The complete solution of the quartic analogue of the Kontsevich model

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based on joint work with
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and work in progress with Johannes Branahl and Alexander Hock

Introduction

- We still admire the straightedge and compass constructions conceived by ancient Greek mathematicians. They were unsuccessful for others (squaring the circle, trisecting an angle).
- Cubic and quartic equations are solvable [Cardano 1539-45]. Led to the invention of $\mathbb{C}$. Higher polynomial equations not solvable by radicals [Ruffini 1799, Abel 1824, Galois 1835].
- The two-body problem is solvable [Newton 1687], the three-body problem is not. This failure pushed the development of perturbation theory.
- Hydrogen atom is solvable [Pauli 1925], other atoms are not.
- Many spin models [critical, 3-state Potts, Heisenberg, 6-vertex, 8-vertex,...] are solvable. But not all.

Common to all these solvable cases is a certain beauty, and a lasting impact in mathematics and physics.

Matrix models

Today we have a long list of such nice examples. Some of them come from matrix models.

- Hermitian 1-matrix model $\mathcal{Z} = \int_{X_N} dM\, e^{-\mathrm{tr}(\text{polynomial}(M))}$ is solvable (X_N – space of self-adjoint $N \times N$-matrices) [Brézin-Kazakov, Douglas-Shenker, Gross-Migdal 89/90]
- It is deeply equivalent to the [Kontsevich 91]-model. It proves a conjecture by [Witten 91] about 2D quantum gravity. We introduce it on the next two slides.
- The Hermitian 2-matrix model is solvable [Staudacher 93]. Understanding the common mechanism behind the 1- and 2-matrix models led to the discovery of topological recursion by Eynard and others.

We are now sure to have another solvable matrix model: the quartic analogue of the Kontsevich model. The talk is about it.

Free Euclidean fields on noncommutative geometries

Let X_N be the real vector space of self-adjoint $N \times N$-matrices, and $(E_1, \dots, E_N)$ be (increasing) positive real numbers.

Theorem [Bochner 1933, Schur 1911]

For any inner product $\langle \, , \, \rangle$ on X_N there exists a unique probability measure $d\mu_0$ on the dual space X'_N with

$$\exp\Big(-\frac{1}{2}\langle M, M\rangle\Big) = \int_{X'_N} d\mu_0(\Phi)\; e^{\mathrm{i}\Phi(M)} \qquad \forall M{=}(M_{kl}) \in X\,.$$

Choose $\langle M, M'\rangle_E = \dfrac{1}{N}\displaystyle\sum_{k,l=1}^{N} \frac{M_{kl}M'_{lk}}{E_k + E_l}$ and corresponding $d\mu_{E,0}$

- Defines the free Euclidean scalar field on N-dimensional approximation of a noncommutative geometry.
- $(E_1, \dots, E_N)$ is truncated spectrum of the Laplacian.

Next: Deform $d\mu_{E,0}(\Phi)$ and study large-N asymptotics.

Two deformations

3. The Kontsevich model $d\mu_{E,\lambda}(\Phi) = \dfrac{e^{-\frac{\lambda N}{3}\mathrm{Tr}(\Phi^3)} d\mu_{E,0}(\Phi)}{\int_{X'_N} e^{-\frac{\lambda N}{3}\mathrm{Tr}(\Phi^3)} d\mu_{E,0}(\Phi)}$
   - Computes intersection numbers of tautological characteristic classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves.
   - It is integrable via a relation (suggested by Witten) to the KdV hierarchy. Its moments satisfy topological recursion.

4. A quartic analogue $d\mu_{E,\lambda}(\Phi) = \dfrac{e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_{E,0}(\Phi)}{\int_{X'_N} e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_{E,0}(\Phi)}$
   - Although perturbatively far apart, we find very similar algebraic geometrical structures.
   - Moments evaluate to explicitly known rational functions.
   - Integrability is not yet proved, but it seems the only reasonable explanation.

Moments and cumulants

$$\int_{X'_N} d\mu_{E,\lambda}(\Phi) \prod_{i=1}^{n} \Phi(e_{k_i l_i}) =: \Big\langle \prod_{i=1}^{n} e_{k_i l_i} \Big\rangle = \sum_{\substack{\text{partitions} \\ \pi \text{ of } \{1,\dots,n\}}} \prod_{\substack{\text{blocks} \\ \beta\in\pi}} \Big\langle \prod_{i\in\beta} e_{k_i l_i} \Big\rangle_c$$

- A culumulant $\big\langle \prod_{i=1}^{n} e_{k_i l_i} \big\rangle_c$ is only non-zero if $(l_1,\dots,l_n) = (k_{\sigma(1)},\dots,k_{\sigma(n)})$ for permutation σ.
- In fact only the conjugacy class of σ matters, i.e. its cycle type $(\ell_1(\sigma),\dots,\ell_n(\sigma))$ with $\sum_i i\ell_i = n$.
- Adapted notation for b cycles of lengths $n_1,\dots,n_b$:

$$N^n \Big\langle \prod_{j=1}^{b} \Big(\prod_{i=1}^{n_j} e_{k_i^j k_{i+1}^j} \Big) \Big\rangle_c =: N^{2-b} G_{|k_1^1\dots k_{n_1}^1|\dots|k_1^b\dots k_{n_b}^b|}, \qquad k_{n_j+1}^j \equiv k_1^j$$

- Expansion $\mathcal{Z}(M) := \int_{X'_N} d\mu_{E,\lambda}(\Phi)\; e^{\mathrm{i}\Phi(M)}$

$$= 1 - \frac{1}{N^2}\sum_{k,l=1}^{N}\Big(N \underbrace{G_{|kl|}}_{\text{cycle type }(0,1)} \frac{M_{kl}M_{lk}}{1!\cdot 2^1} + \underbrace{G_{|k|l|}}_{\text{cycle type }(2,0)} \frac{M_{kk}M_{ll}}{2!\cdot 1^2}\Big) + \mathcal{O}(M^4)$$

Equations of motion

Fourier transform $\mathcal{Z}(M) := \int_{X'_N} d\mu_{E,\lambda}(\Phi)\, e^{\mathrm{i}\Phi(M)}$ satisfies (in quartic case)

$$\frac{1}{\mathrm{i}}\frac{\partial\mathcal{Z}(M)}{\partial M_{ab}} = \frac{\mathrm{i}M_{ba}\mathcal{Z}(M)}{N(E_a+E_b)} - \frac{\lambda}{\mathrm{i}^3(E_a+E_b)}\sum_{k,l=1}^{N}\frac{\partial^3\mathcal{Z}(M)}{\partial M_{ak}\partial M_{kl}\partial M_{lb}}\,.$$

$$\frac{1}{N}\frac{\partial\mathcal{Z}(M)}{\partial E_a} = \Big(\sum_{k=1}^{N}\frac{\partial^2}{\partial M_{ak}\partial M_{ka}} + \frac{1}{N}\sum_{k=1}^{N}G_{|ak|} + \frac{1}{N^2}G_{|a|a|}\Big)\mathcal{Z}(M)$$

They give gise to Dyson-Schwinger equations between the $G_{\dots}$.

In Kontsevich model (cubic case), first equation reads

$$\frac{1}{\mathrm{i}}\frac{\partial\mathcal{Z}(M)}{\partial M_{ab}} = \frac{\mathrm{i}M_{ba}\mathcal{Z}(M)}{N(E_a+E_b)} - \frac{\lambda}{\mathrm{i}^2(E_a+E_b)}\sum_{k=1}^{N}\frac{\partial^2\mathcal{Z}(M)}{\partial M_{ak}\partial M_{kb}}\,.$$

For $N=1$, $\lambda=\mathrm{i}$ this is the Airy differential equation.

Dyson-Schwinger equations [Grosse-W 09, 12]

$$
\begin{aligned}
(E_a+E_b)G_{|ab|} &= 1-\frac{\lambda}{N}\sum_{p=1}^{N} G_{|ab|}G_{|ap|}+\frac{\lambda}{N}\sum_{p=1}^{N}\frac{G_{|pb|}-G_{|ab|}}{E_p-E_a} \\
&\quad -\frac{\lambda}{N^2}\Big(-\frac{G_{|b|b|}-G_{|a|b|}}{E_b-E_a}+G_{|abab|}+G_{|baaa|} \\
&\qquad +G_{|ab|}G_{|a|a|}+\frac{1}{N}\sum_{p=1}^{N}G_{|ab|ap|}\Big)-\frac{\lambda}{N^4}G_{|a|a|ab|}
\end{aligned}
$$

$$
\begin{aligned}
(E_a+E_a)G_{|a|b|} &= -\frac{\lambda}{N}\sum_{p=1}^{N} G_{|ap|}G_{|a|b|}+\frac{\lambda}{N}\sum_{p=1}^{N}\frac{G_{|p|b|}-G_{|a|b|}}{E_p-E_a} \\
&\quad +\lambda\frac{G_{|bb|}-G_{|ab|}}{E_b-E_a}-\frac{\lambda}{N^2}\Big(G_{|b|aaa|}+G_{|a|abb|} \\
&\quad +3G_{|a|b|}G_{|a|a|}+\frac{1}{N}\sum_{p=1}^{N}G_{|a|b|ap|}\Big)-\frac{\lambda}{N^4}G_{|b|a|a|a|}
\end{aligned}
$$

DSEs decouple in formal genus expansion $G_{\dots}=\sum_{g=0}^{\infty}N^{-2g}G^{(g)}_{\dots}$

The planar 2-point function $G^{(0)}_{|ab|}$ (of cycle type (0,1))

- …extends to holomorphic function $G^{(0)} : U \times U \to \mathbb{C}$ on neighbourhood U of $\{E_1, \dots, E_N\}$ with $G^{(0)}(E_a, E_b) = G^{(0)}_{|ab|}$.
- If E_i has multiplicity r_i, with $r_1{+}...{+}r_d{=}N$, then $G^{(0)}$ satisfies

$$\Big(\zeta{+}\eta{+}\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(\zeta, E_k)\Big) G^{(0)}(\zeta,\eta) = 1 + \frac{\lambda}{N}\sum_{k=1}^{d} r_k \frac{G^{(0)}(E_k,\eta){-}G^{(0)}(\zeta,\eta)}{E_k - \zeta}$$

- Alternatively, setting $\varrho_0(t) = \frac{1}{N}\sum_{k=1}^{d} r_k \delta(t - E_k)$,

$$\Big(\zeta{+}\eta{+}\lambda\int dt\, \varrho_0(t) G^{(0)}(\zeta,t)\Big) G^{(0)}(\zeta,\eta) = 1 + \lambda\int dt\, \varrho_0(t) \frac{G^{(0)}(t,\eta){-}G^{(0)}(\zeta,\eta)}{t-\zeta}$$

1. With Erik Panzer we solved this equation for $\varrho_0(t) \equiv 1$.
2. Alexander Hock noticed a remarkable pattern in our solution.
3. His observation gave rise to a general solution method.
4. Connections to algebraic geometry (with Jörg Schürmann).

Solution of the non-linear integral equation

Theorem [Panzer-W 18 for $\varrho_0 = 1$, Grosse-Hock-W 19]

1. Ansatz $G^{(0)}(x,y) = \dfrac{e^{\mathcal{H}_x[\tau_y(\bullet)]}\sin\tau_y(x)}{Z\lambda\pi\varrho_0(x)}$ (Z=renormalisation)
2. $\tau_y(x) = \operatorname{Im}\log\big(y + I(x+\mathrm{i}\epsilon)\big)$ with $I(\zeta) = -R(-\mu^2 - R^{-1}(\zeta))$
3. $$\boxed{R(z) = z - \lambda(-z)^{D/2}\int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)^{D/2}(t+\mu^2+z)}}$$
   $D = 2[\frac{\delta}{2}]$ at spectral dimension $\delta = \inf\Big(p :\ \int \frac{dt\,\varrho_0(t)}{(1+t)^{p/2}} < \infty\Big)$
4. ϱ_λ is implicit solution of $\varrho_0(R(x)) = \varrho_\lambda(x)$.

Proof: [Cauchy 1831] residue theorem,
[Lagrange 1770] inversion theorem, [Bürmann 1799] formula

$D=4$ Moyal space: $\varrho_0(t)=t$ [Grosse-Hock-W 19]

- $\varrho_\lambda(x) \equiv \varrho_0(R(x)) = R(x) = x - \lambda x^2 \int_0^\infty \frac{dt\, \varrho_\lambda(t)}{(\mu^2+t)^2(t+x)}$
- If $\varrho_\lambda(t) \sim \varrho_0(t)=t$, then $R(x)$ bounded above. Consequently, R^{-1} would not be globally defined: triviality!
- Fredholm equation perturbatively solved by iterated integrals: Hyperlogarithms and $\zeta(2n)$; the latter combine to $\alpha_\lambda = \frac{\arcsin(\lambda\pi)}{\pi}$:

$$\begin{aligned}\varrho_\lambda(x) &= \frac{x}{1-\alpha_\lambda}\sum_{n=0}^\infty \alpha_\lambda^{2n}\,\mathrm{Hlog}\big(\tfrac{x}{\mu^2},[\underbrace{0,-1,\dots,0,-1}_{2n}]\big)\\ &\quad - \frac{x+\mu^2}{1-\alpha_\lambda}\sum_{n=0}^\infty \alpha_\lambda^{2n+1}\,\mathrm{Hlog}\big(\tfrac{x}{\mu^2},[\underbrace{-1,0,-1,\dots,0,-1}_{2n+1}]\big)\\ &= x\;{}_2F_1\Big(\begin{matrix}\alpha_\lambda,\,1-\alpha_\lambda\\ 2\end{matrix}\Big|-\frac{x}{\mu^2}\Big), \qquad \alpha_\lambda = \begin{cases}\dfrac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda|\leq \frac{1}{\pi}\\[2mm] \dfrac{1}{2}+\mathrm{i}\,\dfrac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{for } \lambda \geq \frac{1}{\pi}\end{cases}\end{aligned}$$

- The interaction alters the spectral dimension to $4-2\frac{\arcsin(\lambda\pi)}{\pi}$ and thus avoids the triviality problem!

Solution for finite N [Schürmann-W 19]

Recall
$$\Big(\eta+\zeta+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(\zeta,E_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{E_k-\zeta}\Big)G^{(0)}(\zeta,\eta) = 1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{G^{(0)}(E_k,\eta)}{E_k-\zeta}, \qquad r_1+\cdots+r_d=N$$

Assume there is a branched covering $R:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ with

1. R has degree $d+1$ and maps (for $\lambda>0$) neighbourhood of $\mathbb{R}_+$ bijectively to neighbourhood $U\ni\zeta,\eta,E_k$ of $\mathbb{R}_+$
2. $\zeta=R(z)$, $\eta=R(w)$, $E_k=R(\varepsilon_k)$, $G^{(0)}(\zeta,\eta)=\mathcal{G}^{(0)}(z,w)$
3. $$R(z)+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\mathcal{G}^{(0)}(z,\varepsilon_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{R(\varepsilon_k)-R(z)}=-R(-z)$$

Gives $(R(w)-R(-z))\mathcal{G}^{(0)}(z,w)=1+\dfrac{\lambda}{N}\displaystyle\sum_{k=1}^{d} r_k\frac{\mathcal{G}^{(0)}(\varepsilon_k,w)}{R(\varepsilon_k)-R(z)}$

Take $z=-\hat{w}_l$ for other d preimages of $\eta=R(w)=R(\hat{w}^l)$.
Express $\mathcal{G}^{(0)}(\varepsilon_k,w)$ in terms of $R(-\hat{w}^l)$, $R(\varepsilon_k)$; insert back

Rationality

Using formulae for inverses of Cauchy matrices $(\frac{1}{a_k - b_l})_{kl}$ and their row sums [Schechter 59]:

Theorem [Schürmann-W 19, extending Grosse-Hock-W 19]

$$R(z) = z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{z+\varepsilon_k} \quad \text{where } E_k = R(\varepsilon_k),\; r_k = R'(\varepsilon_k)\varrho_k$$

$$\mathcal{G}^{(0)}(z,w) = \frac{1}{(z+w)} \prod_{k,l=1}^{d} \frac{(\varepsilon_k+\varepsilon_l)(-\hat{w}^k - \hat{z}^l)}{(\varepsilon_k - \hat{z}^l)(\varepsilon_l - \hat{w}^k)}$$

$$= \frac{1 - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{r_k}{(R(z)-R(\varepsilon_k))(R(\varepsilon_k)-R(-w))} \prod_{j=1}^{d} \frac{R(w)-R(-\widehat{\varepsilon_k}^j)}{R(w)-R(\varepsilon_j)}}{R(w)-R(-z)}$$

Relation ③ of the ansatz is identically satisfied!!

Thus, planar 2-point function solved by the composition of a rational function $\mathcal{G}^{(0)}$ with inverse of another rational function R.

Planar 2-point function of cycle type $(2,0)$

Dyson-Schwinger equation extends meromorphically to

$$(R(z)-R(-z))\mathcal{G}^{(0)}(z|w) - \frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k\mathcal{G}^{(0)}(\varepsilon_k|w)}{R(\varepsilon_k)-R(z)} = \lambda\frac{\mathcal{G}^{(0)}(z,w)-\mathcal{G}^{(0)}(w,w)}{R(z)-R(w)}$$

Proposition [Schürmann-W 19]

$$\mathcal{G}^{(0)}(z|w) = \frac{\lambda}{(R(z)-R(w))^2}\Bigg\{\mathcal{G}^{(0)}(z,w) - \frac{(R(z)+R(w)-2R(0))\prod_{k=1}^{d}\frac{(R(z)-R(\alpha_k))(R(w)-R(\alpha_k))}{(R(z)-R(\varepsilon_k))(R(w)-R(\varepsilon_k))}}{(R(z)-R(-z))(R(w)-R(-w))}\Bigg\}$$

where $\{0,\pm\alpha_k\}$ are all roots of $R(z)-R(-z)=0$

Higher topologies [Branahl-Hock-W soon]

- $\mathcal{G}^{(0)}(z,w)$ is topologically a disk (sphere with 1 hole)
- $\mathcal{G}^{(0)}(z|w)$ is topologically a cylinder (sphere with 2 holes)
- Does the same construction exhaust all higher topologies?

NO! A new structure is needed [Branahl-Hock-W, w.i.p]

$$\Omega_{a_1,\dots,a_n} = \Big[\int_{X'_N} d\mu_{E,\lambda}(\Phi)\prod_{i=1}^{n}\Big(\sum_{k_i=1}^{d} r_{k_i}\Phi(e_{a_ik_i})\Phi(e_{k_ia_i})\Big)\Big]_c$$

$$T_{a_1,\dots,a_n\|bc|} = \Big[\int_{X'_N} d\mu_{E,\lambda}(\Phi)\Phi(e_{bc})\Phi(e_{cb})\prod_{i=1}^{n}\Big(\sum_{k_i=1}^{d} r_{k_i}\Phi(e_{a_ik_i})\Phi(e_{k_ia_i})\Big)\Big]_c$$

$$T_{a_1,\dots,a_n\|b|c|} = \Big[\int_{X'_N} d\mu_{E,\lambda}(\Phi)\Phi(e_{bb})\Phi(e_{cc})\prod_{i=1}^{n}\Big(\sum_{k_i=1}^{d} r_{k_i}\Phi(e_{a_ik_i})\Phi(e_{k_ia_i})\Big)\Big]_c$$

Complexified $\Omega_n^{(g)}(z_1,\dots,z_n)$ relate to topological recursion!

Dyson-Schwinger equations I: $\mathcal{G}$ from $\mathcal{T}$

$$(R(w)-R(-z))\mathcal{G}^{(g)}(z,w|\mathcal{J}) - \frac{\lambda}{N}\sum_k r_k \frac{\mathcal{G}^{(g)}(\varepsilon_k,w|\mathcal{J})}{R(\varepsilon_k)-R(z)}$$

$$= \delta_{0,|\mathcal{J}|}\delta_{g,0} - \lambda\Big\{ \sum_{\substack{\mathcal{I}\uplus\mathcal{I}'=\mathcal{J},\, h+h'=g \\ (\mathcal{I},h)\neq(\emptyset,0)}} \mathcal{G}^{(h')}(z,w|\mathcal{I}')\mathcal{T}^{(h)}(z\|\mathcal{I}) + \mathcal{T}^{(g-1)}(z\|z,w|\mathcal{J})$$

$$+ \sum_{\mathcal{I}\uplus\mathcal{I}'=\mathcal{J},\; h+h'=g} \mathcal{G}^{(h)}(w|\mathcal{I})\frac{\mathcal{G}^{(h')}(z|\mathcal{I}')-\mathcal{G}^{(h')}(w|\mathcal{I}')}{R(w)-R(z)}$$

$$+ \sum_{\beta=2}^{b}\sum_{j=1}^{N_\beta} \frac{\mathcal{G}^{(g)}(w,z,z_j^\beta,..,z_{N_\beta+j-1}^\beta|\mathcal{J}\backslash\{J^\beta\}) - \mathcal{G}^{(g)}(w,z_j^\beta,..,z_{N_\beta+j}^\beta|\mathcal{J}\backslash\{J^\beta\})}{R(z_j^\beta)-R(z)}$$

$$+ \frac{\mathcal{G}^{(g-1)}(z|w|\mathcal{J})-\mathcal{G}^{(g-1)}(w|w|\mathcal{J})}{R(w)-R(z)}\Big\}$$

- Conditions on cardinalities of $\mathcal{J},\mathcal{I},\mathcal{I}'$
- Inversion of first line via Cauchy matrices
- convention $\mathcal{G}^{(g)}(z,w|\mathcal{J}) = \mathcal{T}^{(g)}(\emptyset\|z,w|\mathcal{J}|)$ on next pages

Dyson-Schwinger equations II: $\mathcal{T}$ from Ω

$$(R(w)-R(-z))\mathcal{T}^{(g)}(u_1,\dots,u_m\|z,w|\mathcal{J}|)-\frac{\lambda}{N}\sum_k r_k\frac{\mathcal{T}^{(g)}(u_1,\dots,u_m\|\varepsilon_k,w|\mathcal{J}|)}{R(\varepsilon_k)-R(z)}$$

$$=\delta_{0,m}\delta_{0,|\mathcal{J}|}\delta_{g,0}-\lambda\Bigg\{\sum_{\substack{\mathcal{K}\uplus\mathcal{K}'=\{u_1,..,u_m\}\\ h+h'=g,\,(\mathcal{K},h)\neq(\emptyset,0)}}\mathcal{T}^{(h')}(\mathcal{K}'\|z,w|\mathcal{J}|)\Omega^{(h)}_{|\mathcal{K}+1|}(\mathcal{K},z)$$

$$+\sum_{\substack{\mathcal{I}\uplus\mathcal{I}'=\mathcal{J},\mathcal{I}\neq\emptyset\\ \mathcal{K}\uplus\mathcal{K}'=\{u_1;..;u_m\},\ h+h'=g}}\mathcal{T}^{(h')}(\mathcal{K}'\|z,w|\mathcal{I}'|)\mathcal{T}^{(h)}(\mathcal{K},z\|\mathcal{I}|)$$

$$+\sum_{i=1}^m\frac{\partial}{\partial R(u_i)}\frac{\mathcal{T}^{(g)}(u_1,..,\check{u}_i,..,u_m\|u_i,w|\mathcal{J}|)}{R(u_i)-R(z)}+\mathcal{T}^{(g-1)}(u_1,\dots,u_m,z\|z,w|\mathcal{J}|)$$

$$+\frac{\mathcal{T}^{(g-1)}(u_1,\dots,u_m\|z|w|\mathcal{J}|)-\mathcal{T}^{(g-1)}(u_1,\dots,u_m\|w|w|\mathcal{J}|)}{R(w)-R(z)}$$

$$+\sum_{\beta=2}^b\sum_{j=1}^{N_\beta}\frac{\mathcal{T}^{(g)}(\mathcal{K}\|w,z,z_j^\beta,..,z^\beta_{N_\beta+j-1}|\mathcal{J}\backslash\{J^\beta\}|)-\mathcal{T}^{(g)}(\mathcal{K}\|w,z_j^\beta,..,z^\beta_{N_\beta+j}|\mathcal{J}\backslash\{J^\beta\}|)}{R(z_j^\beta)-R(z)}\Bigg|_{\mathcal{K}=\{u_1,..,u_m\}}$$

$$+\sum_{\substack{\mathcal{I}\uplus\mathcal{I}'=\mathcal{J}\\ \mathcal{K}\uplus\mathcal{K}'=\{u_1,..,u_m\},\ h+h'=g}}\mathcal{T}^{(h)}(\mathcal{K}\|w|\mathcal{I}|)\frac{\mathcal{T}^{(h')}(\mathcal{K}'\|z|\mathcal{I}'|)-\mathcal{T}^{(h')}(\mathcal{K}'\|w|\mathcal{I}'|)}{R(w)-R(z)}\Bigg\}$$

Dyson-Schwinger equation III: $\Omega_2^{(0)}(v, z)$

$$\Omega_2^{(0)}(v,z)R'(z)\mathfrak{G}_0(z) - \frac{\lambda}{N^2}\sum_{n,k=1}^{d} \frac{r_k r_n \mathcal{T}^{(0)}(v\|\varepsilon_k,\varepsilon_n|)}{(R(\varepsilon_k)-R(z))(R(\varepsilon_n)-R(-z))}$$

$$= -\frac{\partial}{\partial R(v)}\big(\mathcal{G}^{(0)}(v,z)+\mathcal{G}^{(0)}(v,-z)\big)$$

where $\mathfrak{G}_0(z) = \mathrm{Res}_{w\to -z}\,\mathcal{G}^{(0)}(z,w)$.

- Seems to need $\mathcal{T}^{(0)}(v\|\varepsilon_k,\varepsilon_n|)$ which itself needs $\Omega_2^{(0)}$.
- Partial fraction decomposition

$$\mathcal{G}^{(0)}(z,v) = \frac{\mathfrak{G}_0(z)}{v+z} + \frac{\lambda^2}{N^2}\sum_{k,l,m,n=1}^{d} \frac{C_{k,l}^{m,n}}{(z+\widehat{\varepsilon}_l{}^n)(z-\widehat{\varepsilon}_k{}^m)(v-\widehat{\varepsilon}_l{}^n)}$$

separates the poles…

DSE III: $\Omega_2^{(0)}(v,z)$ continued

$$R'(z)\Omega_2^{(0)}(v,z) + \frac{\partial}{\partial R(v)}\Big(\frac{1}{v+z}+\frac{1}{v-z}\Big)$$

$$= \frac{1}{\mathfrak{G}_0(z)}\Big[\frac{\lambda}{N^2}\sum_{n,k=1}^{d}\frac{r_k r_n \mathcal{T}^{(0)}(v\|\varepsilon_k,\varepsilon_n|)}{(R(\varepsilon_k)-R(z))(R(\varepsilon_n)-R(-z))} + \frac{\lambda^2}{N^2}\sum_{k,l,m,n=1}^{d}\frac{C_{k,l}^{m,n}\big(\frac{1}{(v-\widehat{\varepsilon}_l{}^n)^2}+\frac{1}{(v-\widehat{\varepsilon}_k{}^m)^2}\big)}{R'(v)(z+\widehat{\varepsilon}_l{}^n)(z-\widehat{\varepsilon}_k{}^m)}\Big]$$

- We do not know what the rhs is, but we know it has at most poles at the zeros of $\mathfrak{G}_0$.
- Assuming $\Omega_2^{(0)}(v,z)$ is regular there, both sides must be constant in z by Liouville, then zero when $z\to\infty$:

Proposition

The cylinder amplitude is

$$\Omega_2^{(0)}(v,z) = \frac{1}{R'(v)R'(z)}\Big(\frac{1}{(v+z)^2}+\frac{1}{(v-z)^2}\Big)$$

One recognises the Bergmann kernel of topological recursion!

Dyson-Schwinger equation IV: Ω from lower $\mathcal{T}, \Omega$

$$R'(z)\mathfrak{G}_0(z)\Omega^{(g)}_{m+1}(u_1,\dots,u_m,z) + \frac{\lambda}{N^2}\sum_{n,k} r_n r_k \frac{\mathcal{T}^{(g)}(u_1,\dots,u_m\|\varepsilon_k,\varepsilon_n|)}{(R(\varepsilon_k)-R(z))(R(\varepsilon_n)-R(-z))}$$

$$= \frac{\delta_{g,0}\delta_{m,1}}{(R(z)-R(u_1))^2} - \sum_{\substack{\mathcal{K}\uplus\mathcal{K}'=\{u_1;..;u_m\}\\ (\mathcal{K},h)\neq(\emptyset,0)\neq(\mathcal{K}',h')\\ h+h'=g}} \Omega^{(h')}_{|\mathcal{K}'+1|}(\mathcal{K}',z)\frac{\lambda}{N}\sum_n r_n \frac{\mathcal{T}^{(h)}(\mathcal{K}\|z,\varepsilon_n|)}{R(\varepsilon_n)-R(-z)}$$

$$-\sum_{i=1}^m \frac{\partial}{\partial R(u_i)} \frac{\frac{\lambda}{N}\sum_n r_n \frac{\mathcal{T}^{(g)}(u_1,..,\check{u}_i,..,u_m\|u_i,\varepsilon_n|)}{R(\varepsilon_n)-R(-z)}}{R(u_i)-R(z)} - \frac{\lambda}{N}\sum_n r_n \frac{\mathcal{T}^{(g-1)}(u_1,\dots,u_m,z\|z,\varepsilon_n|)}{R(\varepsilon_n)-R(-z)}$$

$$-\frac{\lambda}{N}\sum_n r_n \frac{\mathcal{T}^{(g-1)}(u_1,\dots,u_m\|z|\varepsilon_n|)-\mathcal{T}^{(g-1)}(u_,;\dots,u_m\|\varepsilon_n|\varepsilon_n|)}{(R(\varepsilon_n)-R(z))(R(\varepsilon_n)-R(-z))}$$

$$-\sum_{i=1}^m \frac{\partial}{\partial R(u_i)}\mathcal{T}^{(g)}(u_1,..,\check{u}_i,..,u_m\|u_i,z|) + \mathcal{T}^{(g-1)}(u_1,\dots,u_m\|z|z|)$$

Can be solved without knowing the green term!

Miraculously, all poles on rhs (other than $\pm\widehat{\varepsilon}_n^{\,j}$) have prefactor $\mathfrak{G}_0$!

DSE V: cycle type split $(0,1)$ to $(2,0)$

$$
\begin{aligned}
&(R(z)-R(-z))\mathcal{T}^{(g)}(u_1,...,u_m\|z|\mathcal{J}|) - \frac{\lambda}{N}\sum_k r_k \frac{\mathcal{T}^{(g)}(u_1,...,u_m\|\varepsilon_k|\mathcal{J}|)}{R(\varepsilon_k)-R(z)} \\
&= -\lambda\Big\{ \sum_{\substack{\mathcal{I}\uplus\mathcal{I}'=\mathcal{J},\,\mathcal{I}\neq\emptyset \\ \mathcal{K}\uplus\mathcal{K}'=\{u_1,..,u_m\},\; h+h'=g}} \mathcal{T}^{(h')}(\mathcal{K}'\|z|\mathcal{I}'|)\mathcal{T}^{(h)}(\mathcal{K},z\|\mathcal{I}|) + \sum_{\substack{\mathcal{K}\uplus\mathcal{K}'=\{u_1,..,u_m\} \\ h+h'=g,\,(\mathcal{K},h)\neq(\emptyset,0)}} \mathcal{T}^{(h')}(\mathcal{K}'|z|\mathcal{J}|)\Omega^{(h)}_{|\mathcal{K}+1|}(\mathcal{K},z) \\
&+ \sum_{i=1}^m \frac{\partial}{\partial R(u_i)} \frac{\mathcal{T}^{(g)}(u_1,..,\check{u}_i,..,u_m\|u_i|\mathcal{J}|)}{R(u_i)-R(z)} + \mathcal{T}^{(g-1)}(u_1,...,u_m,z\|z|\mathcal{J}|) \\
&+ \sum_{\beta=2}^b \sum_{j=1}^{N_\beta} \frac{\mathcal{T}^{(g)}(\mathcal{K}\|z,z_j^\beta,..,z^\beta_{N_\beta+j-1}|\mathcal{J}\backslash\{J^\beta\}|) - \mathcal{T}^{(g)}(\mathcal{K}\|z_j^\beta,..,z^\beta_{N_\beta+j}|\mathcal{J}\backslash\{J^\beta\}|)}{R(z_j^\beta)-R(z)}\Big|_{\mathcal{K}=\{u_1,..,u_m\}} \Big\}
\end{aligned}
$$

- $\mathcal{G}^{(g)}(z|\mathcal{J}) = \mathcal{T}^{(g)}(\emptyset\|z|\mathcal{J})$ $\quad (m=0)$
- first line inverted via Cauchy matrices at $z=\alpha_k$
- necessary input for genus increase $\Omega^{(g)}(z) \mapsto \Omega^{(g+1)}(z)$

The solution is complete

- We study the quartic analogue of the Kontsevich model $d\mu_{E,\lambda}$, parametrised by coupling constant λ and spectral values $E_1, \dots, E_d$ of multiplicities $r_1, \dots, r_d$ with $r_1 + \dots + r_d = N$.
- Main obstacle was the non-linear equation for the planar 2-point function obtained with Harald in 2009.
  - The breakthrough was to solve the 2D Moyal case with Erik in Les Houches 2018.
  - After Alex' insight in that particular solution we solved (with Harald & Alex) last year the general continuous case.
  - The even deeper discrete case with its link to algebraic geometry was solved last December with Jörg.
- The solution introduced a rational function

$$R(z) = z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{\varepsilon_k + z} \quad \text{with } R(\varepsilon_k) = E_k \ \& \ \varrho_k R'(\varepsilon_k) = r_k.$$

The solution is complete

- All other correlation functions (or cumulants) satisfy (after meromorphic extension and genus expansion) affine equations.
- To solve them we need partial sums $\mathcal{T}^{(g)}(u_1, ..., u_m \| z, w | .... |)$, $\mathcal{T}^{(g)}(u_1, ..., u_m \| z | w | .... |)$ and $\Omega_n^{(g)}(u_1, ..., u_m)$.
- We know how to invert the linear operators in all these loop equations. This is very easy for $\mathcal{T}^{(g)}$ (unless one wants to simplify the expression and to make the symmetry manifest).
- Extracting $\Omega_n^{(g)}$ in practice becomes difficult for large $2g + n$. So far we have $\Omega_2^{(0)}, \Omega_3^{(0)}, \Omega_4^{(0)}, \Omega_1^{(1)}$ and a third of $\Omega_2^{(1)}$.
- The final results for these $\Omega_n^{(g)}$ are remarkably simple. $\omega_n^{(g)}(z_1, \dots, z_n) := (\prod_{l=1}^n R'(z_l))\Omega_n^{(g)}(z_1, \dots, z_n)$ has poles only at $z_k = -z_l$ and $z_k = \beta_i$ where $R'(\beta_i) = 0$ (the branch points of the curve), but not at $z_k = \pm\widehat{z}_l^j$ or $z_l = \pm\widehat{\varepsilon_k}^j$.

Analogies to the Hermitian 2-matrix model

- It is more or less obvious that the $\omega_n^{(g)}$ are the objects usually studied in topological recursion [Eynard-Orantin 07].
- The need to go beyond TR was previously observed in the Hermitian 2-matrix model where the boundaries of Riemann surfaces are bicoloured (or spins ± 1: Ising model).
- The uniformly coloured sectors $H^{(g)}_{0,m,0}$ and $H^{(g)}_{0,0,n}$ relate to topological recursion. The sectors $H^{(g)}_{0,m,n}$ and $H^{(g)}_{1,m,n}$ are computed in a triangular pattern.
- $H^{(g)}_{0,m,0}(z_1, \dots, z_m)$ and $H^{(g)}_{0,0,n}(z_1, \dots, z_n)$ have poles only at branch points of curve x and $z_k = -z_l$ (not $z_k = 0$). The mixed sectors also have poles at $z_k = \widehat{z}_l^j$.

Blobbed topological recursion?

- At fixed g, our $\Omega_n^{(g)}, \mathcal{T}^{(g)}$ seem to play the same rôle: $\Omega_n^{(g)}$ correspond to uniform sector, $\mathcal{T}^{(g)}$ to mixed sector.
- The part of $\Omega_1^{(1)}$ with poles at branch points is identical to the 2-matrix model when taking the spectral curve

$$\boxed{x(z) = R(z), \quad y(z) = -R(-z)}$$

On top of it we have an additional pole at $z = 0$ from $\mathcal{G}^{(0)}(z|z)$.

It seems that the 2-matrix model and our quartic model are two examples of an extension of topological recursion.

- We currently investigate whether it could be the blobbed topological recursion [Borot-Shadrin 15].
- Our hierarchy of loop equations generates the blobs as additional input for the recursion.

Integrability?

Understanding better our recursion should give access to the partition function itself. It is a function of λ and the spectrum of E.

- At fixed genus it is very likely a polynomial with rational coefficients in two sets of variables

$$t_k^{i+} = \frac{R^{(k+2)}(\beta_i)}{R''(\beta_i)}, \qquad t_k^{i-} = \frac{R^{(k+1)}(-\beta_i)}{R'(-\beta_i)}$$

  These contain the λ-dependence (exactly).
- Because our recursion inserts blobs at every next step (in a controlled way!), it will differ from known examples.

Main question

- Is it nevertheless a τ-function for a Hirota equation, i.e. is it integrable?
- If so, are the rational coefficients intersection numbers of some characteristic classes on a moduli space?

Backup: Topological recursion [Eynard, Orantin 07]

Starting from a spectral curve consisting of

- a branched covering $x : \Sigma \to \Sigma_0$ of Riemann surfaces,
- meromorphic differentials $\omega_{0,1}$ on Σ and $\omega_{0,2}$ on $\Sigma \times \Sigma$,

recursively construct family $\omega_{g,n}$ of meromorphic n-differentials on Σ^n, with poles only at ramification points of x, by

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_a \operatorname*{Res}_{z\to a} K(z_1, z, \sigma_a(z))dz \Big(\omega_{g-1,n+1}(z, \sigma_a(z), z_2, \dots, z_n) + \sum'_{\substack{g_1+g_2=g \\ I_1 \uplus I_2 = \{z_2,\dots,z_n\}}} \omega_{g_1,1+\# I_1}(z, I_1)\omega_{g_2,1+\# I_2}(\sigma_a(z), I_2) \Big)$$

[sum over branch points a of x; local involution $x(z) = x(\sigma_a(z))$ near a; recursion kernel $K(z_1, z_2, z_3) = \frac{\frac{1}{2}\int_{z'=z_3}^{z_2} \omega_{0,2}(z_1, z')}{\omega_{0,1}(z_2) - \omega_{0,1}(z_3)}$]

Examples

one- and two-matrix models, Kontsevich model, Weil-Petersson volumes, Hurwitz numbers, Gromov-Witten numbers, ...

Topological recursion of the Kontsevich model

- branched cover $x : \hat{\mathbb{C}} \ni z \mapsto z^2 \in \hat{\mathbb{C}}$, where $z = (4\zeta^2 + c)^{1/2}$
- $\omega_{0,1}(z) = 2zy(-z)dz$ with $-y(-z) = z + \frac{1}{N}\sum_{k=1}^{N} \frac{1}{2\varepsilon_k(z+\varepsilon_k)}$ (related to planar 1-point function), $\varepsilon_k = (4E_k^2 + c)^{1/2}$
- $\omega_{0,2}(z, z') = \frac{dz\, dz'}{(z-z')^2}$ (related to planar 1+1-point function)

Meromorphic differentials relate to higher correlation functions $\omega_{g,n}(z_1, \dots, z_n) = \mathcal{G}^{(g)}(z_1|\dots|z_n)\prod_{i=1}^{n} d(x(z_i))$, where

$$\mathcal{G}^{(g)}(z_1|\dots|z_n) = (2-t_3)^{2-2g-n}\sum_{l_1,\dots,l_n}\Big\langle \psi_1^{l_1}\cdots\psi_n^{l_n} e^{\sum_k \hat{t}_k\kappa_k}\Big\rangle_{g,n}\prod_{i=1}^{n}\frac{(2l_i+1)!!}{z_i^{2l_i+3}}$$

- ψ_i, κ_k are tautological characteristic classes on $\overline{\mathcal{M}}_{g,n}$ and $\langle\dots\rangle_{g,n}$ their intersection numbers
- $e^{-\sum_k \hat{t}_k u^{-k}} = 1 - \frac{1}{2}\sum_l (2l+1)!!\, t_{2l+1} u^{-l}, \qquad t_l = \frac{1}{N}\sum_{k=1}^{N}\varepsilon_k^{-2l-1}$