From scalar fields on noncommutative geometries to blobbed topological recursion

Raimar Wulkenhaar

Mathematisches Institut der Westfälischen Wilhelms-Universität Münster

Mathematics Münster
Cluster of Excellence

Co-authors: Harald Grosse, Erik Panzer, Alex Hock,
Jörg Schürmann, Johannes Branahl, Maciej Dołęga

Introduction

This project started in 1998 as an attempt to understand quantum field theories on noncommutative geometries.

- No interacting and mathematically consistent QFT is known in 4 dimensions.
- The hope was that the situation could improve on noncommutative spaces. Renormalisation and improvement in β-function were established.

Since 2009, Harald Grosse and I accumulated hints that something special is behind our computations, but we were unable to locate it.

Topological recursion

... is this special structure. It governs a remarkable variety of research lines in mathematics and physics and establishes beautiful connections between different fields.

Dyson-Schwinger equations for NC $\lambda\phi^4$-model

- I assume you know what the $\lambda\phi^4$-model on a noncommutative geometry is.
- In spectral dimension 2 and 4 it needs renormalisation. Its first step is regularisation to a finite dimensional space.
- Finite-dimensional noncommutative algebras are matrix algebras. This means we essentially work with the large-N limit of matrix models.
- Covariance $\langle\phi_{kl}\phi_{mn}\rangle_c = \frac{\delta_{kn}\delta_{lm}}{N(e_k+e_l)}$ of free scalar field deformed by quartic potential.
- Dyson-Schwinger equations are quantum equations of motion; they provide a non-perturbative definition of QFT.
- We are in the happy situation that they decouple in the $1/N$-expansion.

A non-linear equation for the planar 2-point function

Theorem [Grosse-W 09]

The complexified planar 2-point function of the NC $\lambda\Phi^4$-model satisfies the non-linear closed equation

$$\Big(\zeta+\eta+\mu_{bare}^2+\lambda\int_0^\infty dt\;\varrho_0(t)\,ZG^{(0)}(\zeta,t)\Big)ZG^{(0)}(\zeta,\eta)$$
$$=1+\lambda\int_0^\infty dt\;\varrho_0(t)\,\frac{ZG^{(0)}(t,\eta)-ZG^{(0)}(\zeta,\eta)}{t-\zeta}$$

where $\varrho_0(t)=\sum_k \frac{r_k}{N}\delta(t-e_k)$ if $\{e_k\}$ are the spectral values of the NC Laplacian which occur with multiplicities $\{r_k\}$.

- We succeeded to exactly solve this equation in 2018/19.
- All other $\frac{1}{N}$-expanded correlation functions satisfy affine equations. They are always solvable, but possibly not exactly.
- In 2020/21 we found the structure which governs the exact solution: it is blobbed topological recursion.

Solution

Theorem [Panzer-W 18 for $\varrho_0 = 1$, Grosse-Hock-W 19]

1. Ansatz $G^{(0)}(x,y) = \dfrac{e^{\mathcal{H}_x[\tau_y(\bullet)]}\sin\tau_y(x)}{Z\lambda\pi\varrho_0(x)}$ Z=renormalisation $\mathcal{H}_x[f] = \frac{1}{\pi}\fint \frac{dp\, f(p)}{p-x}$
2. $\tau_y(x) = \mathrm{Im}\log\big(y + I(x+\mathrm{i}\epsilon)\big)$ with $I(\zeta) = -R(-\mu^2 - R^{-1}(\zeta))$
3. $$R(z) = z - \lambda(-z)^{D/2}\int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)^{D/2}(t+\mu^2+z)}$$
   $D = 2[\frac{\delta}{2}]$ at spectral dimension $\delta = \inf\big(p : \int \frac{dt\varrho_0(t)}{(1+t)^{p/2}} < \infty\big)$
4. ϱ_λ is implicit solution of $\varrho_0(R(x)) = \varrho_\lambda(x)$.

- Proof: [Cauchy 1831] residue theorem, [Lagrange 1770] inversion theorem, [Bürmann 1799] formula
- $\varrho_0(t) \equiv 1$ (2D Moyal) in terms of Lambert-W, $W(z)e^{W(z)} = z$:
  $I(\zeta) := \lambda W_0\big(\frac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\big) - \lambda\log\big(1 - \lambda W_0\big(\frac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\big)\big)$

$D = 4$ Moyal space: $\varrho_0(t) = t$ [Grosse-Hock-W 19]

- $\varrho_\lambda(x) \equiv \varrho_0(R(x)) = R(x) = x - \lambda x^2 \int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)^2(t+x)}$
- If $\varrho_\lambda(t) \sim \varrho_0(t) = t$, then $R(x)$ bounded above. Consequently, R^{-1} would not be globally defined: triviality!
- Fredholm equation perturbatively solved by iterated integrals: Hyperlogarithms and $\zeta(2n)$ which can be summed to

$$\varrho_\lambda(x) = x \cdot {}_2F_1\left(\begin{matrix}\alpha_\lambda, 1-\alpha_\lambda \\ 2\end{matrix}\,\middle|\, -\frac{x}{\mu^2}\right)$$

$$\alpha_\lambda = \begin{cases} \frac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda| \leq \frac{1}{\pi} \\ \frac{1}{2} + \mathrm{i}\frac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{for } \lambda \geq \frac{1}{\pi} \end{cases}$$

Corollary

The interaction alters the spectral dimension to $4 - 2\frac{\arcsin(\lambda\pi)}{\pi}$ and thus avoids the triviality problem.

Gives non-perturbative integral representation for $G^{(0)}(\xi,\eta)$.

Direct solution for finite N

Theorem ([Schürmann-W 19], inspired by [Hock-Grosse-W 19])

Let $(\varepsilon_k, \varrho_k)$ be implicitly defined by $e_k = R(\varepsilon_k),\ r_k = R'(\varepsilon_k)\varrho_k$

for $R(z) = z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{z+\varepsilon_k}$.

Then $G^{(0)}(\zeta,\eta) = \mathcal{G}^{(0)}(z,w)$ for $R(z) = \zeta$, $R(w) = \eta$ and

$$\mathcal{G}^{(0)}(z,w) = \frac{1 - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{r_k \prod_{j=1}^{d} \frac{R(w)-R(-\widehat{\varepsilon_k}^j)}{R(w)-R(\varepsilon_j)}}{(R(z)-R(\varepsilon_k))(R(\varepsilon_k)-R(-w))}}{R(w)-R(-z)}$$

where $u \in \{z, \hat{z}^1, \dots, \hat{z}^d\}$ are all solutions of $R(u) = R(z)$.

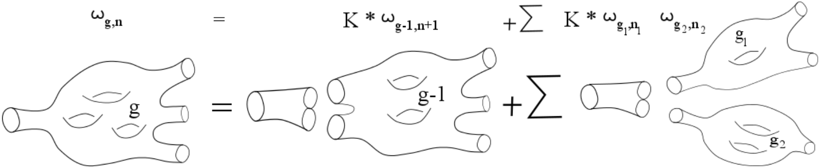
(The symmetry $\mathcal{G}^{(0)}(z,w) = \mathcal{G}^{(0)}(w,z)$ is automatic)

Thus, planar 2-point function solved by the composition of a rational function $\mathcal{G}^{(0)}$ with inverse of another rational function R.

Topological recursion [Eynard-Orantin 07]

- The exact solution by complex geometry suggests a link to topological recursion.
- TR recursively constructs, starting from a spectral curve consisting of
  - a ramified covering $x : \Sigma \to \Sigma_0$ of Riemann surfaces,
  - meromorphic differentials $\omega_{0,1}{=}y dx$ on Σ and $\omega_{0,2}$ on $\Sigma \times \Sigma$,

  a family $\omega_{g,n}$ of meromorphic differentials on Σ^n, with poles at zeros of dx (ramification points), by[1]



[1] https://upload.wikimedia.org/wikipedia/commons/7/74/Topological_recursion_illustration.png, B. Eynard, CC BY-SA 4.0 ⟨https://creativecommons.org/licenses/by-sa/4.0⟩

Selected examples for topological recursion

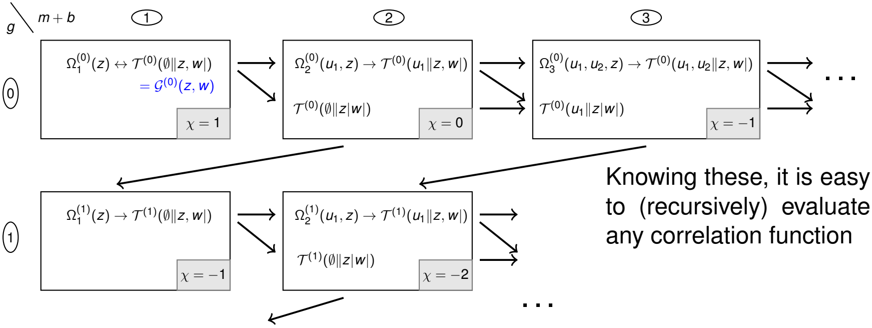
- $(\hat{\mathbb{C}} \to \hat{\mathbb{C}},\ x(z) = \alpha + \gamma(z+\frac{1}{z}),\ y(z) = \sum_j u_j(z^j - z^{-j}), \omega_{0,2} = B)$
  Hermitian one-matrix model, describes 2D quantum gravity, integrable [Gross-Migdal, Brézin-Kazakov, Douglas-Shenker 90]
- $(\hat{\mathbb{C}} \to \hat{\mathbb{C}},\ x(z) = z^2,\ y(z) = z,\ \omega_{0,2} = B)$
  [Kontsevich 92] model, equivalent formulation of 2D quantum gravity, generates intersection numbers on $\overline{\mathcal{M}}_{g,n}$ as conjectured by [Witten 91].
  The $\omega_{g,n}$ computed from TR are complexifications of $\big\langle \int \frac{d\Phi}{\mathcal{Z}} \Phi_{k_1 k_1} \cdots \Phi_{k_n k_n} e^{-N\,\mathrm{Tr}(E^2\Phi^2 + \frac{\lambda}{3}\Phi^3)} \big\rangle_c$ restriced to genus g.
- For $\mathfrak{X}$ a toric Calabi-Yau 3-fold (A-model), let $\mathcal{S}$ be the singular locus (B-model) of its mirror Calabi-Yau. Then $\omega_{g,n}$ of $\mathcal{S}$ generate Gromov-Witten invariants of $\mathfrak{X}$ (which classify stable genus-g maps into $\mathfrak{X}$) [Bouchard-Mariño-Klemm-Pasquetti 07]

Linear $\lambda\Phi^4$-equations [Branahl-Hock-W 20]

Introduce partial sums $\Omega_q := \frac{1}{N}\sum_{k=1}^{d} r_k G_{|qk|} + \frac{1}{N^2} G_{|q|q|}$ of the 2-point functions and derivatives wrt. spectral values:

$$\sum_{g=0}^{\infty} N^{2-2g-n}\Omega^{(g)}_{q_1,\dots,q_n} := \frac{(-N)^{n-1}\partial^{n-1}\Omega_{q_1}}{\partial e_{q_2}\cdots\partial e_{q_n}} + \frac{\delta_{n,2}}{(e_{q_1}-e_{q_2})^2}$$

Their complexification forms with two families $\mathcal{T}(I\|z,w|)$, $\mathcal{T}(I\|z|w|)$ a system of equations to solve in decreasing χ:



Knowing these, it is easy to (recursively) evaluate any correlation function

Contact with topological recursion [BHW 20]

- Pass to meromorphic differentials
  $\omega_{g,m}(z_1, \ldots, z_m) = \lambda^{2-2g-m}\Omega^{(g)}_m(z_1, \ldots, z_m) \prod_{k=1}^m dR(z_i)$
- Intermediate steps of solution scheme extremely lengthy, but final result simple and structured:

$$\omega_{0,2}(u,z) = \frac{du\,dz}{(u-z)^2} + \frac{du\,dz}{(u+z)^2}$$

$$\omega_{0,3}(u_1,u_2,z) = -\sum_{i=1}^{2d} \frac{\Big(\frac{1}{(u_1-\beta_i)^2} + \frac{1}{(u_1+\beta_i)^2}\Big)\Big(\frac{1}{(u_2-\beta_i)^2} + \frac{1}{(u_2+\beta_i)^2}\Big) du_1\,du_2\,dz}{R'(-\beta_i)R''(\beta_i)(z-\beta_i)^2} + \Big[d_{u_1}\Big(\frac{\omega_{0,2}(u_2,u_1)}{(dR)(u_1)}\frac{dz}{R'(-u_1)(z+u_1)^2}\Big) + u_1 \leftrightarrow u_2\Big]$$

where $\beta_1, \ldots, \beta_{2d}$ are the ramification points of R, i.e. $dR(\beta_i) = 0$

Observation

The blue terms are exactly those of topological recursion, the red terms are a consistent extension called blobbed topological recursion [Borot-Shadrin 15].

Conjecture: NC $\lambda\Phi^4$-model obeys BTR!

When trying to prove the conjecture for $g = 0$ we noticed surprising identities between $\omega_{0,m+1}(u_1, ..., u_m, -z)$ and $\omega_{0,k+1}(u_1, ..., u_k, z)$. They are of independent interest:

Definition

Let $x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a ramified covering with ramification points $\beta_1, ..., \beta_r$. For a global involution $\iota : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, which neither fixes nor permutes the β_i, let $y(z) := -x(\iota z)$. Then a family $\{\omega_{0,n}\}_{n\geq 2}$ of meromorphic differentials is introduced by

$$\omega_{0,2}(w,z) = \frac{1}{2}\frac{dw\,dz}{(w-z)^2} + \frac{1}{2}\frac{d(\iota w)\,d(\iota z)}{(\iota w - \iota z)^2} - \frac{1}{2}\frac{dw\,d(\iota z)}{(w - \iota z)^2} - \frac{1}{2}\frac{d(\iota w)\,dz}{(\iota w - z)^2}$$

and for $m \geq 2$ by the involution identity

$$\omega_{0,m+1}(u_1, ..., u_m, z) + \omega_{0,m+1}(u_1, ..., u_m, \iota z) = \sum_{s=2}^{m} \sum_{I_1 \uplus \ldots \uplus I_s = \{u_1,\ldots,u_m\}} \frac{1}{s} \operatorname*{Res}_{w\to z} \left(\frac{dy(z)dx(w)}{(y(z)-y(w))^s} \prod_{i=1}^{s} \frac{\omega_{0,|I_i|+1}(I_i, w)}{dx(w)} \right).$$

Theorem [Hock-W 21]

The involution identity has the unique solution

$$\omega_{0,m+1}(u_1,\dots,u_m,z)$$

$$= \sum_{i=1}^{r} \operatorname*{Res}_{q\to\beta_i} K_i(z,q) \sum_{I_1\uplus I_2=\{u_1,\dots,u_m\}} \omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,\sigma_i(q))$$

$$- \sum_{k=1}^{m} d_{u_k}\Big[\operatorname*{Res}_{q\to\iota u_k} \sum_{I_1\uplus I_2=\{u_1,\dots,u_m\}} \tilde{K}(z,q,u_k) d_{u_k}^{-1}\big(\omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,q)\big)\Big].$$

Here σ_i is the local Galois involution near β_i, i.e. $x(z)=x(\sigma_i(z))$, $\sigma_i(\beta_i)=\beta_i$, $\sigma_i\neq \mathrm{id}$. The recursion kernels are given by

$$K_i(z,q):=\frac{\frac{1}{2}\big(\frac{dz}{z-q}-\frac{dz}{z-\sigma_i(q)}\big)}{dx(\sigma_i(q))(y(q)-y(\sigma_i(q)))}\,,\tilde{K}(z,q,u):=\frac{\frac{1}{2}\big(\frac{d(\iota z)}{\iota z-\iota q}-\frac{d(\iota z)}{\iota z-u}\big)}{dx(q)(y(q)-y(\iota u))}.$$

The solution implies symmetry $z\mapsto\iota z$ of the rhs of the involution identity.

Back to $\lambda\Phi^4$-matrix model

Theorem [Hock-W 21]

For the choice

$$\iota z = -z\,, \qquad x(z) = R(z) := z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{\varepsilon_k + z}\,,$$

the solution of the involution identity coincides with the solution of the system for $(\Omega_n^{(0)}, \mathcal{T}^{(0)})$ found in [Branahl-Hock-W 20].

- There are a few examples for blobbed topological recursion (e.g. multitrace Hermitian matrix model, stuffed maps), but recursion kernel for blob is rather special.
- We have some ideas to extend the involution identity to higher g.

Summary and outlook

- Quantum field theories on noncommutative geometries can be identified with matrix models.
- These are often exactly solvable or even integrable, which is best understood in terms of topological recursion.
- We have shown that these prospects also apply to the noncommutative $\lambda\Phi^4$-model, which relates to blobbed TR.

Intersection numbers [Borot-Shadrin 15]

Forms $\omega_{g,m}$ which satisfy BTR encode intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,m}$ of stable complex curves. We expect that they encode a geometric structure related to $z \mapsto -z$.

Integrability

Is not known in BTR. But this model with involution $z \mapsto -z$ and blob given by recursion kernel is special. We are optimistic.