Solvable Dyson-Schwinger equations

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contributed to “Algebraic Structures in Perturbative Quantum Field Theory”

based on collaboration with

Harald Grosse, Erik Panzer, Alex Hock, Jörg Schürmann & Johannes Branahl

Introduction

- In March 1998 (shortly after I started as postdoc at CPT Marseille), a larger group of us attended a conference on noncommutative geometry in Vietri sul Mare (Italy).
- Alain Connes reported on a ground-breaking result by a physicist Dirk Kreimer who discovered in q-alg/9707029 that renormalisation in quantum field theory is encoded in a Hopf algebra.
- Remarkably, this Hopf algebra is closely related to another Hopf algebra which emerges in the computation of the local index formula for transverse hypoelliptic operators [Connes-Moscovici 98].
- All participants understood that this is a development of greatest importance. In Marseille we stopped all other projects and tried to understand the results.

Overlapping divergences

- With Thomas Krajewski we understood the generic cases, but had problems with overlapping divergences.
- Dirk accepted an invitation to Marseille for the end of May 1998. As a basis for discussion, Thomas and I made our notes available as arXiv:hep-th/9805098:

On Kreimer's Hopf algebra of Feynman graphs

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Abstract. We reinvestigate Kreimer's Hopf algebra structure of perturbative quantum field theories with a special emphasis on overlapping divergences. Kreimer first disentangles overlapping divergences into a linear combination of disjoint and nested ones and then tackles that linear combination by the Hopf algebra operations. We present a formulation where the Hopf algebra operations are directly defined on any type of divergence. We explain the precise relation to Kreimers Hopf algebra and obtain thereby a characterization of their primitive elements.

Dyson-Schwinger equations

Dirk's e-mail from 13 May 1998

'It is actually so that the problem of overlapping divergences can be totally resolved using the construction as given in q-alg/9707029, though the paper is succinct and assumes that the reader digested the use of the Schwinger Dyson equation as indicated in Fig.5 in that paper. This needs reading of section 6 of my Habil Thesis (J.Knot Th.Ram.6 (1997) 479-581).'

I cannot contribute to the Hopf algebra of Feynman graphs and refer to talks by Walter, Alain, Thomas and others.

But I am happy to contribute to Dyson-Schwinger equations. It is true that I hadn't digested them in 1998. In the meantime they became my strongest tool . . .

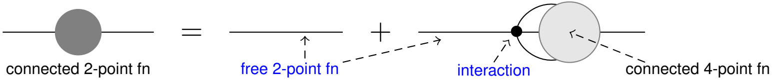
INSPIRE “f a Kreimer and t Dyson”

- D. Kreimer, “Dyson-Schwinger equations: Fix-point equations for quantum fields”
- O. Krüger and D. Kreimer, “Filtrations in Dyson-Schwinger equations: Next-to$^{\{j\}}$-leading log expansions systematically”
- A. Youssef and D. Kreimer, “Resummation of infrared logarithms in de Sitter space via Dyson-Schwinger equations: the ladder-rainbow approximation”
- A. Tanasa and D. Kreimer, “Combinatorial Dyson-Schwinger equations in noncommutative field theory”
- G. van Baalen, D. Kreimer, D. Uminsky and K. Yeats, “The QCD beta-function from global solutions to Dyson-Schwinger equations”
- G. van Baalen, D. Kreimer, D. Uminsky and K. Yeats, “The QED beta-function from global solutions to Dyson-Schwinger equations”
- D. Kreimer, “Dyson Schwinger equations: From Hopf algebras to number theory”
- D. Kreimer and K. Yeats, “An Étude in non-linear Dyson-Schwinger Equations”
- C. Bergbauer and D. Kreimer, “Hopf algebras in renormalization theory: Locality and Dyson-Schwinger equations from Hochschild cohomology”
- D. Kreimer, “What is the trouble with Dyson-Schwinger equations?”
- D. J. Broadhurst and D. Kreimer, “Exact solutions of Dyson-Schwinger equations for iterated one loop integrals and propagator coupling duality”

Dyson-Schwinger equations

...are quantum equations of motion for Green functions in a QFT.

- Can be graphically understood when collecting Feynman graph series of the same external structure into blobs:



- This graphical picture relies on perturbation theory. However, the equations between blobs can be rigorously derived without any reference to formal power series.

Dyson-Schwinger equations thus provide a non-perturbative definition of QFTs — provided we can solve these equations

- Difficulty: n-point function needs $(m>n)$-point function
- Can be resolved in QFT on finite-dim. approximations of noncommutative geometries (matrix models)

Free Euclidean fields on noncommutative geometries

Let H_N be the real vector space of self-adjoint $N \times N$-matrices, and $(E_1, \ldots, E_N)$ be (increasing) positive real numbers.

Theorem [Bochner 1933, Schur 1911]

For any inner product $\langle\ ,\ \rangle$ on H_N there exists a unique probability measure $d\mu_0$ on the dual space H'_N with

$$\exp\Big(-\frac{1}{2}\langle M, M\rangle\Big) = \int_{H'_N} d\mu_0(\Phi)\, e^{\mathrm{i}\Phi(M)} \qquad \forall M{=}(M_{kl}) \in H_N\,.$$

Choose $\langle M, M'\rangle_E = \frac{1}{N}\sum_{k,l=1}^{N} \frac{M_{kl}M'_{lk}}{E_k + E_l}$ and corresponding $d\mu_{E,0}$

- Defines the free Euclidean scalar field on N-dimensional approximation of a noncommutative geometry.
- $(E_1, \ldots, E_N)$ is truncated spectrum of the Laplacian.
- All moments can be described explicitly.

The Kontsevich model and its quartic analogue

3. The Kontsevich model $d\mu_{E,\lambda}(\Phi) = \dfrac{e^{-\frac{\lambda N}{3}\mathrm{Tr}(\Phi^3)} d\mu_{E,0}(\Phi)}{\int_{H'_N} e^{-\frac{\lambda N}{3}\mathrm{Tr}(\Phi^3)} d\mu_{E,0}(\Phi)}$

   - Computes intersection numbers of tautological characteristic classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves [Kontsevich 92].
   - It is integrable via a relation (suggested by [Witten 91]) to the KdV hierarchy. Its moments obey topological recursion.

4. A quartic analogue $d\mu_{E,\lambda}(\Phi) = \dfrac{e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_{E,0}(\Phi)}{\int_{H'_N} e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_{E,0}(\Phi)}$

   - Although perturbatively far apart, we find very similar algebraic geometrical structures. Our solutions are exact in λ.

Equations of motion for quartic Kontsevich model

Fourier transform $\mathcal{Z}(M) := \int_{H'_N} d\mu_{E,\lambda}(\Phi)\, e^{\mathrm{i}\Phi(M)}$ satisfies

1. $$-N(E_p - E_q)\sum_{k=1}^{N} \frac{\partial^2 \mathcal{Z}(M)}{\partial M_{pk}\partial M_{kq}} = \sum_{k=1}^{N}\Big(M_{kp}\frac{\partial \mathcal{Z}(M)}{\partial M_{kq}} - M_{qk}\frac{\partial \mathcal{Z}(M)}{\partial M_{pk}}\Big)$$
2. $$\frac{1}{N}\frac{\partial \mathcal{Z}(M)}{\partial E_p} = \sum_{k=1}^{N} \frac{\partial^2 \mathcal{Z}(M)}{\partial M_{pk}\partial M_{kp}} + \mathcal{Z}(M)\int_{H'_N} d\mu_{E,\lambda}(\Phi)\frac{1}{N}\sum_{k=1}^{N}\Phi_{pk}\Phi_{kp}$$

- They allow to express $\sum_{k=1}^{N} \frac{\mathcal{Z}(M)}{\partial M_{pk}\partial M_{kq}}$ in Dyson-Schwinger equations by fewer derivatives, i.e. of same or lower order.
- Eq. 1 can be used for $p \neq q$, whereas $p = q$ requires 2.
- Dyson-Schwinger equations complexify to equations for meromorphic functions in several complex variables in which we admit multiplicities $(E_1, \dots, E_N) = (\underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_d, \dots, e_d}_{r_d})$

Dyson-Schwinger equation for planar 2-point function

For $p \neq q$, expand $N \int_{H'_N} d\mu_{E,\lambda}(\Phi)\ \Phi_{pq}\Phi_{qp} =: \sum_{g=0}^{\infty} N^{-2g} Z G^{(g)}_{|pq|}$.

Then $G^{(g)}_{|pq|} = G^{(g)}(\zeta,\eta)\big|_{\zeta=e_p,\eta=e_q}$ with initial equation [Grosse-W 09]

$$\Big(\mu^2_{bare} + \xi + \eta + \frac{\lambda}{N}\sum_{k=1}^{d} r_k ZG^{(0)}(\zeta,e_k)\Big) ZG^{(0)}(\zeta,\eta)$$

$$= 1 + \frac{\lambda}{N}\sum_{k=1}^{d} r_k \frac{ZG^{(0)}(e_k,\eta) - ZG^{(0)}(\zeta,\eta)}{e_k - \zeta}$$

Z, μ_{bare}: renormalisation parameters

- In [Panzer-W 18] we solved this equation for $r_k = 1$, $e_k = \frac{k}{N}$ in large-N limit, corresponding to $\lambda\Phi^4$ on 2D-Moyal space.
- Key step was to resum perturbative results (obtained with `HyperInt`) for an auxiliary function to Lambert-W.
- In [Grosse-Hock-W 19] we understood the general solution. Find ${}_2F_1$ for 4D Moyal. See Alex Hock's talk at 15h15.

Solution for finite matrices

Theorem [Grosse-Hock-W 19, Schürmann-W 19]

Let $(\varepsilon_k, \varrho_k)$ be implicitly defined by $e_k = R(\varepsilon_k),\; r_k = R'(\varepsilon_k)\varrho_k$

for $R(z) = z - \dfrac{\lambda}{N}\sum_{k=1}^{d} \dfrac{\varrho_k}{z+\varepsilon_k}$.

Then $G^{(0)}(\zeta,\eta) = \mathcal{G}^{(0)}(z,w)$ for $R(z)=\zeta$, $R(w)=\eta$ and

$$\mathcal{G}^{(0)}(z,w) = \frac{1 - \dfrac{\lambda}{N}\displaystyle\sum_{k=1}^{d} \frac{r_k \displaystyle\prod_{j=1}^{d} \frac{R(w)-R(-\widehat{\varepsilon}_k^{\,j})}{R(w)-R(\varepsilon_j)}}{(R(z)-R(\varepsilon_k))(R(\varepsilon_k)-R(-w))}}{R(w)-R(-z)}$$

where $u \in \{z, \hat{z}^1, \dots, \hat{z}^d\}$ are all solutions of $R(u)=R(z)$.

(The symmetry $\mathcal{G}^{(0)}(z,w)=\mathcal{G}^{(0)}(w,z)$ is automatic)

Thus, planar 2-point function solved by the composition of a rational function $\mathcal{G}^{(0)}$ with inverse of another rational function R.

Remarks

We succeeded in solving a non-linear (D-S) equation.

- First (with Erik) by brute force and luck in a special case, later by the beauty of complex analysis.
- There must be a hidden algebraic structure which made this possible. We are confident to find it in the affine equations [with Johannes Branahl & Alex Hock].

Message to retain

Original model had spectrum $(\underbrace{e_1,\dots,e_1}_{r_1},\dots,\underbrace{e_d,\dots,e_d}_{r_d})$, coupling λ.

But in these variables the structure is completely obscure!

- The structure emerges when transforming via R^{-1}, with $R(z)=z-\frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{z+\varepsilon_k}$
- Q: Is something analogous true in familar QFT, i.e. can we possiby uncover some deep structure after transformation (to discover) to more appropriate variables?

The affine equations

- All other correlations functions satisfy affine equations. They are always solvable, but no path seemed to exist.
- Alex Hock: need first to look at auxiliary functions!

Recall that $d\mu_{E,\lambda}$ depends on given family $E_1, \dots, E_N$. Introduce

$$\sum_{g=0}^{\infty} N^{2-2g-n}\Omega^{(g)}_{a_1,\dots,a_n} := \frac{\partial^{n-1}\Big(N\sum_{k=1}^{N}\int_{H'_N} d\mu_{E,\lambda}(\Phi)\Phi_{a_1k}\Phi_{ka_1}\Big)}{\partial E_{a_2}\cdots\partial E_{a_n}} + \frac{\delta_{n,2}}{(E_{a_1}-E_{a_2})^2}$$

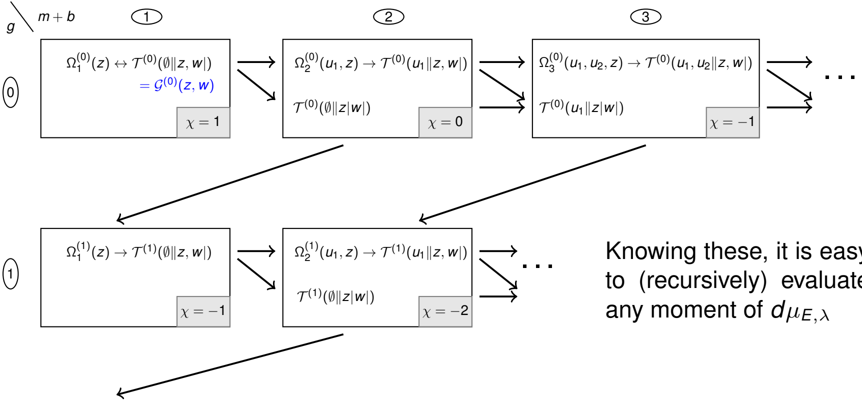
- We derive and solve Dyson-Schwinger equations for (meromorphic continuation of) $\Omega^{(g)}$.
- This needs R and $\mathcal{G}^{(0)}$, but no prior knowledge of its E-derivatives and of 2-point functions of higher topology.

Unexpected result: The $\Omega^{(g)}$ translate to differential forms which obey blobbed topological recursion [Borot-Shadrin 15]!

Solution procedure [Branahl-Hock-W 20]

Three types of functions involved:

- $\Omega^{(g)}_m(u_1, \dots, u_m)$ objects of BTR, most difficult to compute
- $\mathcal{T}^{(g)}(u_1, \dots, u_m \| z, w|)$ auxiliary functions, easy to compute
- $\mathcal{T}^{(g)}(u_1, \dots, u_m \| z|w|)$ auxiliary functions, easy to compute



Knowing these, it is easy to (recursively) evaluate any moment of $d\mu_{E,\lambda}$

Results

Proposition

$$\Omega_2^{(0)}(u,z) = \frac{1}{R'(u)R'(z)}\Big(\frac{1}{(u-z)^2} + \frac{1}{(u+z)^2}\Big)$$

One recognises the Bergman kernel of topological recursion!

Suggests $\omega_{g,m}(z_1,...,z_m) = \lambda^{2-2g-m}\Omega_m^{(g)}(z_1,...,z_m)\prod_{k=1}^{m} dR(z_i)$

Proposition ($g=0$) / Conjecture ($g>0$)

$z \mapsto \omega_{g,m}(u_1,...,u_{m-1},z)$ is meromorphic with poles at $z \in \{0,-u_1,...,-u_{m-1},\beta_1,..,\beta_{2d}\}$ where $R'(\beta_i)=0$ (ramification points)

Gives residue formula for $\omega_{g,m}$ into which solutions of the Dyson-Schwinger equations for $\mathcal{T}^{(g)}(u_1,...,u_m\|z,w|)$ and $\mathcal{T}^{(g)}(u_1,\dots,u_m\|z|w|)$ are inserted. Many cancellations arise.

Solution at low $-\chi = 2g + m - 2$

$$\omega_{0,3}(u_1,u_2,z) = -\sum_{i=1}^{2d} \frac{\Big(\frac{1}{(u_1-\beta_i)^2}+\frac{1}{(u_1+\beta_i)^2}\Big)\Big(\frac{1}{(u_2-\beta_i)^2}+\frac{1}{(u_2+\beta_i)^2}\Big)du_1\,du_2\,dz}{R'(-\beta_i)R''(\beta_i)(z-\beta_i)^2} + \Big[d_{u_1}\Big(\frac{\omega_{0,2}(u_2,u_1)}{(dR)(u_1)}\frac{dz}{R'(-u_1)(z+u_1)^2}\Big) + u_1 \leftrightarrow u_2\Big]$$

$$\omega_{1,1}(z) = \sum_{i=1}^{2d} \frac{dz}{R'(-\beta_i)R''(\beta_i)}\Big\{-\frac{1}{8(z-\beta_i)^4}+\frac{R'''(\beta_i)}{24R''(\beta_i)(z-\beta_i)^3} + \frac{\frac{R''''(\beta_i)}{48R''(\beta_i)} - \frac{(R'''(\beta_i))^2}{48(R''(\beta_i))^2} + \frac{R''(-\beta_i)R'''(\beta_i)}{48R'(-\beta_i)R''(\beta_i)} + \frac{(R''(-\beta_i))^2}{48(R'(-\beta_i))^2} - \frac{1}{8\beta_i^2}}{(z-\beta_i)^2}\Big\} - \frac{dz}{8(R'(0))^2z^3} + \frac{R''(0)dz}{16(R'(0))^3z^2}$$

- Reflect (convergent!) summation of infinite series of Feynman (ribbon) graphs of fixed external structure and topology.
- The λ-series results by solving the system

  $$R(\varepsilon_k) = e_k,\ R'(\varepsilon_k)\varrho_k = r_k,\ R'(\beta_i) = 0 \text{ and } z = R^{-1}(\zeta)$$

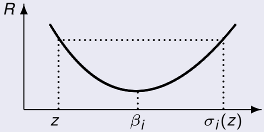
  via Taylor approach to the implicit function theorem.

Abstract loop equations [Borot-Eynard-Orantin 13]

Proposition $(g,m) \in \{(0,2),\dots,(0,5),(1,1)\}$ / Conjecture

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the ramified cover identified in the solution of $\mathcal{G}^{(0)}(z,w)$.

Let $\beta_1, \dots, \beta_{2d}$ be the ramification points of R and σ_i be the corresponding local Galois involution in the vicinity of β_i.



Define $\omega_{0,1}(z) = -R(-z)R'(z)dz$ and for $2-2g-m \leq 0$ the $\omega_{g,m}$ as before. Then:

1. linear loop equation:
$$\omega_{g,m}(u_1,\dots,u_{m-1},z) + \omega_{g,m}(u_1,\dots,u_{m-1},\sigma_i(z)) = \mathcal{O}(z-\beta_i)dz$$
2. quadratic loop equation:
$$\omega_{g-1,m+1}(u_1,\dots,u_{m-1},z,\sigma_i(z)) + \sum_{\substack{I_1 \uplus I_2 = \{u_1,\dots,u_{m-1}\} \\ g_1+g_2=g}} \omega_{g_1,|I_1|+1}(I_1,z)\,\omega_{g_2,|I_2|+1}(I_2,\sigma_i(z)) = \mathcal{O}((z-\beta_i)^2)(dz)^2$$

Blobbed topological recursion [Borot-Shadrin 15]

Theorem

Let $\{\omega_{g,m}\}_{g\geq 0,m>0}$ be a family of meromorphic differential forms which satisfy the abstract loop equations. Then their parts $\mathcal{P}\omega_{g,m}$ containing the poles at ramification points are given by

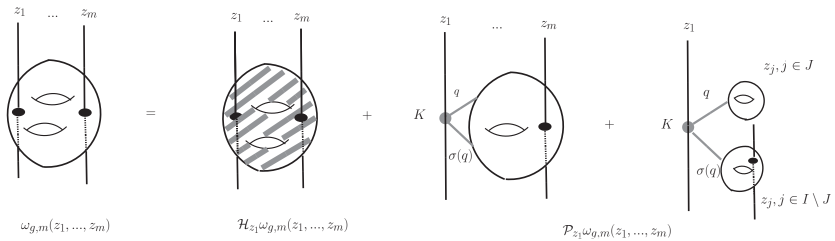
$$\mathcal{P}_z\omega_{g,m}(u_1,\dots,u_{m-1},z)$$

$$= \sum_{i=1}^{2d} \operatorname*{Res}_{q\to\beta_i} \frac{\frac{1}{2}\int_{q'=\sigma(q)}^{q'=q} B(z,q')}{\omega_{0,1}(q)-\omega_{0,1}(\sigma_i(q))}\Big(\omega_{g-1,m+1}(u_1,\dots,u_{m-1},q,\sigma_i(q)) + \sum_{\substack{I_1\uplus I_2=\{u_1,\dots,u_{m-1}\}\\ g_1+g_2=g\\ (I_1,g_1)\neq(\emptyset,0)\neq(I_2,g_2)}} \omega_{g_1,|I_1|+1}(I_1,q)\omega_{g_2,|I_2|+1}(I_2,\sigma_i(q))\Big)$$

where $B(u,z)=\frac{du\,dz}{(u-z)^2}$ is the Bergman kernel (for $x:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$).

$\mathcal{H}_z\omega_{g,m}(\dots,z) := \omega_{g,m}(\dots,z)-\mathcal{P}_z\omega_{g,m}(\dots,z)$ is made of blobs.

A picture



$\omega_{g,m}$ = meromorphic forms on space of compactified complex lines through the marked points on a genus-g Riemann surface.

- The universal formula of topological recursion produces the parts $\mathcal{P}\omega_{g,m}$ from the entire $\omega_{g',m'}$ of smaller degree.
- The parts $\mathcal{H}\omega_{g,m}$ are additional input at every recursion step. We are confident to understand them soon.

The quartic analogue of the Kontsevich model distinguishes a unique such form $\omega_{g,m}$ for every (g, m). What is its significance?

Intersection numbers and integrability

Fact [Borot-Shadrin 15]

- Forms $\omega_{g,m}$ which satisfy BTR encode intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,m}$ of stable complex curves.
- These are several copies of the same intersections of ψ, κ-classes as in the Kontsevich model, coupled via blobs.

These coupled intersections could be interesting or not. Since the global involution $z \to -z$ is very natural we expect that blobs about its fixed point $z = 0$ could be significant.

Integrability

- Understanding better our recursion should give access to the partition function itself, a function of λ and (E_i).
- Is it a τ-function for a Hirota equation, i.e. is it integrable? [not known in general BTR]

Summary

- Dyson-Schwinger equations resolve the problem of overlapping divergences in the Hopf algebra of Feynman graphs.
- They are a central research topic for Dirk and for me.
- I have tried to convince you that, at least for some QFT toy models, Dyson-Schwinger equations provide the best non-perturbative approach. They can lead to a complete understanding.

@Dirk

I wish you a lot of pleasure and success with your work on Dyson-Schwinger equations.

Happy Birthday!