

# Blobbed topological recursion of the quartic analogue of the Kontsevich model

Raimar Wolkenhaar

Mathematisches Institut der Westfälischen Wilhelms-Universität Münster



arXiv:2008.12201 with Johannes Branahl & Alex Hock  
building on previous collaborations with  
Harald Grosse, Erik Panzer & Jörg Schürmann

# Introduction

This project started in 1998 as an attempt to understand **quantum field theories on noncommutative geometries**.

- Many nice results were obtained on renormalisation,  $\beta$ -function, extension to tensor models.
- We aimed at an **analytic construction** of these models (of course combined with combinatorics).

During the last two years we understood: **the constructive path is misleading!**

- It is about miracles in algebra, about **amazing solutions of non-linear problems**.
- It is another example for **topological recursion**, thus links to structures in complex algebraic geometry.

We just entered a new world and admire the structures seen so far. But **much more remains to explore**.

# Interlude: Topological recursion [Eynard-Orantin 07]

Universal structure that governs, e.g.

one- and two-matrix models, Kontsevich model, Weil-Petersson volumes, Hurwitz numbers, Gromov-Witten numbers, ...

Starting from a **spectral curve** consisting of

- a branched covering  $x : \Sigma \rightarrow \Sigma_0$  of Riemann surfaces,
- meromorphic differentials  $\omega_{0,1}$  on  $\Sigma$  and  $\omega_{0,2}$  on  $\Sigma \times \Sigma$ ,

recursively construct family  $\omega_{g,n}$  of meromorphic differentials on  $\Sigma^n$ , with poles at zeros of  $dx$  (ramification points), by

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_a \operatorname{Res}_{q \rightarrow a} K(z_1, q, \sigma_a(q)) dz \left( \omega_{g-1, n+1}(q, \sigma_a(q), z_2, \dots, z_n) \right. \\ \left. + \sum'_{\substack{g_1+g_2=g \\ l_1 \uplus l_2 = \{z_2, \dots, z_n\}}} \omega_{g_1, |l_1|+1}(q, l_1) \omega_{g_2, |l_2|+1}(\sigma_a(q), l_2) \right)$$

[sum over ramification points  $a$  of  $x$ ; local involution  $x(q) = x(\sigma_a(q))$ ]

near  $a$ ; recursion kernel  $K(z_1, z_2, z_3) = \frac{\frac{1}{2} \int_{z'=z_3}^{z_2} \omega_{0,2}(z_1, z')}{\omega_{0,1}(z_2) - \omega_{0,1}(z_3)}$

# Free Euclidean fields on noncommutative geometries

Let  $X_N$  be the real vector space of self-adjoint  $N \times N$ -matrices, and  $(E_1, \dots, E_N)$  be (increasing) positive real numbers.

## Theorem [Bochner 1933, Schur 1911]

For any inner product  $\langle \cdot, \cdot \rangle$  on  $X_N$  there exists a unique probability measure  $d\mu_0$  on the dual space  $X'_N$  with

$$\exp\left(-\frac{1}{2}\langle M, M \rangle\right) = \int_{X'_N} d\mu_0(\Phi) e^{i\Phi(M)} \quad \forall M = (M_{kl}) \in X.$$

Choose  $\langle M, M' \rangle_E = \frac{1}{N} \sum_{k,l=1}^N \frac{M_{kl} M'_{lk}}{E_k + E_l}$  and corresponding  $d\mu_{E,0}$

- Defines the **free Euclidean scalar field** on  $N$ -dimensional approximation of a noncommutative geometry.
- $(E_1, \dots, E_N)$  is truncated spectrum of the Laplacian.
- All moments can be described explicitly.

# Two deformations

- ③ The **Kontsevich model**  $d\mu_{E,\lambda}(\Phi) = \frac{e^{-\frac{\lambda N}{3}\text{Tr}(\Phi^3)} d\mu_{E,0}(\Phi)}{\int_{X'_N} e^{-\frac{\lambda N}{3}\text{Tr}(\Phi^3)} d\mu_{E,0}(\Phi)}$
- Computes **intersection numbers** of tautological characteristic classes on the **moduli space**  $\overline{\mathcal{M}}_{g,n}$  of **stable complex curves**.
  - It is **integrable** via a relation (suggested by Witten) to the **KdV hierarchy**. Its moments satisfy **topological recursion**.

- ④ A quartic analogue  $d\mu_{E,\lambda}(\Phi) = \frac{e^{-\frac{\lambda N}{4}\text{Tr}(\Phi^4)} d\mu_{E,0}(\Phi)}{\int_{X'_N} e^{-\frac{\lambda N}{4}\text{Tr}(\Phi^4)} d\mu_{E,0}(\Phi)}$
- Although perturbatively far apart, we find **very similar algebraic geometrical structures**. Our solutions are exact in  $\lambda$ .

# Overview

Aim: Say something about moments  $\int d\mu_{E,\lambda}(\Phi) \Phi_{k_1 l_1} \cdots \Phi_{k_n l_n}$

- As in all matrix models, best approach is **1/N-expansion**.
- After 10 years of work, we found in [Grosse-Hock-W 19] the 1/N-leading contribution to the 2-point function **as exact function of  $\lambda, E_1, \dots, E_N$** .
- This involved the solution of a **non-linear problem** (identified in [Grosse-W 09], main step is [Panzer-W 18]).

Exact solutions of non-linear problems are only possible in presence of a **profound algebraic structure**

- By analogy with the Kontsevich model we expected that this structure consists in a **topological recursion** which governs the entirety of moments of  $d\mu_{E,\lambda}$ .
- Recent discovery [Alex Hock]: The structure *is* present, but at an unexpected place!

# The objects for blobbed topological recursion

Recall that  $d\mu_{E,\lambda}$  depends on given family  $E_1, \dots, E_N$ . Introduce

$$\sum_{g=0}^{\infty} N^{2-2g-n} \Omega_{a_1, \dots, a_n}^{(g)} := \frac{\partial^{n-1} \left( N \sum_{k=1}^N \int d\mu_{E,\lambda}(\Phi) \Phi_{a_1 k} \Phi_{k a_1} \right)}{\partial E_{a_2} \cdots \partial E_{a_n}} + \frac{\delta_{n,2}}{(E_{a_1} - E_{a_2})^2}$$

- As substitute for unavailable direct  $E$ -derivatives we derive **Dyson-Schwinger equations** for the  $\Omega$ .
- They extend naturally (not uniquely) to (very complicated) equations for **meromorphic functions in several complex variables**.
- Their solutions are strikingly simple and structured: The  $\Omega$  extend to **meromorphic forms**  $\omega_{g,n}$  which exactly follow the rules of **blobbed topological recursion [Borot-Shadrin 15]**.

# Equations of motion

Fourier transform  $\mathcal{Z}(M) := \int_{X'_N} d\mu_{E,\lambda}(\Phi) e^{i\Phi(M)}$  satisfies

$$\textcircled{1} \quad -N(E_p - E_q) \sum_{k=1}^N \frac{\partial^2 \mathcal{Z}(M)}{\partial M_{pk} \partial M_{kq}} = \sum_{k=1}^N \left( M_{kp} \frac{\partial \mathcal{Z}(M)}{\partial M_{kq}} - M_{qk} \frac{\partial \mathcal{Z}(M)}{\partial M_{pk}} \right)$$

$$\textcircled{2} \quad \frac{1}{N} \frac{\partial \mathcal{Z}(M)}{\partial E_p} = \sum_{k=1}^N \frac{\partial^2 \mathcal{Z}(M)}{\partial M_{pk} \partial M_{kp}} + \mathcal{Z}(M) \int d\mu_{E,\lambda}(\Phi) \frac{1}{N} \sum_{k=1}^N \Phi_{pk} \Phi_{kp}$$

They allow to express  $\sum_{k=1}^N \frac{\mathcal{Z}(M)}{\partial M_{pk} \partial M_{kq}}$  in Dyson-Schwinger equations by **fewer derivatives**, i.e. of same or lower order.

Eq.  $\textcircled{1}$  (essentially due [Disertori-Gurau-Magnen-Rivasseau 06]) can be used for  $p \neq q$ , whereas  $p = q$  requires  $\textcircled{2}$ .



# Meromorphic continuation

These Dyson-Schwinger equations complexify to equations for meromorphic functions in several complex variables in which we can admit multiplicities  $(E_1, \dots, E_N) = (\underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_d, \dots, e_d}_{r_d})$

## Example: The two-point function

For  $p \neq q$ , set  $\sum_{g=0}^{\infty} N^{-2g} G_{|pq|}^{(g)} = N \int d\mu_{E,\lambda}(\Phi) \Phi_{pq} \Phi_{qp}$ .

Then  $G_{|pq|}^{(g)} = G^{(g)}(e_p, e_q)$  with initial equation [Grosse-W 09]

$$\begin{aligned}
 & (\xi + \eta) G^{(0)}(\zeta, \eta) \\
 &= 1 - \lambda G^{(0)}(\zeta, \eta) \Omega_1^{(0)}(\zeta) + \frac{\lambda}{N} \sum_{k=1}^d r_k \frac{G^{(0)}(e_k, \eta) - G^{(0)}(\zeta, \eta)}{e_k - \zeta}, \\
 & \Omega_1^{(0)}(\zeta) = \frac{1}{N} \sum_{k=1}^d r_k G^{(0)}(\zeta, e_k)
 \end{aligned}$$

# Main achievement: We can guess $\Omega_1^{(0)}$ !

$$\left( \eta + \zeta + \frac{\lambda}{N} \sum_{k=1}^d r_k G^{(0)}(\zeta, \mathbf{e}_k) + \frac{\lambda}{N} \sum_{k=1}^d \frac{r_k}{\mathbf{e}_k - \zeta} \right) G^{(0)}(\zeta, \eta) = 1 + \frac{\lambda}{N} \sum_{k=1}^d r_k \frac{G^{(0)}(\mathbf{e}_k, \eta)}{\mathbf{e}_k - \zeta}$$

Ansatz [Schürmann-W 19, building on Grosse-Hock-W 19]

Assume there is a **branched cover**  $R : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  with

- ①  $R$  has degree  $d + 1$  and maps (for  $\lambda > 0$ ) some domain  $\mathcal{U}$  bijectively to neighbourhood of  $\mathbf{e}_1, \dots, \mathbf{e}_d$
- ②  $\zeta = R(z), \eta = R(w), \mathbf{e}_k = R(\varepsilon_k), G^{(0)}(\zeta, \eta) = \mathcal{G}^{(0)}(z, w)$

- ③ 
$$R(z) + \frac{\lambda}{N} \sum_{k=1}^d r_k \mathcal{G}^{(0)}(z, \varepsilon_k) + \frac{\lambda}{N} \sum_{k=1}^d \frac{r_k}{R(\varepsilon_k) - R(z)} = -R(-z)$$

Gives 
$$(R(w) - R(-z)) \mathcal{G}^{(0)}(z, w) = 1 + \frac{\lambda}{N} \sum_{k=1}^d r_k \frac{\mathcal{G}^{(0)}(\varepsilon_k, w)}{R(\varepsilon_k) - R(z)}$$

# Rationality

With techniques for Lagrange interpolation polynomials [Schechter 59] it is easy to establish:

Theorem [Schürmann-W 19]

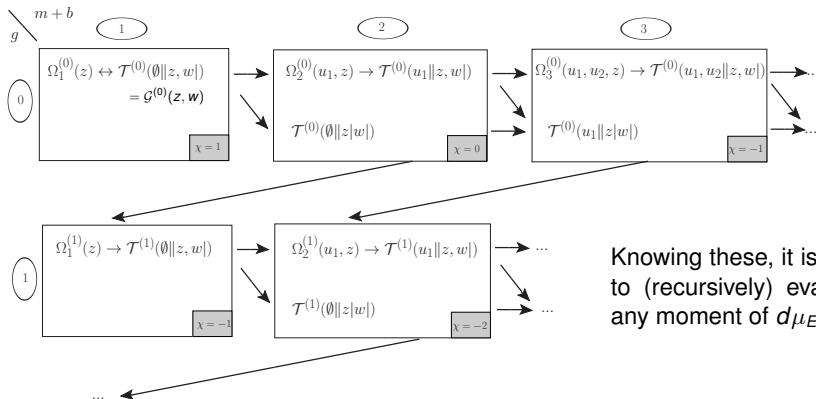
- $R(z) = z - \frac{\lambda}{N} \sum_{k=1}^d \frac{\varrho_k}{z + \varepsilon_k}$  where  $e_k = R(\varepsilon_k)$ ,  $r_k = R'(\varepsilon_k)\varrho_k$
- $\mathcal{G}^{(0)}(z, w) = \frac{1 - \frac{\lambda}{N} \sum_{k=1}^d \frac{r_k \prod_{j=1}^d \frac{R(w) - R(-\hat{\varepsilon}_k^j)}{R(w) - R(\varepsilon_j)}}{(R(z) - R(\varepsilon_k))(R(\varepsilon_k) - R(-w))}}{R(w) - R(-z)}$
- where  $u \in \{z, \hat{z}^1, \dots, \hat{z}^d\}$  are all solutions of  $R(u) = R(z)$ .
- $\mathcal{G}^{(0)}(z, w) = \mathcal{G}^{(0)}(w, z)$
- The ansatz ③ is identically satisfied!

Thus, planar 2-point function solved by the **composition of a rational function  $\mathcal{G}^{(0)}$  with inverse of another rational function  $R$ .**

# Solution procedure

Three types of functions involved:

- $\Omega_m^{(g)}(u_1, \dots, u_m)$  objects of BTR, most difficult to compute
- $\mathcal{T}^{(g)}(u_1, \dots, u_m \| z, w)$  auxiliary functions, easy to compute
- $\mathcal{T}^{(g)}(u_1, \dots, u_m \| z | w)$  auxiliary functions, easy to compute



$$\Omega_2^{(0)}(u, z)$$

$$\begin{aligned} & \Omega_2^{(0)}(u, z) R'(z) \mathfrak{G}_0(z) - \frac{\lambda}{N^2} \sum_{n,k=1}^d \frac{r_k r_n \mathcal{T}^{(0)}(u \parallel \varepsilon_k, \varepsilon_n)}{(R(\varepsilon_k) - R(z))(R(\varepsilon_n) - R(-z))} \\ &= -\frac{\partial}{\partial R(u)} (\mathcal{G}^{(0)}(u, z) + \mathcal{G}^{(0)}(u, -z)) \end{aligned}$$

where  $\mathfrak{G}_0(z) = \text{Res}_{w \rightarrow -z} \mathcal{G}^{(0)}(z, w)$ .

- Seems to need  $\mathcal{T}^{(0)}(u \parallel \varepsilon_k, \varepsilon_n)$  which itself needs  $\Omega_2^{(0)}$ .
- But poles separate by partial fraction decomposition

$$\mathcal{G}^{(0)}(z, u) = \frac{\mathfrak{G}_0(z)}{u+z} + \frac{\lambda^2}{N^2} \sum_{k,l,m,n=1}^d \frac{C_{k,l}^{m,n}}{(z + \hat{\varepsilon}_l^n)(z - \hat{\varepsilon}_k^m)(u - \hat{\varepsilon}_l^n)}$$

### Proposition

$$\Omega_2^{(0)}(u, z) = \frac{1}{R'(u)R'(z)} \left( \frac{1}{(u-z)^2} + \frac{1}{(u+z)^2} \right)$$

One recognises the **Bergman kernel** of topological recursion!

# Proposition: $\mathcal{T}^{(g)}(u_1, \dots, u_m \parallel z, w \parallel)$

The solution is easier than the equation itself:

$$\begin{aligned}
 & \mathcal{T}^{(g)}(u_1, \dots, u_m \parallel z, w \parallel) \\
 &= \lambda \mathcal{G}^{(0)}(z, w) \operatorname{Res}_{t \rightarrow z, -\hat{w}^j} \frac{R'(t) dt}{(R(z) - R(t))(R(w) - R(-t))\mathcal{G}^{(0)}(t, w)} \\
 & \times \left[ \sum_{\substack{I_1 \uplus I_2 = \{u_1, \dots, u_m\} \\ g_1 + g_2 = g \\ (I_1, g_1) \neq (\emptyset, 0)}} \Omega_{|I_1|+1}^{(g_1)}(I_1, t) \mathcal{T}^{(g_2)}(I_2 \parallel t, w \parallel) \right. \\
 & + \sum_{i=1}^m \frac{\partial}{\partial R(u_i)} \frac{\mathcal{T}^{(g)}(u_1, \dots, \check{u}_i, \dots, u_m \parallel u_i, w \parallel)}{R(u_i) - R(t)} \\
 & + \frac{\mathcal{T}^{(g-1)}(u_1, \dots, u_m \parallel t \parallel w \parallel) - \mathcal{T}^{(g-1)}(u_1, \dots, u_m \parallel w \parallel w \parallel)}{R(w) - R(t)} \\
 & \left. + \mathcal{T}^{(g-1)}(u_1, \dots, u_m, t \parallel t, w \parallel) \right]
 \end{aligned}$$

# Proposition: $\mathcal{T}^{(g)}(u_1, \dots, u_m \parallel z \parallel w \parallel)$

$$\begin{aligned}
 & \mathcal{T}^{(g)}(u_1, \dots, u_m \parallel z \parallel w \parallel) \\
 &= \frac{\lambda \prod_{j=1}^d \frac{R(z) - R(\alpha_j)}{R(z) - R(\varepsilon_j)}}{(R(z) - R(-z))} \operatorname{Res}_{t \rightarrow z, \alpha_j} \frac{R'(t) dt}{(R(z) - R(t)) \prod_{j=1}^d \frac{R(t) - R(\alpha_j)}{R(t) - R(\varepsilon_j)}} \\
 & \times \left[ \sum_{\substack{l_1 \uplus l_2 = \{u_1, \dots, u_m\} \\ g_1 + g_2 = g \\ (l_1, g_1) \neq (\emptyset, 0)}} \Omega_{|l_1|+1}^{(g_1)}(l_1, t) \mathcal{T}^{(g_2)}(l_2 \parallel t \parallel w \parallel) + \mathcal{T}^{(g-1)}(u_1, \dots, u_m, t \parallel t \parallel w \parallel) \right. \\
 & + \sum_{i=1}^m \frac{\partial}{\partial R(u_i)} \frac{\mathcal{T}^{(g)}(u_1, \dots, \check{u}_i, \dots, u_m \parallel u_i \parallel u_i \parallel w \parallel)}{R(u_i) - R(t)} \\
 & \left. + \frac{\mathcal{T}^{(g)}(u_1, \dots, u_m \parallel t, w \parallel) - \mathcal{T}^{(g)}(u_1, \dots, u_m \parallel w, w \parallel)}{R(w) - R(t)} \right],
 \end{aligned}$$

where  $\{0, \pm\alpha_1, \dots, \pm\alpha_d\}$  are all solutions of  $R(z) - R(-z) = 0$ .

# $\Omega^{(g)}(u_1, \dots, u_m)$ for $2 - 2g - m < 0$

## Proposition ( $g = 0$ ) / Conjecture ( $g \geq 1$ )

$$\begin{aligned}
 R'(z)\Omega_{m+1}^{(g)}(u_1, \dots, u_m, z) = & \operatorname{Res}_{q \rightarrow 0, -u_i, \beta_i} \frac{dq}{(q-z)\mathfrak{G}_0(q)} \left[ \right. \\
 & \sum_{\substack{l_1 \uplus l_2 = \{u_1, \dots, u_m\} \\ g_1 + g_2 = g \\ (l_1, g_1) \neq (\emptyset, 0) \neq (l_2, g_2)}} \Omega_{|l_1|+1}^{(g_1)}(l_1, q) \frac{\lambda}{N} \sum_{n=1}^d \frac{r_n \mathcal{T}^{(g_2)}(l_2 \| q, \varepsilon_n)}{R(\varepsilon_n) - R(-q)} \\
 & + \sum_{i=1}^m \frac{\partial}{\partial R(u_i)} \mathcal{T}^{(g)}(u_1, \dots, \check{u}_i, \dots, u_m \| u_i, q) \\
 & + \frac{\lambda}{N} \sum_{n=1}^d \frac{r_n \mathcal{T}^{(g-1)}(u_1, \dots, u_m, q \| q, \varepsilon_n)}{R(\varepsilon_n) - R(-q)} \\
 & \left. + \frac{\lambda}{N} \sum_{n=1}^d \frac{r_n \mathcal{T}^{(g-1)}(u_1, \dots, u_m \| q | \varepsilon_n)}{(R(\varepsilon_n) - R(q))(R(\varepsilon_n) - R(-q))} - \mathcal{T}^{(g-1)}(u_1, \dots, u_m \| q | q) \right]
 \end{aligned}$$

$\beta_1, \dots, \beta_{2d}$  – ramification points,  $R'(\beta_i) = 0$



Differential forms  $\omega_{g,m}(z_1, \dots, z_m) = \lambda^{2-2g-m} \Omega_m^{(g)}(z_1, \dots, z_m) \prod_{k=1}^m dR(z_k)$

There is not much to improve for the  $\mathcal{T}^{(g)}$ . In contrast, the  $\Omega_m^{(g)}$  simplify tremendously (by a factor  $> 100$ ) compared with the formula. Their structure is only visible after these simplifications.

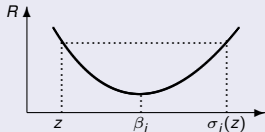
$$\begin{aligned} \omega_{0,3}(u_1, u_2, z) &= - \sum_{i=1}^{2d} \frac{\left(\frac{1}{(u_1-\beta_i)^2} + \frac{1}{(u_1+\beta_i)^2}\right) \left(\frac{1}{(u_2-\beta_i)^2} + \frac{1}{(u_2+\beta_i)^2}\right) du_1 du_2 dz}{R'(-\beta_i)R''(\beta_i)(z-\beta_i)^2} \\ &\quad + \left[ d_{u_1} \left( \frac{\omega_{0,2}(u_2, -u_1)}{(dR)(-u_1)} \frac{dz}{R'(u_1)(z+u_1)^2} \right) + u_1 \leftrightarrow u_2 \right] \\ \omega_{1,1}(z) &= \sum_{i=1}^{2d} \frac{dz}{R'(-\beta_i)R''(\beta_i)} \left\{ -\frac{1}{8(z-\beta_i)^4} + \frac{R'''(\beta_i)}{24R''(\beta_i)(z-\beta_i)^3} \right. \\ &\quad \left. + \frac{\frac{R''''(\beta_i)}{48R''(\beta_i)} - \frac{(R'''(\beta_i))^2}{48(R''(\beta_i))^2} + \frac{R''(-\beta_i)R'''(\beta_i)}{48R'(-\beta_i)R''(\beta_i)} + \frac{(R'(-\beta_i))^2}{48(R'(-\beta_i))^2} - \frac{1}{8\beta_i^2}}{(z-\beta_i)^2} \right\} \\ &\quad - \frac{dz}{8(R'(0))^2 z^3} + \frac{R''(0)dz}{16(R'(0))^3 z^2} \end{aligned}$$

# Abstract loop equations [Borot-Eynard-Orantin 13]

Proposition  $(g, m) \in \{(0, 2), (0, 3), (0, 4), (1, 1)\}$  / Conjecture

Let  $R : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  be the ramified cover identified in the solution of  $\mathcal{G}^{(0)}(z, w)$ .

Let  $\beta_1, \dots, \beta_{2d}$  be the **ramification points** of  $R$  and  $\sigma_i$  be the corresponding **local Galois involution** in the vicinity of  $\beta_i$ .



Define  $\omega_{0,1}(z) = -R(-z)R'(z)dz$  and for  $2 - 2g - m \leq 0$  the  $\omega_{g,m}$  as before. Then:

- 1 linear loop equation:

$$\omega_{g,m}(u_1, \dots, u_{m-1}, z) + \omega_{g,m}(u_1, \dots, u_{m-1}, \sigma_i(z)) = \mathcal{O}(z - \beta_i) dz$$

- 2 quadratic loop equation:

$$\begin{aligned} & \omega_{g-1, m+1}(u_1, \dots, u_{m-1}, z, \sigma_i(z)) \\ & + \sum_{\substack{l_1 \uplus l_2 = \{u_1, \dots, u_{m-1}\} \\ g_1 + g_2 = g}} \omega_{g_1, |l_1|+1}(l_1, z) \omega_{g_2, |l_2|+1}(l_2, \sigma_i(z)) \\ & = \mathcal{O}((z - \beta_i)^2) (dz)^2 \end{aligned}$$

# Blobbed topological recursion [Borot-Shadrin 15]

## Theorem

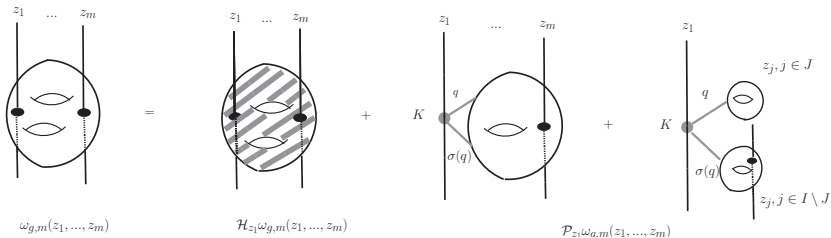
Let  $\{\omega_{g,m}\}_{g \geq 0, m > 0}$  be a family of meromorphic differential forms which satisfy the abstract loop equations. Then their parts  $\mathcal{P}\omega_{g,m}$  containing the poles at ramification points are given by

$$\begin{aligned} & \mathcal{P}_z \omega_{g,m}(u_1, \dots, u_{m-1}, z) \\ &= \sum_{i=1}^{2d} \operatorname{Res}_{q \rightarrow \beta_i} \frac{\frac{1}{2} \int_{q'=\sigma(q)}^{q'=q} B(z, q')}{\omega_{0,1}(q) - \omega_{0,1}(\sigma_i(q))} \left( \omega_{g-1, m+1}(u_1, \dots, u_{m-1}, q, \sigma_i(q)) \right. \\ & \quad \left. + \sum_{\substack{l_1 \uplus l_2 = \{u_1, \dots, u_{m-1}\} \\ g_1 + g_2 = g \\ (l_1, g_1) \neq (\emptyset, 0) \neq (l_2, g_2)}} \omega_{g_1, |l_1|+1}(l_1, q) \omega_{g_2, |l_2|+1}(l_2, \sigma_i(q)) \right) \end{aligned}$$

where  $B(u, z) = \frac{du dz}{(u-z)^2}$  is the Bergman kernel (for  $x : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ ).

$\mathcal{H}_z \omega_{g,m}(\dots, z) := \omega_{g,m}(\dots, z) - \mathcal{P}_z \omega_{g,m}(\dots, z)$  is made of **blobs**.

# A picture



$\omega_{g,m}$  = meromorphic forms on space of compactified complex lines through the marked points on a genus- $g$  Riemann surface.

- The universal formula of topological recursion produces the parts  $\mathcal{P}\omega_{g,m}$  from the entire  $\omega_{g',m'}$  of smaller degree.
- The part  $\mathcal{H}\omega_{g,m}$  is an additional input at every recursion step. We have a residue formula for them.

The quartic analogue of the Kontsevich model distinguishes a unique such form  $\omega_{g,m}$  for every  $(g, m)$ . **What is its significance?**

# Intersection numbers and integrability

## Fact [Borot-Shadrin 15]

- Forms  $\omega_{g,m}$  which satisfy BTR encode **intersection numbers on the moduli space  $\overline{\mathcal{M}}_{g,m}$**  of stable complex curves.
- These are several copies of the same intersections of  $\psi, \kappa$ -classes as in the Kontsevich model, **coupled via blobs**.

We do not know whether this coupling function encoded in the quartic Kontsevich model is interesting or not.

## Integrability

- Understanding better our recursion should give access to the **partition function** itself, a function of  $\lambda$  and  $(E_i)$ .
- Is it a  $\tau$ -function for a **Hirota equation**, i.e. **is it integrable?** [not known in general BTR]

# Outlook: $\lambda\Phi^4$ on noncommutative Moyal space

- ② Moyal in  $2D$  and  $\theta \rightarrow \infty$  described by a **Lambert curve**  
 $x(z) = R(z) = z + \lambda \log(1+z)$ ,  $y(z) = -R(-z)$  [Panzer-W 18]
  - It has a single ramification point of infinite order.
  - According to the [Bouchard-Mariño 07] conjecture, *another* Lambert curve generates **simple Hurwitz numbers**.
  - Is this more than a formal analogy?  
See [Borot-Eynard-Mulase-Safnuk 09]

- ④ Moyal in  $4D$  and  $\theta \rightarrow \infty$  according to [Grosse-Hock-W 20] described by
 
$$R(z) = z \cdot {}_2F_1\left(\alpha_\lambda, 1-\alpha_\lambda \mid -z\right), \alpha_\lambda = \begin{cases} \frac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda| \leq \frac{1}{\pi} \\ \frac{1}{2} + i \frac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{for } \lambda \geq \frac{1}{\pi} \end{cases}$$
  - Ramification points accumulate to a curve. Can one turn the TR-formula into an integral representation?
  - Can one generalise this to finite  $\theta$ ?