The complete solution of the quartic analogue of the Kontsevich model

Raimar Wulkenhaar

Mathematisches Institut der Westfälischen Wilhelms-Universität Münster

MM Mathematics Münster Cluster of Excellence

based on joint work with
Harald Grosse, Erik Panzer, Alexander Hock, Jörg Schürmann
and work in progress with Johannes Branahl and Alexander Hock

Introduction

This project started in 1998 as an attempt to understand quantum field theories on noncommutative geometries.

- No interacting and mathematically consistent QFT is known in 4 dimensions.
- The hope was that the situation could improve on noncommutative spaces.
- Many nice results were obtained on renormalisation, β-function, construction in 2D, extension to tensor models.

We know since 2003 that there are parallels with the Kontsevich model (which beautifully links mathematics and physics).

Recent progress

when we understood that topological recursion, a universal structure behind many developments in algebraic geometry (including the Kontsevich model), is also present in our case.

Solvable matrix models

...provide deep relations between many fields of mathematics and physics:

- Hermitian 1-matrix model $\mathcal{Z} = \int_{X_N} dM\, e^{-\mathrm{tr}(\text{polynomial}(M))}$ is solvable (X_N – space of self-adjoint $N \times N$-matrices) [Brézin-Kazakov, Douglas-Shenker, Gross-Migdal 89/90]
- It is deeply equivalent to the [Kontsevich 91]-model. It proves a conjecture by [Witten 91] about 2D quantum gravity.
- The Hermitian 2-matrix model is solvable [many people].
- Understanding the common mechanism led to the discovery of topological recursion by Eynard and others.

We are now sure to have another solvable matrix model: the quartic analogue of the Kontsevich model.

Free Euclidean fields on noncommutative geometries

Let X_N be the real vector space of self-adjoint $N \times N$-matrices, and $(E_1, \dots, E_N)$ be (increasing) positive real numbers.

Theorem [Bochner 1933, Schur 1911]

For any inner product $\langle\ ,\ \rangle$ on X_N there exists a unique probability measure $d\mu_0$ on the dual space X'_N with

$$\exp\Big(-\frac{1}{2}\langle M, M\rangle\Big) = \int_{X'_N} d\mu_0(\Phi)\; e^{i\Phi(M)} \qquad \forall M{=}(M_{kl}) \in X\ .$$

Choose $\langle M, M'\rangle_E = \frac{1}{N}\sum_{k,l=1}^{N} \frac{M_{kl}M'_{lk}}{E_k + E_l}$ and corresponding $d\mu_{E,0}$

- Defines the free Euclidean scalar field on N-dimensional approximation of a noncommutative geometry.
- $(E_1, \dots, E_N)$ is truncated spectrum of the Laplacian.

Next: Deform $d\mu_{E,0}(\Phi)$ and study large-N asymptotics.

Two deformations

3. The Kontsevich model $d\mu_{E,\lambda}(\Phi) = \dfrac{e^{-\frac{\lambda N}{3}\mathrm{Tr}(\Phi^3)} d\mu_{E,0}(\Phi)}{\int_{X'_N} e^{-\frac{\lambda N}{3}\mathrm{Tr}(\Phi^3)} d\mu_{E,0}(\Phi)}$
   - Computes intersection numbers of tautological characteristic classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves.
   - It is integrable via a relation (suggested by Witten) to the KdV hierarchy. Its moments satisfy topological recursion.

4. A quartic analogue $d\mu_{E,\lambda}(\Phi) = \dfrac{e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_{E,0}(\Phi)}{\int_{X'_N} e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_{E,0}(\Phi)}$
   - Although perturbatively far apart, we find very similar algebraic geometrical structures. Our solutions are exact in λ.

Moments and cumulants

$$\int_{X'_N} d\mu_{E,\lambda}(\Phi) \prod_{i=1}^{n} \Phi(e_{k_i l_i}) =: \Big\langle \prod_{i=1}^{n} e_{k_i l_i} \Big\rangle = \sum_{\substack{\text{partitions} \\ \pi \text{ of } \{1,\dots,n\}}} \prod_{\substack{\text{blocks} \\ \beta\in\pi}} \Big\langle \prod_{i\in\beta} e_{k_i l_i} \Big\rangle_c$$

- A culumulant $\big\langle \prod_{i=1}^{n} e_{k_i l_i} \big\rangle_c$ is only non-zero if $(l_1,\dots,l_n) = (k_{\sigma(1)},\dots,k_{\sigma(n)})$ for permutation σ, and only the cycle type matters.
- Adapted notation for b cycles of lengths $n_1,\dots,n_b$:

$$N^{n_1+..+n_b} \Big\langle \prod_{j=1}^{b} \Big(\prod_{i=1}^{n_j} e_{k_i^j k_{i+1}^j} \Big) \Big\rangle_c =: N^{2-b} G_{|k_1^1 \dots k_{n_1}^1|\dots|k_1^b\dots k_{n_b}^b|}$$

- Expansion $\mathcal{Z}(M) := \int_{X'_N} d\mu_{E,\lambda}(\Phi)\, e^{\mathrm{i}\Phi(M)}$

cycle type (0,1) cycle type (2,0)

$$= 1 - \frac{1}{N^2} \sum_{k,l=1}^{N} \Big(N G_{|kl|} \frac{M_{kl}M_{lk}}{1!\cdot 2^1} + G_{|k|l|} \frac{M_{kk}M_{ll}}{2!\cdot 1^2} \Big) + \mathcal{O}(M^4)$$

Equations of motion

Fourier transform $\mathcal{Z}(M) := \int_{X'_N} d\mu_{E,\lambda}(\Phi)\, e^{\mathrm{i}\Phi(M)}$ satisfies (in quartic case)

$$\frac{1}{\mathrm{i}}\frac{\partial \mathcal{Z}(M)}{\partial M_{ab}} = \frac{\mathrm{i}M_{ba}\mathcal{Z}(M)}{N(E_a+E_b)} - \frac{\lambda}{\mathrm{i}^3(E_a+E_b)}\sum_{k,l=1}^{N}\frac{\partial^3 \mathcal{Z}(M)}{\partial M_{ak}\partial M_{kl}\partial M_{lb}}\,.$$

$$\frac{1}{N}\frac{\partial \mathcal{Z}(M)}{\partial E_a} = \Big(\sum_{k=1}^{N}\frac{\partial^2}{\partial M_{ak}\partial M_{ka}} + \frac{1}{N}\sum_{k=1}^{N} G_{|ak|} + \frac{1}{N^2} G_{|a|a|}\Big)\mathcal{Z}(M)$$

They give rise to Dyson-Schwinger equations between the $G_{\dots}$.

Remark: In Kontsevich model (cubic case), first equation reads

$$\frac{1}{\mathrm{i}}\frac{\partial \mathcal{Z}(M)}{\partial M_{ab}} = \frac{\mathrm{i}M_{ba}\mathcal{Z}(M)}{N(E_a+E_b)} - \frac{\lambda}{\mathrm{i}^2(E_a+E_b)}\sum_{k=1}^{N}\frac{\partial^2 \mathcal{Z}(M)}{\partial M_{ak}\partial M_{kb}}\,.$$

For $N=1$, $\lambda=\mathrm{i}$ this is the Airy differential equation.

Dyson-Schwinger equations [Grosse-W 09, 12]

$$(E_a+E_b)G_{|ab|} = 1 - \frac{\lambda}{N}\sum_{p=1}^{N} G_{|ab|}G_{|ap|} + \frac{\lambda}{N}\sum_{p=1}^{N}\frac{G_{|pb|}-G_{|ab|}}{E_p-E_a}$$

$$-\frac{\lambda}{N^2}\Big(-\frac{G_{|b|b|}-G_{|a|b|}}{E_b-E_a} + G_{|abab|} + G_{|baaa|}$$

$$+G_{|ab|}G_{|a|a|} + \frac{1}{N}\sum_{p=1}^{N} G_{|ab|ap|}\Big) - \frac{\lambda}{N^4}G_{|a|a|ab|}$$

$$(E_a+E_a)G_{|a|b|} = -\frac{\lambda}{N}\sum_{p=1}^{N} G_{|ap|}G_{|a|b|} + \frac{\lambda}{N}\sum_{p=1}^{N}\frac{G_{|p|b|}-G_{|a|b|}}{E_p-E_a}$$

$$+\lambda\frac{G_{|bb|}-G_{|ab|}}{E_b-E_a} - \frac{\lambda}{N^2}\Big(G_{|b|aaa|}+G_{|a|abb|}$$

$$+3G_{|a|b|}G_{|a|a|} + \frac{1}{N}\sum_{p=1}^{N} G_{|a|b|ap|}\Big) - \frac{\lambda}{N^4}G_{|b|a|a|a|}$$

DSEs decouple in formal genus expansion $G_{\dots} = \sum_{g=0}^{\infty} N^{-2g}G^{(g)}_{\dots}$

The planar 2-point function $G^{(0)}_{|ab|}$ (of cycle type (0,1))

- ... extends to holomorphic function $G^{(0)} : U \times U \to \mathbb{C}$ on neighbourhood U of $\{E_1, \dots, E_N\}$ with $G^{(0)}(E_a, E_b) = G^{(0)}_{|ab|}$.
- If E_i has multiplicity r_i, with $r_1+...+r_d=N$, then $G^{(0)}$ satisfies

$$\Big(\zeta+\eta+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(\zeta, E_k)\Big) G^{(0)}(\zeta,\eta) = 1 + \frac{\lambda}{N}\sum_{k=1}^{d} r_k \frac{G^{(0)}(E_k,\eta)-G^{(0)}(\zeta,\eta)}{E_k - \zeta}$$

- Alternatively, setting $\varrho_0(t) = \frac{1}{N}\sum_{k=1}^{d} r_k \delta(t - E_k)$,

$$\Big(\zeta+\eta+\lambda\int dt\, \varrho_0(t) G^{(0)}(\zeta, t)\Big) G^{(0)}(\zeta,\eta) = 1 + \lambda\int dt\, \varrho_0(t) \frac{G^{(0)}(t,\eta)-G^{(0)}(\zeta,\eta)}{t - \zeta}$$

1. With Erik Panzer we solved this equation for $\varrho_0(t) \equiv 1$.
2. Alexander Hock noticed a remarkable pattern in our solution.
3. His observation gave rise to a general solution method.
4. Connections to algebraic geometry (with Jörg Schürmann).

Solution of the non-linear integral equation

Theorem [Panzer-W 18 for $\varrho_0 = 1$, Grosse-Hock-W 19]

1. Ansatz $G^{(0)}(x,y) = \dfrac{e^{\mathcal{H}_x[\tau_y(\bullet)]}\sin\tau_y(x)}{Z\lambda\pi\varrho_0(x)}$

   Z=renormalisation

   $\mathcal{H}_x[f] = \frac{1}{\pi}\fint \frac{dp\, f(p)}{p-x}$

2. $\tau_y(x) = \mathrm{Im}\log\big(y + I(x+\mathrm{i}\epsilon)\big)$ with $I(\zeta) = -R(-\mu^2 - R^{-1}(\zeta))$

3. $$R(z) = z - \lambda(-z)^{D/2}\int_0^\infty \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)^{D/2}(t+\mu^2+z)}$$

   $D = 2[\frac{\delta}{2}]$ at spectral dimension $\delta = \inf\big(p:\ \int \frac{dt\varrho_0(t)}{(1+t)^{p/2}}<\infty\big)$

4. ϱ_λ is implicit solution of $\varrho_0(R(x)) = \varrho_\lambda(x)$.

Proof: [Cauchy 1831] residue theorem,
[Lagrange 1770] inversion theorem, [Bürmann 1799] formula

$D=4$ Moyal space: $\varrho_0(t)=t$ [Grosse-Hock-W 19]

- $\varrho_\lambda(x) \equiv \varrho_0(R(x)) = R(x) = x - \lambda x^2 \int_0^\infty \frac{dt\, \varrho_\lambda(t)}{(\mu^2+t)^2(t+x)}$
- If $\varrho_\lambda(t) \sim \varrho_0(t) = t$, then $R(x)$ bounded above. Consequently, R^{-1} would not be globally defined: triviality!
- Fredholm equation perturbatively solved by iterated integrals: Hyperlogarithms and $\zeta(2n)$ which can be summed to

$$\varrho_\lambda(x) = x \cdot {}_2F_1\left(\begin{matrix}\alpha_\lambda, 1-\alpha_\lambda \\ 2\end{matrix}\middle| -\frac{x}{\mu^2}\right)$$

$$\alpha_\lambda = \begin{cases} \frac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda| \leq \frac{1}{\pi} \\ \frac{1}{2} + \mathrm{i}\frac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{for } \lambda \geq \frac{1}{\pi} \end{cases}$$

Corollary

The interaction alters the spectral dimension to $4 - 2\frac{\arcsin(\lambda\pi)}{\pi}$ and thus avoids the triviality problem.

Solution for finite N [Schürmann-W 19]

Recall
$$\Big(\eta+\zeta+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(\zeta,E_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{E_k-\zeta}\Big)G^{(0)}(\zeta,\eta)$$
$$=1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{G^{(0)}(E_k,\eta)}{E_k-\zeta}, \qquad r_1+\cdots+r_d=N$$

Assume there is a branched covering $R:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ with

1. R has degree $d+1$ and maps (for $\lambda>0$) neighbourhood of $\mathbb{R}_+$ bijectively to neighbourhood $U\ni\zeta,\eta,E_k$ of $\mathbb{R}_+$
2. $\zeta=R(z)$, $\eta=R(w)$, $E_k=R(\varepsilon_k)$, $G^{(0)}(\zeta,\eta)=\mathcal{G}^{(0)}(z,w)$
3. $R(z)+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\mathcal{G}^{(0)}(z,\varepsilon_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{R(\varepsilon_k)-R(z)}=-R(-z)$

Gives $(R(w)-R(-z))\mathcal{G}^{(0)}(z,w)=1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{\mathcal{G}^{(0)}(\varepsilon_k,w)}{R(\varepsilon_k)-R(z)}$

Take $z=-\hat{w}_l$ for other d preimages of $\eta=R(w)=R(\hat{w}^l)$.
Express $\mathcal{G}^{(0)}(\varepsilon_k,w)$ in terms of $R(-\hat{w}^l)$, $R(\varepsilon_k)$; insert back

Rationality

Using formulae for inverses of Cauchy matrices $(\frac{1}{a_k - b_l})_{kl}$ and their row sums [Schechter 59]:

Theorem [Schürmann-W 19, extending Grosse-Hock-W 19]

$$R(z) = z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{z+\varepsilon_k} \quad \text{where } E_k = R(\varepsilon_k),\; r_k = R'(\varepsilon_k)\varrho_k$$

$$\mathcal{G}^{(0)}(z,w) = \frac{1 - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{r_k}{(R(z)-R(\varepsilon_k))(R(\varepsilon_k)-R(-w))} \prod_{j=1}^{d} \frac{R(w)-R(-\widehat{\varepsilon}_k^{\,j})}{R(w)-R(\varepsilon_j)}}{R(w)-R(-z)}$$

$\mathcal{G}^{(0)}(z,w) = \mathcal{G}^{(0)}(w,z)$, and ③ of the ansatz identically satisfied!

Thus, planar 2-point function solved by the composition of a rational function $\mathcal{G}^{(0)}$ with inverse of another rational function R.

Planar 2-point function of cycle type $(2,0)$

Dyson-Schwinger equation extends meromorphically to

$$(R(z)-R(-z))\mathcal{G}^{(0)}(z|w) - \frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k\mathcal{G}^{(0)}(\varepsilon_k|w)}{R(\varepsilon_k)-R(z)} = \lambda\frac{\mathcal{G}^{(0)}(z,w)-\mathcal{G}^{(0)}(w,w)}{R(z)-R(w)}$$

Proposition [Schürmann-W 19]

$$\mathcal{G}^{(0)}(z|w) = \frac{\lambda}{(R(z)-R(w))^2}\left\{\mathcal{G}^{(0)}(z,w) - \frac{(R(z)+R(w)-2R(0))\prod_{k=1}^{d}\frac{(R(z)-R(\alpha_k))(R(w)-R(\alpha_k))}{(R(z)-R(\varepsilon_k))(R(w)-R(\varepsilon_k))}}{(R(z)-R(-z))(R(w)-R(-w))}\right\}$$

where $\{0,\pm\alpha_k\}$ are all roots of $R(z)-R(-z)=0$

Higher topologies [Branahl-Hock-W soon]

- $\mathcal{G}^{(0)}(z, w)$ is topologically a disk (sphere with 1 hole)
- $\mathcal{G}^{(0)}(z|w)$ is topologically a cylinder (sphere with 2 holes)
- Does the same construction exhaust all higher topologies?

NO! A new structure is needed [Branahl-Hock-W, w.i.p]

$$\Omega_{a_1,\dots,a_n} = \Big[\int_{X'_N} d\mu_{E,\lambda}(\Phi) \prod_{i=1}^{n}\Big(\sum_{k_i=1}^{d} r_{k_i}\Phi(e_{a_i k_i})\Phi(e_{k_i a_i})\Big)\Big]_c$$

$$T_{a_1,\dots,a_n\|bc|} = \Big[\int_{X'_N} d\mu_{E,\lambda}(\Phi)\Phi(e_{bc})\Phi(e_{cb}) \prod_{i=1}^{n}\Big(\sum_{k_i=1}^{d} r_{k_i}\Phi(e_{a_i k_i})\Phi(e_{k_i a_i})\Big)\Big]_c$$

$$T_{a_1,\dots,a_n\|b|c|} = \Big[\int_{X'_N} d\mu_{E,\lambda}(\Phi)\Phi(e_{bb})\Phi(e_{cc}) \prod_{i=1}^{n}\Big(\sum_{k_i=1}^{d} r_{k_i}\Phi(e_{a_i k_i})\Phi(e_{k_i a_i})\Big)\Big]_c$$

Complexified $\Omega_n^{(g)}(z_1, \dots, z_n)$ relate to topological recursion!

Dyson-Schwinger equations I: $\mathcal{G}$ from $\mathcal{T}$

$$
\begin{aligned}
&(R(w)-R(-z))\mathcal{G}^{(g)}(z,w|\mathcal{J}) - \frac{\lambda}{N}\sum_k r_k \frac{\mathcal{G}^{(g)}(\varepsilon_k,w|\mathcal{J})}{R(\varepsilon_k)-R(z)} \\
&= \delta_{0,|\mathcal{J}|}\delta_{g,0} - \lambda\Big\{ \sum_{\substack{\mathcal{I}\uplus\mathcal{I}'=\mathcal{J},\ h+h'=g \\ (\mathcal{I},h)\neq(\emptyset,0)}} \mathcal{G}^{(h')}(z,w|\mathcal{I}')\mathcal{T}^{(h)}(z\|\mathcal{I}) + \mathcal{T}^{(g-1)}(z\|z,w|\mathcal{J}) \\
&+ \sum_{\mathcal{I}\uplus\mathcal{I}'=\mathcal{J},\ h+h'=g} \mathcal{G}^{(h)}(w|\mathcal{I})\frac{\mathcal{G}^{(h')}(z|\mathcal{I}')-\mathcal{G}^{(h')}(w|\mathcal{I}')}{R(w)-R(z)} \\
&+ \sum_{\beta=2}^{b}\sum_{j=1}^{n_\beta} \frac{\mathcal{G}^{(g)}(w,z,z^\beta_j,..,z^\beta_{n_\beta+j-1}|\mathcal{J}\backslash\{J^\beta\}) - \mathcal{G}^{(g)}(w,z^\beta_j,..,z^\beta_{n_\beta+j}|\mathcal{J}\backslash\{J^\beta\})}{R(z^\beta_j)-R(z)} \\
&+ \frac{\mathcal{G}^{(g-1)}(z|w|\mathcal{J})-\mathcal{G}^{(g-1)}(w|w|\mathcal{J})}{R(w)-R(z)}\Big\}
\end{aligned}
$$

- Conditions on cardinalities of $\mathcal{J},\mathcal{I},\mathcal{I}'$
- Inversion of first line via Cauchy matrices
- convention $\mathcal{G}^{(g)}(z,w|\mathcal{J}) = \mathcal{T}^{(g)}(\emptyset\|z,w|\mathcal{J}|)$ on next pages

Dyson-Schwinger equations II: $\mathcal{T}$ from Ω

$$(R(w)-R(-z))\mathcal{T}^{(g)}(u_1,...,u_m\|z,w|\mathcal{J}|)-\frac{\lambda}{N}\sum_k r_k\frac{\mathcal{T}^{(g)}(u_1,...,u_m\|\varepsilon_k,w|\mathcal{J}|)}{R(\varepsilon_k)-R(z)}$$

$$=\delta_{0,m}\delta_{0,|\mathcal{J}|}\delta_{g,0}-\lambda\Big\{\sum_{\substack{\mathcal{K}\uplus\mathcal{K}'=\{u_1,..,u_m\}\\ h+h'=g,\,(\mathcal{K},h)\neq(\emptyset,0)}}\mathcal{T}^{(h')}(\mathcal{K}'\|z,w|\mathcal{J}|)\Omega^{(h)}_{|\mathcal{K}+1|}(\mathcal{K},z)$$

$$+\sum_{\substack{\mathcal{I}\uplus\mathcal{I}'=\mathcal{J},\mathcal{I}\neq\emptyset\\ \mathcal{K}\uplus\mathcal{K}'=\{u_1;..;u_m\},\ h+h'=g}}\mathcal{T}^{(h')}(\mathcal{K}'\|z,w|\mathcal{I}'|)\mathcal{T}^{(h)}(\mathcal{K},z\|\mathcal{I}|)$$

$$+\sum_{i=1}^m\frac{\partial}{\partial R(u_i)}\frac{\mathcal{T}^{(g)}(u_1,..,\check{u}_i,..,u_m\|u_i,w|\mathcal{J}|)}{R(u_i)-R(z)}+\mathcal{T}^{(g-1)}(u_1,...,u_m,z\|z,w|\mathcal{J}|)$$

$$+\frac{\mathcal{T}^{(g-1)}(u_1,...,u_m\|z|w|\mathcal{J}|)-\mathcal{T}^{(g-1)}(u_1,...,u_m\|w|w|\mathcal{J}|)}{R(w)-R(z)}$$

$$+\sum_{\beta=2}^b\sum_{j=1}^{n_\beta}\frac{\mathcal{T}^{(g)}(\mathcal{K}\|w,z,z^\beta_j,..,z^\beta_{n_\beta+j-1}|\mathcal{J}\backslash\{J^\beta\}|)-\mathcal{T}^{(g)}(\mathcal{K}\|w,z^\beta_j,..,z^\beta_{n_\beta+j}|\mathcal{J}\backslash\{J^\beta\}|)}{R(z^\beta_j)-R(z)}\Big|_{\mathcal{K}=\{u_1,..,u_m\}}$$

$$+\sum_{\substack{\mathcal{I}\uplus\mathcal{I}'=\mathcal{J}\\ \mathcal{K}\uplus\mathcal{K}'=\{u_1,..,u_m\},\ h+h'=g}}\mathcal{T}^{(h)}(\mathcal{K}\|w|\mathcal{I}|)\frac{\mathcal{T}^{(h')}(\mathcal{K}'\|z|\mathcal{I}'|)-\mathcal{T}^{(h')}(\mathcal{K}'\|w|\mathcal{I}'|)}{R(w)-R(z)}\Big\}$$

Dyson-Schwinger equation III: $\Omega_2^{(0)}(v,z)$

$$\Omega_2^{(0)}(v,z)R'(z)\mathfrak{G}_0(z) - \frac{\lambda}{N^2}\sum_{n,k=1}^{d}\frac{r_k r_n \mathcal{T}^{(0)}(v\|\varepsilon_k,\varepsilon_n|)}{(R(\varepsilon_k)-R(z))(R(\varepsilon_n)-R(-z))}$$
$$= -\frac{\partial}{\partial R(v)}\big(\mathcal{G}^{(0)}(v,z)+\mathcal{G}^{(0)}(v,-z)\big)$$

where $\mathfrak{G}_0(z) = \mathrm{Res}_{w\to -z}\,\mathcal{G}^{(0)}(z,w)$.

- Seems to need $\mathcal{T}^{(0)}(v\|\varepsilon_k,\varepsilon_n|)$ which itself needs $\Omega_2^{(0)}$.
- But poles separate by partial fraction decomposition

$$\mathcal{G}^{(0)}(z,v) = \frac{\mathfrak{G}_0(z)}{v+z} + \frac{\lambda^2}{N^2}\sum_{k,l,m,n=1}^{d}\frac{C_{k,l}^{m,n}}{(z+\widehat{\varepsilon}_l{}^n)(z-\widehat{\varepsilon}_k{}^m)(v-\widehat{\varepsilon}_l{}^n)}$$

Proposition

$$\Omega_2^{(0)}(v,z) = \frac{1}{R'(v)R'(z)}\Big(\frac{1}{(v-z)^2}+\frac{1}{(v+z)^2}\Big)$$

One recognises the Bergman kernel of topological recursion!

Dyson-Schwinger equation IV: Ω from lower $\mathcal{T}, \Omega$

$$R'(z)\mathfrak{G}_0(z)\Omega^{(g)}_{m+1}(u_1,\dots,u_m,z) + \frac{\lambda}{N^2}\sum_{n,k} r_n r_k \frac{\mathcal{T}^{(g)}(u_1,\dots,u_m\|\varepsilon_k,\varepsilon_n|)}{(R(\varepsilon_k)-R(z))(R(\varepsilon_n)-R(-z))}$$

$$= \frac{\delta_{g,0}\delta_{m,1}}{(R(z)-R(u_1))^2} - \sum_{\substack{\mathcal{K}\uplus\mathcal{K}'=\{u_1;..;u_m\}\\ (\mathcal{K},h)\neq(\emptyset,0)\neq(\mathcal{K}',h')\\ h+h'=g}} \Omega^{(h')}_{|\mathcal{K}'+1|}(\mathcal{K}',z)\frac{\lambda}{N}\sum_n r_n \frac{\mathcal{T}^{(h)}(\mathcal{K}\|z,\varepsilon_n|)}{R(\varepsilon_n)-R(-z)}$$

$$-\sum_{i=1}^m \frac{\partial}{\partial R(u_i)} \frac{\frac{\lambda}{N}\sum_n r_n \frac{\mathcal{T}^{(g)}(u_1,..,\check{u}_i,..,u_m\|u_i,\varepsilon_n|)}{R(\varepsilon_n)-R(-z)}}{R(u_i)-R(z)} - \frac{\lambda}{N}\sum_n r_n \frac{\mathcal{T}^{(g-1)}(u_1,\dots,u_m,z\|z,\varepsilon_n|)}{R(\varepsilon_n)-R(-z)}$$

$$-\frac{\lambda}{N}\sum_n r_n \frac{\mathcal{T}^{(g-1)}(u_1,\dots,u_m\|z|\varepsilon_n|) - \mathcal{T}^{(g-1)}(u_;\dots,u_m\|\varepsilon_n|\varepsilon_n|)}{(R(\varepsilon_n)-R(z))(R(\varepsilon_n)-R(-z))}$$

$$-\sum_{i=1}^m \frac{\partial}{\partial R(u_i)} \mathcal{T}^{(g)}(u_1,..,\check{u}_i,..,u_m\|u_i,z|) + \mathcal{T}^{(g-1)}(u_1,\dots,u_m\|z|z|)$$

Can be solved without knowing the green term!

Miraculously, all poles on rhs (other than $\pm\widehat{\varepsilon}_n^j$) have prefactor $\mathfrak{G}_0$!

DSE V: cycle type split $(0, 1)$ to $(2, 0)$

$$(R(z)-R(-z))\mathcal{T}^{(g)}(u_1,...,u_m\|z|\mathcal{J}|) - \frac{\lambda}{N}\sum_k r_k \frac{\mathcal{T}^{(g)}(u_1,...,u_m\|\varepsilon_k|\mathcal{J}|)}{R(\varepsilon_k)-R(z)}$$

$$= -\lambda\Big\{ \sum_{\substack{\mathcal{I}\uplus\mathcal{I}'=\mathcal{J},\,\mathcal{I}\neq\emptyset \\ \mathcal{K}\uplus\mathcal{K}'=\{u_1,..,u_m\},\; h+h'=g}} \mathcal{T}^{(h')}(\mathcal{K}'\|z|\mathcal{I}'|)\mathcal{T}^{(h)}(\mathcal{K},z\|\mathcal{I}|) + \sum_{\substack{\mathcal{K}\uplus\mathcal{K}'=\{u_1,..,u_m\} \\ h+h'=g,\,(\mathcal{K},h)\neq(\emptyset,0)}} \mathcal{T}^{(h')}(\mathcal{K}'|z|\mathcal{J}|)\Omega^{(h)}_{|\mathcal{K}+1|}(\mathcal{K},z)$$

$$+\sum_{i=1}^{m}\frac{\partial}{\partial R(u_i)}\frac{\mathcal{T}^{(g)}(u_1,..,\check{u}_i,..,u_m\|u_i|\mathcal{J}|)}{R(u_i)-R(z)} + \mathcal{T}^{(g-1)}(u_1,...,u_m,z\|z|\mathcal{J}|)$$

$$+\sum_{\beta=2}^{b}\sum_{j=1}^{n_\beta}\frac{\mathcal{T}^{(g)}(\mathcal{K}\|z,z_j^\beta,..,z^\beta_{n_\beta+j-1}|\mathcal{J}\backslash\{J^\beta\}|) - \mathcal{T}^{(g)}(\mathcal{K}\|z_j^\beta,..,z^\beta_{n_\beta+j}|\mathcal{J}\backslash\{J^\beta\}|)}{R(z_j^\beta)-R(z)}\Big|_{\mathcal{K}=\{u_1,..,u_m\}}\Big\}$$

- $\mathcal{G}^{(g)}(z|\mathcal{J}) = \mathcal{T}^{(g)}(\emptyset\|z|\mathcal{J}) \quad (m=0)$
- first line inverted via Cauchy matrices at $z=\alpha_k$
- necessary input for genus increase $\Omega^{(g)}(z)\mapsto\Omega^{(g+1)}(z)$

The solution is complete

- We study the quartic analogue of the Kontsevich model $d\mu_{E,\lambda}$, parametrised by coupling constant λ and spectral values $E_1, \dots, E_d$ of multiplicities $r_1, \dots, r_d$ with $r_1 + \dots + r_d = N$.
- Main obstacle was the non-linear equation for the planar 2-point function obtained with Harald Grosse in 2009.
- The breakthrough was to solve the 2D Moyal case with Erik Panzer in Les Houches 2018.
- We found soon the solution in full generality. The discrete case links to algebraic geometry.
- The solution introduced a rational function

$$R(z) = z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{\varepsilon_k + z} \text{ with } R(\varepsilon_k) = E_k \ \& \ \varrho_k R'(\varepsilon_k) = r_k.$$

The solution is complete

- All other correlation functions (or cumulants) satisfy (after meromorphic extension and genus expansion) affine equations.
- To solve them we need partial sums $\mathcal{T}^{(g)}(u_1,\dots,u_m\|z,w|\dots|)$, $\mathcal{T}^{(g)}(u_1,\dots,u_m\|z|w|\dots|)$ and $\Omega_n^{(g)}(u_1,\dots,u_m)$.
- We know how to invert the linear operators in all these loop equations. This is very easy for $\mathcal{T}^{(g)}$ (unless one wants to simplify the expression and to make the symmetry manifest).
- Extracting $\Omega_n^{(g)}$ in practice becomes difficult for large $2g+n$. So far we have $\Omega_2^{(0)},\Omega_3^{(0)},\Omega_4^{(0)},\Omega_1^{(1)}$ and a third of $\Omega_2^{(1)}$.

Blobbed topological recursion?

Final results for these $\Omega^{(g)}_n$ are remarkably simple. Define

$$\omega_{0,1}(z) := -R(-z)R'(z)dz$$
$$\omega_{g,n}(z_1,\dots,z_n) := (-\lambda)^{2-2g-n}\Omega^{(g)}_n(z_1,\dots,z_n)\prod_{l=1}^{n} R'(z_i)dz_i$$

For $2g+n-2<0$, these $\omega_{g,n}$ seem to have poles only at $z_k=-z_l$ and $z_k=\beta_i$ where $R'(\beta_i)=0$ (the ramification points of the curve), but not at $z_k=\pm\widehat{z}_l^{\,j}$ or $z_l=\pm\widehat{\varepsilon}_k^{\,j}$.

- The parts of $\omega_{g,n}$ with poles at ramification points seem to follow the universal formula of topological recursion.
- The other poles could signal an extension of TR.

We currently investigate whether it could be the blobbed topological recursion [Borot-Shadrin 15]. Our hierarchy of loop equations generates the blobs as additional input for the recursion.

Integrability?

Understanding better our recursion should give access to the partition function itself. It is a function of λ and the spectrum of E.

- At fixed genus it is very likely a polynomial with rational coefficients in two sets of variables
$$x_{k,i} = \frac{R^{(k+2)}(\beta_i)}{R''(\beta_i)}, \quad y_{k,i} = \frac{R^{(k+1)}(-\beta_i)}{R'(-\beta_i)}$$
These contain the λ-dependence (exactly).
- Because our recursion inserts blobs at every next step (in a controlled way!), it will differ from known examples.

Main question

- Is it nevertheless a τ-function for a Hirota equation, i.e. is it integrable?
- If so, are the rational coefficients intersection numbers of some characteristic classes on a moduli space?

Backup: Topological recursion [Eynard, Orantin 07]

Starting from a spectral curve consisting of

- a branched covering $x : \Sigma \to \Sigma_0$ of Riemann surfaces,
- meromorphic differentials $\omega_{0,1}$ on Σ and $\omega_{0,2}$ on $\Sigma \times \Sigma$,

recursively construct family $\omega_{g,n}$ of meromorphic n-differentials on Σ^n, with poles only at ramification points of x, by

$$\omega_{g,n}(z_1,...,z_n) = \sum_a \operatorname*{Res}_{z\to a} K(z_1,z,\sigma_a(z))dz\Big(\omega_{g-1,n+1}(z,\sigma_a(z),z_2,...,z_n) + \sideset{}{'}\sum_{\substack{g_1+g_2=g\\ I_1\uplus I_2=\{z_2,\dots,z_n\}}} \omega_{g_1,1+\#I_1}(z,I_1)\omega_{g_2,1+\#I_2}(\sigma_a(z),I_2)\Big)$$

[sum over ramification points a of x; local involution $x(z)=x(\sigma_a(z))$ near a; recursion kernel $K(z_1,z_2,z_3) = \frac{\frac{1}{2}\int_{z'=z_3}^{z_2}\omega_{0,2}(z_1,z')}{\omega_{0,1}(z_2)-\omega_{0,1}(z_3)}$]

Examples

one- and two-matrix models, Kontsevich model, Weil-Petersson volumes, Hurwitz numbers, Gromov-Witten numbers, ...

Topological recursion of the Kontsevich model

- branched cover $x : \hat{\mathbb{C}} \ni z \mapsto z^2 \in \hat{\mathbb{C}}$, where $z = (4\zeta^2 + c)^{1/2}$
- $\omega_{0,1}(z) = 2zy(-z)dz$ with $-y(-z) = z + \frac{1}{N}\sum_{k=1}^{N} \frac{1}{2\varepsilon_k(z+\varepsilon_k)}$ (related to planar 1-point function), $\varepsilon_k = (4E_k^2 + c)^{1/2}$
- $\omega_{0,2}(z,z') = \frac{dz\,dz'}{(z-z')^2}$ (related to planar 1+1-point function)

Meromorphic differentials relate to higher correlation functions $\omega_{g,n}(z_1,\dots,z_n) = \mathcal{G}^{(g)}(z_1|\dots|z_n)\prod_{i=1}^{n} d(x(z_i))$, where

$$\mathcal{G}^{(g)}(z_1|\dots|z_n) = (2-t_3)^{2-2g-n}\sum_{l_1,\dots,l_n}\Big\langle \psi_1^{l_1}\cdots\psi_n^{l_n} e^{\sum_k \hat{t}_k\kappa_k}\Big\rangle_{g,n}\prod_{i=1}^{n}\frac{(2l_i+1)!!}{z_i^{2l_i+3}}$$

- ψ_i, κ_k are tautological characteristic classes on $\overline{\mathcal{M}}_{g,n}$ and $\langle\dots\rangle_{g,n}$ their intersection numbers
- $e^{-\sum_k \hat{t}_k u^{-k}} = 1 - \frac{1}{2}\sum_l (2l+1)!!\,t_{2l+1}u^{-l}, \qquad t_l = \frac{1}{N}\sum_{k=1}^{N}\varepsilon_k^{-2l-1}$