Towards integrability of a quartic analogue of the Kontsevich model

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based on joint work with

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Free Euclidean fields on noncommutative geometries

Let X_N be the real vector space of self-adjoint $N \times N$-matrices, and $(E_1, \dots, E_N)$ be (increasing) positive real numbers.

Theorem [Bochner 1933, Schur 1911]

There exists a unique probability measure $d\mu_0$ on the dual space X'_N with

$$\exp\Big(-\frac{1}{2N}\sum_{k,l=1}^{N}\frac{M_{kl}M_{lk}}{E_k+E_l}\Big) = \int_{X'_N} d\mu_0(\Phi)\, e^{\mathrm{i}\Phi(M)} \qquad \forall M=(M_{kl}) \in X$$

- Defines the free Euclidean scalar field on N-dimensional approximation of a noncommutative geometry.
- $(E_1, \dots, E_N)$ is truncated spectrum of the Laplacian.

Next: Deform $d\mu_0(\Phi)$ and study large-N behaviour.

This shares many aspects with QFT: Feynman graphs, divergences, renormalisation, special numbers (periods).

Two deformations

3. The Kontsevich model $d\mu_\lambda(\Phi) = \dfrac{e^{-\frac{\lambda N}{3}\mathrm{Tr}(\Phi^3)} d\mu_0(\Phi)}{\int_{X'_N} e^{-\frac{\lambda N}{3}\mathrm{Tr}(\Phi^3)} d\mu_0(\Phi)}$
   - Computes intersection numbers of tautological characteristic classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves.
   - It is integrable via a relation (suggested by Witten) to the KdV hierarchy. Its moments satisfy topological recursion.

4. A quartic analogue $d\mu_\lambda(\Phi) = \dfrac{e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_0(\Phi)}{\int_{X'_N} e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_0(\Phi)}$
   - Although perturbatively far apart, we find very similar algebraic geometrical structures.
   - Moments evaluate to explicitly known rational functions.
   - Integrability is not yet proved, but it seems the only reasonable explanation.

Moments and cumulants

$$\int_{X'_N} d\mu_\lambda(\Phi) \prod_{i=1}^{n} \Phi(e_{k_i l_i}) =: \Big\langle \prod_{i=1}^{n} e_{k_i l_i} \Big\rangle = \sum_{\substack{\text{partitions} \\ \pi \text{ of } \{1,\dots,n\}}} \prod_{\substack{\text{blocks} \\ \beta\in\pi}} \Big\langle \prod_{i\in\beta} e_{k_i l_i} \Big\rangle_c$$

- A culumulant $\langle \prod_{i=1}^{n} e_{k_i l_i} \rangle_c$ is only non-zero if $(l_1, \dots, l_n) = (k_{\sigma(1)}, \dots, k_{\sigma(n)})$ for permutation σ.
- In fact only the conjugacy class of σ matters, i.e. its cycle type $(\ell_1(\sigma), \dots, \ell_n(\sigma))$ with $\sum_i i\ell_i = n$.
- Adapted notation for b cycles of lengths $n_1, \dots, n_b$:

$$N^n \Big\langle \prod_{j=1}^{b} \Big(\prod_{i=1}^{n_j} e_{k_i^j k_{i+1}^j} \Big) \Big\rangle_c =: N^{2-b} G_{|k_1^1 \dots k_{n_1}^1| \dots |k_1^b \dots k_{n_b}^b|}, \qquad k_{n_j+1}^j \equiv k_1^j$$

- Expansion $\mathcal{Z}(M) := \int_{X'_N} d\mu_\lambda(\Phi)\, e^{\mathrm{i}\Phi(M)}$

$$= 1 - \frac{1}{N^2} \sum_{k,l=1}^{N} \Big(N G_{|kl|} \underbrace{\frac{M_{kl} M_{lk}}{1! \cdot 2^1}}_{\text{cycle type }(0,1)} + G_{|k|l|} \underbrace{\frac{M_{kk} M_{ll}}{2! \cdot 1^2}}_{\text{cycle type }(2,0)} \Big) + \mathcal{O}(M^4)$$

Equations of motion

Fourier transform $\mathcal{Z}(M) := \int_{X'_N} d\mu_\lambda(\Phi)\; e^{\mathrm{i}\Phi(M)}$ satisfies (in quartic case)

$$\frac{1}{\mathrm{i}}\frac{\partial\mathcal{Z}(M)}{\partial M_{ab}} = \frac{\mathrm{i}M_{ba}\mathcal{Z}(M)}{N(E_a+E_b)} - \frac{\lambda}{\mathrm{i}^3(E_a+E_b)}\sum_{k,l=1}^{N}\frac{\partial^3\mathcal{Z}(M)}{\partial M_{ak}\partial M_{kl}\partial M_{lb}}\;.$$

$$\frac{1}{N}\frac{\partial\mathcal{Z}(M)}{\partial E_a} = \Big(\sum_{k=1}^{N}\frac{\partial^2}{\partial M_{ak}\partial M_{ka}} + \frac{1}{N}\sum_{k=1}^{N}G_{|ak|} + \frac{1}{N^2}G_{|a|a|}\Big)\mathcal{Z}(M)$$

They give gise to Dyson-Schwinger equations between the $G_{\dots}$.

In Kontsevich model (cubic case), first equation reads

$$\frac{1}{\mathrm{i}}\frac{\partial\mathcal{Z}(M)}{\partial M_{ab}} = \frac{\mathrm{i}M_{ba}\mathcal{Z}(M)}{N(E_a+E_b)} - \frac{\lambda}{\mathrm{i}^2(E_a+E_b)}\sum_{k=1}^{N}\frac{\partial^2\mathcal{Z}(M)}{\partial M_{ak}\partial M_{kb}}\;.$$

For $N=1$, $\lambda=\mathrm{i}$ this is the Airy differential equation.

Dyson-Schwinger equations

$$(E_a+E_b)G_{|ab|} = 1 - \frac{\lambda}{N}\sum_{p=1}^{N} G_{|ab|}G_{|ap|} + \frac{\lambda}{N}\sum_{p=1}^{N}\frac{G_{|pb|}-G_{|ab|}}{E_p-E_a}$$
$$- \frac{\lambda}{N^2}\Big(-\frac{G_{|b|b|}-G_{|a|b|}}{E_b-E_a} + G_{|abab|} + G_{|baaa|}$$
$$+ G_{|ab|}G_{|a|a|} + \frac{1}{N}\sum_{p=1}^{N} G_{|ab|ap|}\Big) - \frac{\lambda}{N^4}G_{|a|a|ab|}$$

$$(E_a+E_a)G_{|a|b|} = -\frac{\lambda}{N}\sum_{p=1}^{N} G_{|ap|}G_{|a|b|} + \frac{\lambda}{N}\sum_{p=1}^{N}\frac{G_{|p|b|}-G_{|a|b|}}{E_p-E_a}$$
$$+\lambda\frac{G_{|bb|}-G_{|ab|}}{E_b-E_a} - \frac{\lambda}{N^2}\Big(G_{|b|aaa|} + G_{|a|abb|}$$
$$+3G_{|a|b|}G_{|a|a|} + \frac{1}{N}\sum_{p=1}^{N} G_{|a|b|ap|}\Big) - \frac{\lambda}{N^4}G_{|b|a|a|a|}$$

DSEs decouple in formal genus expansion $G_{\dots} = \sum_{g=0}^{\infty} N^{-2g} G^{(g)}_{\dots}$

The planar 2-point function $G^{(0)}_{|ab|}$ (of cycle type (0,1))

- ... extends to holomorphic function $G^{(0)} : U \times U \to \mathbb{C}$ on neighbourhood U of $\{E_1, \dots, E_N\}$ with $G^{(0)}(E_a, E_b) = G^{(0)}_{|ab|}$.
- If E_i has multiplicity r_i, with $r_1 + ... + r_d = N$, then $G^{(0)}$ satisfies

$$\Big(\zeta+\eta+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(\zeta,E_k)\Big)G^{(0)}(\zeta,\eta) = 1 + \frac{\lambda}{N}\sum_{k=1}^{d} r_k \frac{G^{(0)}(E_k,\eta)-G^{(0)}(\zeta,\eta)}{E_k-\zeta}$$

- Alternatively, setting $\varrho(t) = \frac{1}{N}\sum_{k=1}^{d} r_k \delta(t-E_k)$,

$$\Big(\zeta+\eta+\lambda\int dt\, \varrho_0(t) G^{(0)}(\zeta,t)\Big)G^{(0)}(\zeta,\eta) = 1 + \lambda\int dt\, \varrho_0(t)\frac{G^{(0)}(t,\eta)-G^{(0)}(\zeta,\eta)}{t-\zeta}$$

1. With Erik Panzer we solved this equation for $\rho(t) \equiv 1$.
2. Alexander Hock noticed a remarkable pattern in our solution.
3. His observation gave rise to a general solution method.
4. Connections to algebraic geometry (with Jörg Schürmann).

The non-linear integral equation

- Renormalisation $G^{(0)} \mapsto ZG^{(0)}$ and $\zeta \mapsto \frac{\tilde{\mu}^2}{2} + x, \eta \mapsto \frac{\tilde{\mu}^2}{2} + y$ with the new $x, y \in [0, \Lambda^2]$
- Ansatz $ZG^{(0)}(x,y) = \dfrac{e^{\mathcal{H}_x[\tau_y(\bullet)]} \sin \tau_y(x)}{\lambda\pi\varrho_0(x)} = \dfrac{e^{\mathcal{H}_y[\tau_x(\bullet)]} \sin \tau_y(x)}{\lambda\pi\varrho_0(y)}$

  with Hilbert transform $\mathcal{H}_y[f(\bullet)] := \frac{1}{\pi} \lim_{\epsilon\to 0} \big(\int_0^{y-\epsilon} + \int_{y+\epsilon}^{\Lambda^2}\big) \frac{dt\ f(t)}{t-y}$
- Gives

$$\Big(\tilde{\mu}^2 + x + y + \lambda\pi\mathcal{H}_x[\varrho_0(\bullet)] + \frac{1}{\pi}\int_0^{\Lambda^2} dt\ e^{\mathcal{H}_t[\tau_x(\bullet)]} \sin \tau_x(t)\Big) ZG^{(0)}(x,y)$$
$$= 1 + \mathcal{H}_x\big[e^{\mathcal{H}_\bullet[\tau_y]} \sin \tau_y(\bullet)\big]$$

- [Tricomi 57] $\qquad \mathcal{H}_x\big[e^{\mathcal{H}_\bullet[f]} \sin f(\bullet)\big] = e^{\mathcal{H}_x[f]} \cos f(x) - 1$
- [Panzer, W 18] $\qquad \int_0^{\Lambda^2} dt\ e^{\mathcal{H}_t[f(\bullet)]} \sin f(t) = \int_0^{\Lambda^2} dt\ f(t)$

The τ-equation

$$\tau_y(x) = \arctan\left(\frac{\lambda\pi\varrho_0(x)}{\tilde{\mu}^2 + y + x + \lambda\pi\mathcal{H}_x[\varrho_0(\bullet)] + \frac{1}{\pi}\int_0^{\Lambda^2} dt\ \tau_x(t)}\right)$$

Solution in case of $\varrho_0(t) \equiv 1$ [Panzer, W 18]

$$\tau_y(x) = \mathrm{Im}\log\big(y + I(x+\mathrm{i}\epsilon)\big)$$

$$I(\zeta) := \lambda W_0\Big(\frac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\Big) - \lambda\log\Big(1 - \lambda W_0\Big(\frac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\Big)\Big)$$

where W_0 is the principal branch of Lambert-W and $\tilde{\mu}^2 = 1 - 2\lambda\log(1+\Lambda^2)$.

Observation [Alexander Hock]

I and W are related:

$$I(\zeta) = -\big(-(1 + z(\zeta)) + \lambda\log(-z(\zeta))\big)$$

where $z(\zeta) = \lambda W_0\big(\frac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\big) - 1$ solves $\zeta = z + \lambda\log(1+z)$.

There is a reason for this coincidence: topological recursion.

Solution for general ϱ_0

Ansatz

$\tau_y(x) = \mathrm{Im}\log\big(y + I(x+\mathrm{i}\epsilon)\big)$ with $I(\zeta) = -R(-\mu^2 - R^{-1}(\zeta))$,
where

$$\boxed{R(z) = z - \lambda(-z)^{D/2}\int_{\tilde{\nu}}^{\tilde{\Lambda}^2} \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)^{D/2}(t+\mu^2+z)}}$$

- If $\delta = \inf\big(p : \int \frac{dt\varrho_0(t)}{(1+t)^{p/2}} < \infty\big)$ is the spectral dimension, take $D = 2[\frac{\delta}{2}]$
- For $\varrho_\lambda(t) \equiv 1$ and $D = 2$, get $R(z) = z + \lambda\log(1 + \frac{z}{\mu^2})$.
- For $\varrho_0 \not\equiv 1$, ϱ_λ is NOT the same as ϱ_0.
- $\mu = \tilde{\mu}$ for $D = 0$, otherwise a free parameter
- $R : \{\mathrm{Re}(z) > -\frac{2\mu^2}{3}\} \to U \subseteq \mathbb{C}$ is biholomorphic

Solution of all quartic matrix models

As in [Panzer, W 18], use

- [Cauchy 1831] residue theorem
- [Lagrange 1770] inversion theorem
- [Bürmann 1799] formula

Theorem [Grosse, Hock, W 19]

The ansatz $R(z) = z - \lambda(-z)^{D/2} \int_{\tilde{\nu}}^{\tilde{\Lambda}^2} \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)^{D/2}(t+\mu^2+z)}$ solves the τ-equation provided that

- ϱ_λ is implicit solution of $\varrho_0(R(x)) = \varrho_\lambda(x)$.
- $\tilde{\nu} = R^{-1}(0)$, $\tilde{\Lambda}^2 = R^{-1}(\Lambda^2)$,
- $\tilde{\mu}^2 = \mu^2 - 2\lambda \int_{\tilde{\nu}}^{\tilde{\Lambda}^2} \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)}$ for $D = 2$,
  $\tilde{\mu}^2 = \mu^2\Big(1 - \lambda \int_{\tilde{\nu}}^{\tilde{\Lambda}^2} \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)^2}\Big) - 2\lambda \int_{\tilde{\nu}}^{\tilde{\Lambda}^2} \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)}$ for $D = 4$.

Evaluating the Hilbert transform

Remains to evaluate $G^{(0)}_{ren}(\frac{\mu^2}{2}+x, \frac{\mu^2}{2}+y) = Z^{-1}\dfrac{e^{\mathcal{H}_y[\tau_x(\bullet)]}\sin\tau_x(y)}{\lambda\pi\varrho_0(y)}$.

For $D=4$ need $Z = Z_0 e^{\mathcal{H}_r[\tau_r(\bullet)]}$ to remove divergences.

Integrals very similar to [Panzer, W 18]:

Proposition [Grosse, Hock, W 19]

$$G^{(0)}_{ren}(\tfrac{\mu^2}{2}+x, \tfrac{\mu^2}{2}+y) := \frac{(\mu^2)^{\delta_{D,4}}(\mu^2+x+y)\exp(N(x,y))}{(\mu^2+y+R^{-1}(x))(\mu^2+x+R^{-1}(y))},$$

$$N(x,y) := \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} dt\Big\{\log\Big(\frac{x-R(-\frac{\mu^2}{2}-\mathrm{i}t)}{x-(-\frac{\mu^2}{2}-\mathrm{i}t)}\Big)\frac{d}{dt}\log\Big(\frac{y-R(-\frac{\mu^2}{2}+\mathrm{i}t)}{y-(-\frac{\mu^2}{2}+\mathrm{i}t)}\Big)$$
$$-\delta_{D,4}\log\Big(\frac{R(-\frac{\mu^2}{2}-\mathrm{i}t)}{(-\frac{\mu^2}{2}-\mathrm{i}t)}\Big)\frac{d}{dt}\log\Big(\frac{R(-\frac{\mu^2}{2}+\mathrm{i}t)}{(-\frac{\mu^2}{2}+\mathrm{i}t)}\Big)\Big\}$$

For $\varrho_0(t)=1$, $N(x,y)$ expands into Nielsen polylogarithms and $\zeta(n)$

$D=4$ Moyal space: $\varrho_0(t)=t$

- $\varrho_\lambda(x) \equiv \varrho_0(R(x)) = R(x) = x - \lambda x^2 \int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)^2(t+x)}$
- If $\varrho_\lambda(t) \sim \varrho_0(t)=t$, then $R(x)$ bounded above. Consequently, R^{-1} would not be globally defined: triviality!
- Fredholm equation perturbatively solved by iterated integrals: Hyperlogarithms and $\zeta(2n)$; the latter combine to $\alpha_\lambda = \frac{\arcsin(\lambda\pi)}{\pi}$:

$$
\begin{aligned}
\varrho_\lambda(x) &= \frac{x}{1-\alpha_\lambda}\sum_{n=0}^\infty \alpha_\lambda^{2n}\,\mathrm{Hlog}\big(\tfrac{x}{\mu^2},[\underbrace{0,-1,\dots,0,-1}_{2n}]\big)\\
&\quad -\frac{x+\mu^2}{1-\alpha_\lambda}\sum_{n=0}^\infty \alpha_\lambda^{2n+1}\,\mathrm{Hlog}\big(\tfrac{x}{\mu^2},[\underbrace{-1,0,-1,\dots,0,-1}_{2n+1}]\big)\\
&= x\,{}_2F_1\Big(\begin{matrix}\alpha_\lambda,1-\alpha_\lambda\\ 2\end{matrix}\Big|-\frac{x}{\mu^2}\Big), \qquad \alpha_\lambda = \begin{cases} \frac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda|\leq\frac{1}{\pi}\\ \frac{1}{2}+\mathrm{i}\frac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{for } \lambda\geq\frac{1}{\pi}\end{cases}
\end{aligned}
$$

- The interaction alters the spectral dimension to $4-2\frac{\arcsin(\lambda\pi)}{\pi}$ and thus avoids the triviality problem!

Summary I

We have a toy model which captures typical aspects of QFT:

- Divergences and their renormalisation.
- Perturbative expansion into Feynman graphs which evaluation into number-theoretical quantities (periods).

But in contrast to usual QFT, we know exact (non-perturbative) results which we can expand and compare. . . .

Maybe we can learn something for other QFTs

- Organise by topology, not number of vertices.
- Variables of the free theory (say p^2) are inconvenient. Deform them to z implicitly defined by $p^2 = z + f(z)$.
- Periods arise by inverting to $z(p^2)$.
- The deformation $p^2 \mapsto z$ could have drastic consequences: $\lambda\phi^4_4$ and QED_4 could escape triviality.

Solution for finite N

Recall
$$\Big(\eta+\zeta+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(\zeta,E_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{E_k-\zeta}\Big)G^{(0)}(\zeta,\eta)$$
$$=1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{G^{(0)}(E_k,\eta)}{E_k-\zeta},\qquad r_1+\dots+r_d=N$$

Assume there is a branched covering $R:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ with

1. R has degree $d+1$ and maps (for $\lambda>0$) neighbourhood of $\mathbb{R}_+$ bijectively to neighbourhood $U\ni\zeta,\eta,E_k$ of $\mathbb{R}_+$
2. $\zeta=R(z)$, $\eta=R(w)$, $E_k=R(\varepsilon_k)$, $G^{(0)}(\zeta,\eta)=\mathcal{G}^{(0)}(z,w)$
3. $R(z)+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\mathcal{G}^{(0)}(z,\varepsilon_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{R(\varepsilon_k)-R(z)}=-R(-z)$

Gives $(R(w)-R(-z))\mathcal{G}^{(0)}(z,w)=1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{\mathcal{G}^{(0)}(\varepsilon_k,w)}{R(\varepsilon_k)-R(z)}$

Take $z=-\hat{w}_l$ for other d preimages of $\eta=R(w)=R(\hat{w}^l)$.
Express $\mathcal{G}^{(0)}(\varepsilon_k,w)$ in terms of $R(-\hat{w}^l)$, $R(\varepsilon_k)$; insert back

Rationality

Using formulae for inverses of Cauchy matrices $(\frac{1}{a_k-b_l})_{kl}$ and their row sums [Schechter 59]:

Theorem [Schürmann-W 19]

$$R(z) = z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{z+\varepsilon_k} \quad \text{where } E_k = R(\varepsilon_k),\ r_k = R'(\varepsilon_k)\varrho_k$$

$$\mathcal{G}^{(0)}(z,w) = \frac{1}{(z+w)} \prod_{k,l=1}^{d} \frac{(\varepsilon_k+\varepsilon_l)(-\hat{w}^k-\hat{z}^l)}{(\varepsilon_k-\hat{z}^l)(\varepsilon_l-\hat{w}^k)}$$

$$= \frac{1 - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{r_k}{(R(z)-R(\varepsilon_k))(R(\varepsilon_k)-R(-w))} \prod_{j=1}^{d} \frac{R(w)-R(-\widehat{\varepsilon}_k^{\,j})}{R(w)-R(\varepsilon_j)}}{R(w)-R(-z)}$$

Relation 3 of the ansatz is identically satisfied!!

Thus, planar 2-point function solved by the composition of a rational function $\mathcal{G}^{(0)}$ with inverse of another rational function R.

Integrability?

- All other correlation functions solve affine Dyson-Schwinger equations with recursively known inhomogeneity.
- As functions of $z_i = R^{-1}(\zeta_i)$, all correlation functions are expected to be rational, which is rare for QFT-models.

This rationality and comparison with [Kontsevich 92]

- defined by $d\mu_\lambda(\Phi) = \frac{1}{\mathcal{Z}} d\mu_0(\Phi) \exp(-\frac{N\lambda}{3}\mathrm{Tr}(\Phi^3))$
- $\log \mathcal{Z}$ is τ-function of KdV-hierarchy [Witten 90].
- It generates intersection numbers of tautological characteristic classes on $\overline{\mathcal{M}}_{g,n}$.

give strong support for

Conjecture

The quartic analogue of the Kontsevich model is integrable and computes topological invariants.

We try to prove it via topological recursion.

Topological recursion [Eynard, Orantin 07]

Starting from a spectral curve consisting of

- a branched covering $x : \Sigma \to \Sigma_0$ of Riemann surfaces,
- meromorphic differentials $\omega_{0,1}$ on Σ and $\omega_{0,2}$ on $\Sigma \times \Sigma$,

recursively construct family $\omega_{g,n}$ of meromorphic n-differentials on Σ^n, with poles only at ramification points of x, by

$$\omega_{g,n}(z_1,...,z_n) = \sum_a \operatorname*{Res}_{z\to a} K(z_1,z,\sigma_a(z))dz\Big(\omega_{g-1,n+1}(z,\sigma_a(z),z_2,...,z_n) + \sum_{\substack{g_1+g_2=g \\ I_1\uplus I_2=\{z_2,\dots,z_n\}}}{}^{\prime} \omega_{g_1,1+\#I_1}(z,I_1)\omega_{g_2,1+\#I_2}(\sigma_a(z),I_2)\Big)$$

[sum over branch points a of x; local involution $x(z) = x(\sigma_a(z))$ near a; recursion kernel $K(z_1,z_2,z_3) = \frac{\frac{1}{2}\int_{z'=z_3}^{z_2}\omega_{0,2}(z_1,z')}{\omega_{0,1}(z_2)-\omega_{0,1}(z_3)}$]

Examples

one- and two-matrix models, Kontsevich model, Weil-Petersson volumes, Hurwitz numbers, Gromov-Witten numbers, ...

Topological recursion of the Kontsevich model

- branched cover $x : \hat{\mathbb{C}} \ni z \mapsto z^2 \in \hat{\mathbb{C}}$, where $z = (4\zeta^2 + c)^{1/2}$
- $\omega_{0,1}(z) = 2zy(-z)dz$ with $-y(-z) = z + \frac{1}{N}\sum_{k=1}^{N} \frac{1}{2\varepsilon_k(z+\varepsilon_k)}$ (related to planar 1-point function), $\varepsilon_k = (4E_k^2 + c)^{1/2}$
- $\omega_{0,2}(z, z') = \frac{dz\, dz'}{(z-z')^2}$ (related to planar 1+1-point function)

Meromorphic differentials relate to higher correlation functions $\omega_{g,n}(z_1, \dots, z_n) = \mathcal{G}^{(g)}(z_1|\dots|z_n)\prod_{i=1}^{n} d(x(z_i))$, where

$$\mathcal{G}^{(g)}(z_1|\dots|z_n) = (2-t_3)^{2-2g-n}\sum_{l_1,\dots,l_n}\Big\langle \psi_1^{l_1}\cdots\psi_n^{l_n} e^{\sum_k \hat{t}_k\kappa_k}\Big\rangle_{g,n}\prod_{i=1}^{n}\frac{(2l_i+1)!!}{z_i^{2l_i+3}}$$

- ψ_i, κ_k are tautological characteristic classes on $\overline{\mathcal{M}}_{g,n}$ and $\langle\dots\rangle_{g,n}$ their intersection numbers
- $e^{-\sum_k \hat{t}_k u^{-k}} = 1 - \frac{1}{2}\sum_l (2l+1)!!\, t_{2l+1}u^{-l}, \quad t_l = \frac{1}{N}\sum_{k=1}^{N}\varepsilon_k^{-2l-1}$

Outlook

- In striking parallel to the Kontsevich model, also its quartic analogue $d\mu_\lambda(\Phi) = \frac{1}{\mathcal{Z}} d\mu_0(\Phi) \exp(-\frac{N\lambda}{4}\mathrm{Tr}(\Phi^4))$ is exactly solvable in terms of rational functions.
- We can recursively compute all (g,n)-cumulants of $d\mu_\lambda(\Phi)$.

We expect more ...

- By determining poles, branch points, rôle of involution, symmetries, we hope to relate our cumulants to meromorphic differentials in topological recursion.
- Can their rational coefficients be related to intersection numbers on a moduli space?
- We expect to have holomorphicity in $\mathrm{Re}(z_i) > 0$. Hence, cumulants are Laplace transforms of something. Of what?
- Is $\log(\mathcal{Z})$ a τ-function of an integrable model?