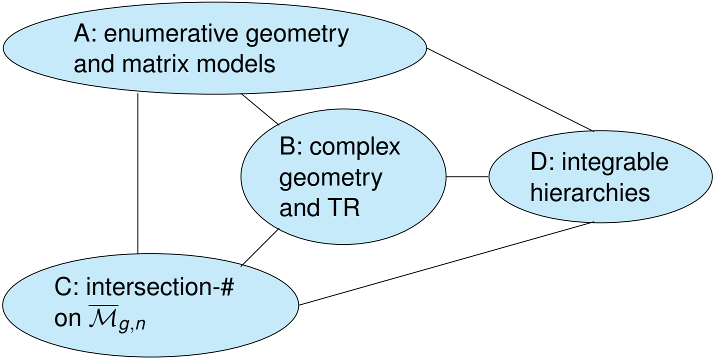
  a family $\omega_{g,n}$ of meromorphic differentials on Σ^n, with poles at zeros of dx (ramification points), schematically[1] by



[1] picture made by Johannes Branahl

Connections

Topological recursion establishes amazing connections between mathematical fields:



see: R. Belliard, S. Charbonnier, B. Eynard & E. Garcia-Failde: *Topological recursion for generalised Kontsevich graphs and r-spin intersection numbers*, arXiv:2105.08035

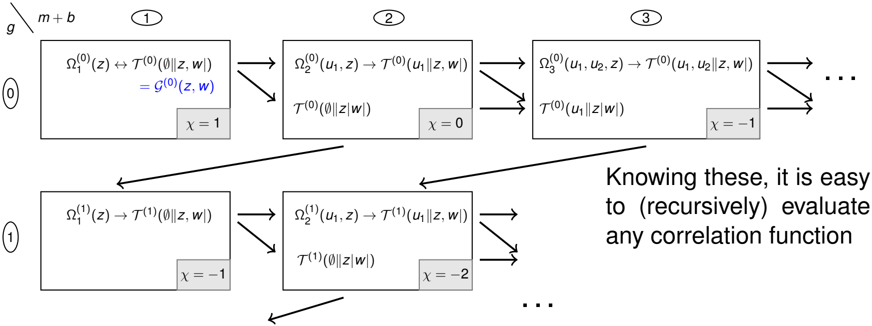
Selected examples for topological recursion

- $(\hat{\mathbb{C}} \to \hat{\mathbb{C}},\ x(z) = \alpha + \gamma(z+\frac{1}{z}),\ y(z) = \sum_j u_j(z^j - z^{-j}), \omega_{0,2} = B)$
  Hermitian one-matrix model, describes 2D quantum gravity, integrable [Gross-Migdal, Brézin-Kazakov, Douglas-Shenker 90]
- $(\hat{\mathbb{C}} \to \hat{\mathbb{C}},\ x(z) = z^2,\ y(z) = z,\ \omega_{0,2} = B)$
  [Kontsevich 92] model, equivalent formulation of 2D quantum gravity, generates intersection numbers on $\overline{\mathcal{M}}_{g,n}$ as conjectured by [Witten 91].
  The $\omega_{g,n}$ computed from TR are complexifications of $\Big\langle \int \frac{d\Phi}{\mathcal{Z}} \Phi_{k_1 k_1} \cdots \Phi_{k_n k_n} e^{-N \mathrm{Tr}(E^2\Phi^2 + \frac{\lambda}{3}\Phi^3)} \Big\rangle_c$ restriced to genus g.
- For $\mathfrak{X}$ a toric Calabi-Yau 3-fold (A-model), let $\mathcal{S}$ be the singular locus (B-model) of its mirror Calabi-Yau. Then $\omega_{g,n}$ of $\mathcal{S}$ generate Gromov-Witten invariants of $\mathfrak{X}$ (which classify stable genus-g maps into $\mathfrak{X}$) [Bouchard-Mariño-Klemm-Pasquetti 07]

Affine $\lambda\Phi^4$-equations [Branahl-Hock-W 20]

There are meromorphic functions $\Omega_m^{(g)}$ which interpolate $\Omega^{(g)}_{q_1,\dots,q_m} = \Omega_m^{(g)}(\varepsilon_{q_1},\dots,\varepsilon_{q_m})$ and which form with two families $\mathcal{T}^{(g)}(I\|z,w|)$, $\mathcal{T}^{(g)}(I\|z|w|)$ of auxiliary functions a system of affine Dyson-Schwinger equations.

This system is explicitly solvable in decreasing χ:



Knowing these, it is easy to (recursively) evaluate any correlation function

Contact with topological recursion [BHW 20]

- Pass to meromorphic differentials
  $\omega_{g,m}(z_1,\dots,z_m) = \lambda^{2-2g-m}\Omega^{(g)}_m(z_1,\dots,z_m)\prod_{k=1}^m dR(z_i)$
- Intermediate steps of solution scheme extremely lengthy, but final result simple and structured:

$$\omega_{0,2}(u,z) = \frac{du\,dz}{(u-z)^2} + \frac{du\,dz}{(u+z)^2}$$

$$\omega_{0,3}(u_1,u_2,z) = -\sum_{i=1}^{2d} \frac{\Big(\frac{1}{(u_1-\beta_i)^2}+\frac{1}{(u_1+\beta_i)^2}\Big)\Big(\frac{1}{(u_2-\beta_i)^2}+\frac{1}{(u_2+\beta_i)^2}\Big)du_1\,du_2\,dz}{R'(-\beta_i)R''(\beta_i)(z-\beta_i)^2} + \Big[d_{u_1}\Big(\frac{\omega_{0,2}(u_2,u_1)}{(dR)(u_1)}\frac{dz}{R'(-u_1)(z+u_1)^2}\Big) + u_1\leftrightarrow u_2\Big]$$

where $\beta_1,\dots,\beta_{2d}$ are the ramification points of R, i.e. $dR(\beta_i)=0$

Observation

The blue terms are exactly those of topological recursion, the red terms are a consistent extension called blobbed topological recursion [Borot-Shadrin 15].

Conjecture: NC $\lambda\Phi^4$-model obeys BTR!

When trying to prove the conjecture for $g = 0$ we noticed surprising identities between $\omega_{0,m+1}(u_1, \dots, u_m, -z)$ and $\omega_{0,k+1}(u_1, \dots, u_k, z)$. They are of independent interest:

Definition

Let $x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a ramified covering with ramification points $\beta_1, \dots, \beta_r$. For a global involution $\iota : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, which neither fixes nor permutes the β_i, let $y(z) := -x(\iota z)$. Then a family $\{\omega_{0,n}\}_{n\geq 2}$ of meromorphic differentials is introduced by

$$\omega_{0,2}(w,z) = \frac{1}{2}\frac{dw\,dz}{(w-z)^2} + \frac{1}{2}\frac{d(\iota w)\,d(\iota z)}{(\iota w - \iota z)^2} - \frac{1}{2}\frac{dw\,d(\iota z)}{(w-\iota z)^2} - \frac{1}{2}\frac{d(\iota w)\,dz}{(\iota w - z)^2}$$

and for $m \geq 2$ by the involution identity

$$\omega_{0,m+1}(u_1, \dots, u_m, z) + \omega_{0,m+1}(u_1, \dots, u_m, \iota z)$$
$$= \sum_{s=2}^{m} \sum_{I_1 \uplus \dots \uplus I_s = \{u_1,\dots,u_m\}} \frac{1}{s} \operatorname*{Res}_{w\to z} \left(\frac{dy(z)dx(w)}{(y(z)-y(w))^s} \prod_{i=1}^{s} \frac{\omega_{0,|I_i|+1}(I_i, w)}{dx(w)} \right).$$

Theorem [Hock-W 21]

The involution identity has the unique solution

$$\omega_{0,m+1}(u_1,\dots,u_m,z)$$
$$=\sum_{i=1}^{r}\operatorname*{Res}_{q\to\beta_i}K_i(z,q)\sum_{I_1\uplus I_2=\{u_1,\dots,u_m\}}\omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,\sigma_i(q))$$
$$-\sum_{k=1}^{m}d_{u_k}\Big[\operatorname*{Res}_{q\to\iota u_k}\sum_{I_1\uplus I_2=\{u_1,\dots,u_m\}}\tilde{K}(z,q,u_k)d_{u_k}^{-1}\big(\omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,q)\big)\Big].$$

Here σ_i is the local Galois involution near β_i, i.e. $x(z)=x(\sigma_i(z))$, $\sigma_i(\beta_i)=\beta_i$, $\sigma_i\neq\mathrm{id}$. The recursion kernels are given by

$$K_i(z,q):=\frac{\frac{1}{2}\big(\frac{dz}{z-q}-\frac{dz}{z-\sigma_i(q)}\big)}{dx(\sigma_i(q))(y(q)-y(\sigma_i(q)))}\,,\tilde{K}(z,q,u):=\frac{\frac{1}{2}\big(\frac{d(\iota z)}{\iota z-\iota q}-\frac{d(\iota z)}{\iota z-u}\big)}{dx(q)(y(q)-y(\iota u))}.$$

The solution implies symmetry $z\mapsto\iota z$ of the rhs of the involution identity.

Back to $\lambda\Phi^4$-matrix model

Theorem [Hock-W 21]

For the choice

$$\iota z = -z\,, \qquad x(z) = R(z) := z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{\varepsilon_k + z}\,,$$

the solution of the involution identity coincides with the solution of the system for $(\Omega_n^{(0)}, \mathcal{T}^{(0)})$ found in [Branahl-Hock-W 20].

- There are a few examples for blobbed topological recursion (e.g. multitrace Hermitian matrix model, stuffed maps), but recursion kernel for blob is rather special.
- Work in progress: Extend everything to higher g.

Summary and outlook

- Quantum field theories on noncommutative geometries can be identified with matrix models.
- These are often exactly solvable or even integrable, which is best understood in terms of topological recursion.
- We have shown that these prospects also apply to the noncommutative $\lambda\Phi^4$-model, which relates to blobbed TR.

Intersection numbers [Borot-Shadrin 15]

Forms $\omega_{g,m}$ which satisfy BTR encode intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,m}$ of stable complex curves. We expect that they encode a geometric structure related to $z \mapsto -z$.

Integrability

Is not known in BTR. But this model with involution $z \mapsto -z$ and blob given by recursion kernel is special. We are optimistic.

Can we learn anything for QFT?

Maybe this: The model we presented is much much simpler than any QFT of interest. Nevertheless we had to do two unusual steps to uncover its secrets:

1. A deformation of variables $z(\lambda) = R^{-1}(\zeta)$ away from the free theory: functions of interest are simple in terms of the (complicated) inverse z of a simple function R.
2. Focus on auxiliary functions $\Omega_n^{(g)}$ of algebraic-geometrical significance. They are much simpler than the correlation functions themselves.

It seems unlikely to me that one can understand a realistic QFT without such steps.

There is no general recipe. My only advice is: Try to be open-minded. The free theory is too extreme to be any guide.