Blobbed topological recursion of the quartic analogue of the Kontsevich model

Raimar Wulkenhaar

Mathematisches Institut der Westfälischen Wilhelms-Universität Münster

Mathematics Münster
Cluster of Excellence

2008.12201 & 2012.02622 with Johannes Branahl & Alex Hock
building on previous collaborations with
Harald Grosse, Erik Panzer & Jörg Schürmann

Prehistory: $\lambda\phi^{\star 4}$ on Moyal space

This project started as an attempt to understand quantum field theories on noncommutative geometries (e.g. Moyal space).

$$S(\phi) = \int_{\mathbb{R}^D} \frac{dx}{(8\pi)^{D/2}} \Big(\frac{1}{2}\phi\star(-\Delta+\Omega^2\|2\Theta^{-1}x\|^2+m^2)\phi + \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

where $(f \star g)(x) = \int_{\mathbb{R}^{2D}} \frac{dydk}{(2\pi)^D} f(x+\frac{1}{2}\Theta k)g(x+y)e^{\mathrm{i}\langle k,y\rangle}$

- $\exists$ matrix basis $(e_{kl}(x))_{k,l\in\mathbb{N}}$ with $(e_{kl}\star e_{mn})(x) = \delta_{lm}e_{kn}(x)$, $\int dx\; e_{kl}(x) = \sqrt{\det(2\pi\Theta)}\delta_{kl}$
- Expand $\phi(x) = \sum_{k,l=1}^{\infty} \Phi_{kl}e_{kl}(x)$ to get $\int_{\mathbb{R}^D} dx(\phi\star\phi\star\phi\star\phi)(x) = \sqrt{\det(2\pi\Theta)}\mathrm{Tr}(\Phi^4)$
- For $\Omega = 1$ also the kinetic term is matrix product:

$$S(\phi) = \sqrt{\det(\Theta/4)}\,\mathrm{Tr}\Big(E\Phi^2 + \frac{\lambda}{4}\Phi^4\Big)$$

$E = c_0\mathrm{id} + c_1\mathrm{diag}(0,1,1,2,2,2,3,3,3,3,\dots)$ in 4D

- need renormalisation $c_0 \mapsto m^2_{bare}$, $\Phi \mapsto \sqrt{Z}\Phi$

Free Euclidean fields on noncommutative geometries

Let H_N be the real vector space of self-adjoint $N \times N$-matrices, and $(E_1, \dots, E_N)$ be pairwise different positive real numbers.

Theorem [Bochner 1933, Schur 1911]

For any inner product $\langle\ ,\ \rangle$ on H_N there exists a unique probability measure $d\mu_0$ on the dual space H'_N with

$$\exp\Big(-\frac{1}{2}\langle M, M\rangle\Big) = \int_{H'_N} d\mu_0(\Phi)\, e^{\mathrm{i}\Phi(M)} \qquad \forall M=(M_{kl}) \in H_N\ .$$

Choose $\langle M, M'\rangle_E = \frac{1}{N}\sum_{k,l=1}^{N} \frac{M_{kl}M'_{lk}}{E_k + E_l}$ and corresponding $d\mu_{E,0}$

- Defines the free Euclidean scalar field on N-dimensional approximation of a noncommutative geometry.
- $(E_1, \dots, E_N)$ is truncated spectrum of the Laplacian.

The Kontsevich model and its quartic analogue

3. The Kontsevich model $d\mu_{E,\lambda}(\Phi) = \dfrac{e^{-\frac{\lambda N}{3}\mathrm{Tr}(\Phi^3)} d\mu_{E,0}(\Phi)}{\int_{H'_N} e^{-\frac{\lambda N}{3}\mathrm{Tr}(\Phi^3)} d\mu_{E,0}(\Phi)}$
   - Computes intersection numbers of tautological characteristic classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves [Kontsevich 92].
   - It is integrable via a relation (suggested by [Witten 91]) to the KdV hierarchy. Its moments obey topological recursion.
4. A quartic analogue $d\mu_{E,\lambda}(\Phi) = \dfrac{e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_{E,0}(\Phi)}{\int_{H'_N} e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_{E,0}(\Phi)}$
   - Passing from $\mathrm{Tr}(\Phi^3)$ to $\mathrm{Tr}(\Phi^4)$ is a minor step for QFT, but it destroys connections to mathematics.
   - The cubic vertices $\mathrm{Tr}(\Phi^3)$ encode simple zeros of the Strebel differential. One cannot make them disappear.

We show: both deformations share similar mathematical structures.

Cumulants/Connected correlation functions

derivatives of log of Fourier transform $\mathcal{Z}(M) := \int_{H'_N} d\mu_{E,\lambda}(\Phi)\, e^{\mathrm{i}\Phi(M)}$

with respect to M_{kl}; these are zero unless organised in cylces:

$$\sum_{g=0}^{\infty} N^{2-2g-b} G^{(g)}_{|k_1^1\dots k_{n_1}^1|\dots|k_1^b\dots k_{n_b}^b|} = \frac{\partial^{n_1+\dots+n_n} \log \mathcal{Z}(M)}{\partial\mathbb{M}_{k^b_{n_b}\dots k^b_1} \cdots \partial\mathbb{M}_{k^1_{n_1}\dots k^1_1}}\Big|_{M\equiv 0}$$

where $\frac{\partial^n}{\partial\mathbb{M}_{k_n\dots k_1}} = \frac{N^n(-\mathrm{i})^n\partial^n}{\partial M_{k_nk_{n-1}}\cdots\partial M_{k_2k_1}\partial M_{k_1k_n}}$ and k_i^j pairwise different

- $\frac{\partial}{\partial M_{kl}}$ produces $\mathrm{i}\Phi_{kl} := \mathrm{i}\Phi(e_{kl})$ which contracts to

$$\frac{\partial\mathcal{Z}(M)}{\partial M_{kl}} = \frac{\mathrm{i}}{E_k+E_l}\int_{H'_N} d\mu_{E,\lambda}(\Phi)\Big(\mathrm{i}M_{lk} - \lambda N\sum_{p,q=1}^{N}\Phi_{kp}\Phi_{pq}\Phi_{ql}\Big)e^{\mathrm{i}\Phi(M)}$$

- leads to identities between multiple M-derivatives: Dyson-Schwinger equations

Equations of motion for quartic Kontsevich model

Fourier transform $\mathcal{Z}(M) := \int_{H'_N} d\mu_{E,\lambda}(\Phi)\, e^{\mathrm{i}\Phi(M)}$ satisfies

1. $$-N(E_p - E_q)\sum_{k=1}^{N} \frac{\partial^2 \mathcal{Z}(M)}{\partial M_{pk}\partial M_{kq}} = \sum_{k=1}^{N}\Big(M_{kp}\frac{\partial \mathcal{Z}(M)}{\partial M_{kq}} - M_{qk}\frac{\partial \mathcal{Z}(M)}{\partial M_{pk}}\Big)$$
2. $$\frac{1}{N}\frac{\partial \mathcal{Z}(M)}{\partial E_p} = \sum_{k=1}^{N} \frac{\partial^2 \mathcal{Z}(M)}{\partial M_{pk}\partial M_{kp}} + \mathcal{Z}(M)\int_{H'_N} d\mu_{E,\lambda}(\Phi)\frac{1}{N}\sum_{k=1}^{N}\Phi_{pk}\Phi_{kp}$$

- They allow to express $\sum_{k=1}^{N} \frac{\mathcal{Z}(M)}{\partial M_{pk}\partial M_{kq}}$ in Dyson-Schwinger equations by fewer derivatives, i.e. of same or lower order.
- Dyson-Schwinger equations complexify to equations for meromorphic functions in several complex variables $G^{(g)}(E_{k_1^1},\dots,E_{k_{n_1}^1}|\dots|E_{k_1^b},\dots,E_{k_{n_b}^b}) := G^{(g)}_{|k_1^1\dots k_{n_1}^1|\dots|k_1^b\dots k_{n_b}^b|}$
- We admit multiplicities $(E_1,\dots,E_N) = (\underbrace{e_1,\dots,e_1}_{r_1},\dots,\underbrace{e_d,\dots,e_d}_{r_d})$

Examples of Dyson-Schwinger equations (I)

1. Non-linear equation for planar 2-point function $G^{(0)}(\zeta,\eta)$ alone [Grosse-W 09]. Needs nenormalisation for $N\to\infty$.

$$\Big(\zeta+\eta+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(\zeta,e_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{e_k-\zeta}\Big)G^{(0)}(\zeta,\eta) -\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{G^{(0)}(e_k,\eta)}{e_k-\zeta}=1$$

2. Planar (1+1)-point function $G^{(0)}(\zeta|\eta)$:

$$\Big(2\zeta+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(\zeta,e_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{e_k-\zeta}\Big)G^{(0)}(\zeta|\eta)-\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k G^{(0)}(e_k|\eta)}{e_k-\zeta} =\lambda\frac{G^{(0)}(\zeta,\eta)-G^{(0)}(\eta,\eta)}{\zeta-\eta}$$

(easy to solve if $G^{(0)}(\zeta,\eta)$ was known)

Examples of Dyson-Schwinger equations (II)

3. direct solution for planar 4-point function

$$G^{(0)}(\zeta_1,\zeta_2,\zeta_3,\zeta_4) = (-\lambda)\frac{G^{(0)}(\zeta_1,\zeta_2)G^{(0)}(\zeta_3,\zeta_4) - G^{(0)}(\zeta_1,\zeta_4)G^{(0)}(\zeta_3,\zeta_2)}{(\zeta_1-\zeta_3)(\zeta_2-\zeta_4)}$$

(nested structures counted by Catalan numbers [de Jong-Hock-W 19])

4. planar (2+2)-point function

$$\Big(\zeta+\eta_1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(\zeta,e_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{e_k-\zeta}\Big)G^{(0)}(\zeta,\eta_1|\eta_2,\eta_3)$$

$$-\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{G^{(0)}(e_k,\eta_1|\eta_2,\eta_3)}{e_k-\zeta}-\frac{\lambda}{N}\sum_{l=1}^{d} r_l G^{(0)}(\zeta,e_l|\eta_2,\eta_3)G^{(0)}(\zeta,\eta_1)$$

$$=\lambda\frac{G^{(0)}(\eta_1,\zeta,\eta_3,\eta_2)-G^{(0)}(\eta_1,\eta_2,\eta_3,\eta_2)}{\zeta-\eta_2}-\lambda G^{(0)}(\zeta,\eta_1)G^{(0)}(\zeta,\eta_2,\eta_3,\eta_2)$$

$$+\lambda\frac{G^{(0)}(\eta_1,\zeta,\eta_2,\eta_3)-G^{(0)}(\eta_1,\eta_3,\eta_2,\eta_3)}{\zeta-\eta_3}-\lambda G^{(0)}(\zeta,\eta_1)G^{(0)}(\zeta,\eta_3,\eta_2,\eta_3)$$

Note: Linear operator is of new type!

Back to the DSE for planar 2-point function

We pass with $\varrho_0(t) = \frac{1}{N}\sum_{k=1}^d r_k\delta(t-e_k)$ to integral equation

$$\Big(\zeta+\eta+\lambda\int dt\, \varrho_0(t) G^{(0)}(\zeta,t)\Big) G^{(0)}(\zeta,\eta) = 1 + \lambda\int dt\, \varrho_0(t) \frac{G^{(0)}(t,\eta)-G^{(0)}(\zeta,\eta)}{t-\zeta}$$

- Renormalisation $G^{(0)} \mapsto ZG^{(0)}$ and $\zeta \mapsto \frac{\tilde{\mu}^2}{2}+x, \eta \mapsto \frac{\tilde{\mu}^2}{2}+y$ with the new $x,y \in [0,\Lambda^2]$
- Ansatz $ZG^{(0)}(x,y) = \frac{e^{\mathcal{H}_x[\tau_y(\bullet)]}\sin\tau_y(x)}{\lambda\pi\varrho_0(x)} = \frac{e^{\mathcal{H}_y[\tau_x(\bullet)]}\sin\tau_x(y)}{\lambda\pi\varrho_0(y)}$

  with Hilbert transform $\mathcal{H}_y[f(\bullet)] := \frac{1}{\pi}\lim_{\epsilon\to 0}\big(\int_0^{y-\epsilon}+\int_{y+\epsilon}^{\Lambda^2}\big)\frac{dt\, f(t)}{t-y}$
- Gives with identites for Hilbert transform

$$\tau_y(x) = \arctan\left(\frac{\lambda\pi\varrho_0(x)}{\tilde{\mu}^2+y+x+\lambda\pi\mathcal{H}_x[\varrho_0(\bullet)]+\frac{1}{\pi}\int_0^{\Lambda^2} dt\, \tau_x(t)}\right)$$

- Case $\varrho_0(t)\equiv 1$ solved with Erik Panzer in Les Houches 2018

Lambert-W and its lessons

Theorem [Panzer-W 18]

$$\tau_y(x) = \mathrm{Im}\log\big(y + I(x+\mathrm{i}\epsilon)\big)$$

$$I(\zeta) := \lambda W_0\big(\tfrac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\big) - \lambda\log\big(1 - \lambda W_0\big(\tfrac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\big)\big)$$

$W_0 =$ principal branch of Lambert-W, $\tilde{\mu}^2 = 1-2\lambda\log(1+\Lambda^2)$

Observation [Alex Hock]: I and W_0 are related

$$I(\zeta) = -\big(-(1+z(\zeta)) + \lambda\log(-z(\zeta))\big)$$

where $z(\zeta) = \lambda W_0\big(\tfrac{1}{\lambda}e^{\frac{1+\zeta}{\lambda}}\big) - 1$ solves $\zeta = z + \lambda\log(1+z)$.

Theorem [Grosse-Hock-W 19]

The ansatz $I(\zeta) = -R(-\mu^2 - R^{-1}(\zeta))$ solves τ-equation if

- $R(z) = z - \lambda(-z)^{D/2}\int \frac{dt\ \varrho_\lambda(t)}{(\mu^2+t)^{D/2}(t+\mu^2+z)}$
- ϱ_λ is implicit solution of $\varrho_0(R(x)) = \varrho_\lambda(x)$

Solution for finite matrices

Theorem [Grosse-Hock-W 19, Schürmann-W 19]

Let $(\varepsilon_k, \varrho_k)$ be implicitly defined by $e_k = R(\varepsilon_k),\ r_k = R'(\varepsilon_k)\varrho_k$

for $R(z) = z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{z+\varepsilon_k}$.

Then $G^{(0)}(\zeta,\eta) = \mathcal{G}^{(0)}(z,w)$ for $R(z)=\zeta$, $R(w)=\eta$ and

$$\mathcal{G}^{(0)}(z,w) = \frac{1 - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{r_k \prod_{j=1}^{d} \frac{R(w)-R(-\widehat{\varepsilon}_k^{\,j})}{R(w)-R(\varepsilon_j)}}{(R(z)-R(\varepsilon_k))(R(\varepsilon_k)-R(-w))}}{R(w)-R(-z)}$$

where $u \in \{z, \hat{z}^1, \dots, \hat{z}^d\}$ are all solutions of $R(u)=R(z)$.

(The symmetry $\mathcal{G}^{(0)}(z,w) = \mathcal{G}^{(0)}(w,z)$ is automatic)

Thus, planar 2-point function solved by the composition of a rational function $\mathcal{G}^{(0)}$ with inverse of another rational function R.

The affine equations

- All other correlations functions satisfy affine equations. They are always solvable, but no path seemed to exist.
- Alex Hock: need first to look at auxiliary functions!

Recall that $d\mu_{E,\lambda}$ depends on given family $E_1, \dots, E_N$. Introduce

$$\sum_{g=0}^{\infty} N^{2-2g-n} \Omega^{(g)}_{q_1,\dots,q_n} := \frac{\partial^{n-1}\Big(N\sum_{k=1}^{N}\int_{H'_N} d\mu_{E,\lambda}(\Phi)\Phi_{q_1k}\Phi_{kq_1}\Big)}{\partial E_{q_2}\cdots\partial E_{q_n}} + \frac{\delta_{n,2}}{(E_{q_1}-E_{q_2})^2}$$

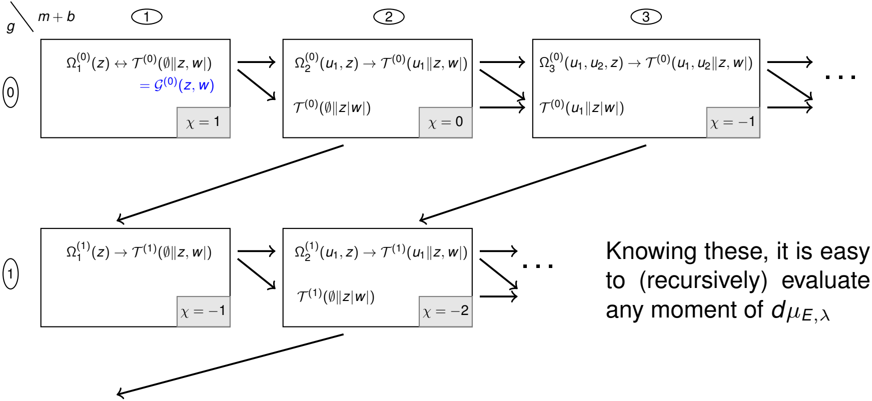
- We derive and solve Dyson-Schwinger equations for (meromorphic continuation of) $\Omega^{(g)}$.
- This needs R and $\mathcal{G}^{(0)}$, but no prior knowledge of its E-derivatives and of 2-point functions of higher topology.

Unexpected result: The $\Omega^{(g)}$ translate to differential forms which obey blobbed topological recursion [Borot-Shadrin 15]!

Solution procedure [Branahl-Hock-W 20]

Three types of functions involved:

- $\Omega^{(g)}_m(u_1, ..., u_m)$ objects of BTR, most difficult to compute
- $\mathcal{T}^{(g)}(u_1, ..., u_m \| z, w|)$ auxiliary functions, easy to compute
- $\mathcal{T}^{(g)}(u_1, \dots, u_m \| z|w|)$ auxiliary functions, easy to compute



Knowing these, it is easy to (recursively) evaluate any moment of $d\mu_{E,\lambda}$

Dyson-Schwinger equation for $\Omega_2^{(0)}(u,z)$

$$\Omega_2^{(0)}(u,z)R'(z)\mathfrak{G}_0(z) - \frac{\lambda}{N^2}\sum_{n,k=1}^{d} \frac{r_k r_n \mathcal{T}^{(0)}(u\|\varepsilon_k,\varepsilon_n|)}{(R(\varepsilon_k)-R(z))(R(\varepsilon_n)-R(-z))}$$
$$= -\frac{\partial}{\partial R(u)}\big(\mathcal{G}^{(0)}(u,z)+\mathcal{G}^{(0)}(u,-z)\big)$$

where $\mathfrak{G}_0(z) = \mathrm{Res}_{w\to -z}\,\mathcal{G}^{(0)}(z,w)dw$.

- Seems to need $\mathcal{T}^{(0)}(u\|\varepsilon_k,\varepsilon_n|)$ which itself needs $\Omega_2^{(0)}$.
- But poles separate by partial fraction decomposition

$$\mathcal{G}^{(0)}(z,u) = \frac{\mathfrak{G}_0(z)}{u+z} + \frac{\lambda^2}{N^2}\sum_{k,l,m,n=1}^{d} \frac{C_{k,l}^{m,n}}{(z+\widehat{\varepsilon}_l^{\,n})(z-\widehat{\varepsilon}_k^{\,m})(u-\widehat{\varepsilon}_l^{\,n})}$$

Proposition

$$\Omega_2^{(0)}(u,z) = \frac{1}{R'(u)R'(z)}\Big(\frac{1}{(u-z)^2}+\frac{1}{(u+z)^2}\Big)$$

One recognises the Bergman kernel of topological recursion!

Proposition: $\mathcal{T}^{(g)}(u_1, \dots, u_m \| z, w|)$

The solution is easier than the equation itself:

$$
\begin{aligned}
&\mathcal{T}^{(g)}(u_1, \dots, u_m \| z, w|) \\
&= \lambda \mathcal{G}^{(0)}(z,w) \operatorname*{Res}_{t \to z, -\hat{w}^j} \frac{R'(t)\,dt}{(R(z)-R(t))(R(w)-R(-t))\mathcal{G}^{(0)}(t,w)} \\
&\times \Bigg[\sum_{\substack{I_1 \uplus I_2 = \{u_1,\dots,u_m\} \\ g_1+g_2=g \\ (I_1,g_1)\neq(\emptyset,0)}} \Omega^{(g_1)}_{|I_1|+1}(I_1,t)\mathcal{T}^{(g_2)}(I_2\|t,w|) \\
&+ \sum_{i=1}^{m} \frac{\partial}{\partial R(u_i)} \frac{\mathcal{T}^{(g)}(u_1,..\check{u}_i..u_m\|u_i,w|)}{R(u_i)-R(t)} \\
&+ \frac{\mathcal{T}^{(g-1)}(u_1,\dots,u_m\|t|w|) - \mathcal{T}^{(g-1)}(u_1,\dots,u_m\|w|w|)}{R(w)-R(t)} \\
&+ \mathcal{T}^{(g-1)}(u_1,\dots,u_m,t\|t,w|) \Bigg]
\end{aligned}
$$

Proposition: $\mathcal{T}^{(g)}(u_1,\dots,u_m\|z|w|)$

$$
\begin{aligned}
&\mathcal{T}^{(g)}(u_1,\dots,u_m\|z|w|)\\
&= \frac{\lambda\prod_{j=1}^{d}\frac{R(z)-R(\alpha_j)}{R(z)-R(\varepsilon_j)}}{(R(z)-R(-z))} \operatorname*{Res}_{t\to z,\alpha_j} \frac{R'(t)\,dt}{(R(z)-R(t))\prod_{j=1}^{d}\frac{R(t)-R(\alpha_j)}{R(t)-R(\varepsilon_j)}}\\
&\times\Bigg[\sum_{\substack{I_1\uplus I_2=\{u_1,\dots,u_m\}\\ g_1+g_2=g\\ (I_1,g_1)\neq(\emptyset,0)}} \Omega^{(g_1)}_{|I_1|+1}(I_1,t)\mathcal{T}^{(g_2)}(I_2\|t|w|)+\mathcal{T}^{(g-1)}(u_1,\dots,u_m,t\|t|w|)\\
&+\sum_{i=1}^{m}\frac{\partial}{\partial R(u_i)}\frac{\mathcal{T}^{(g)}(u_1,..\check{u}_i..,u_m\backslash u_i\|u_i|w|)}{R(u_i)-R(t)}\\
&+\frac{\mathcal{T}^{(g)}(u_1,\dots,u_m\|t,w|)-\mathcal{T}^{(g)}(u_1,\dots,u_m\|w,w|)}{R(w)-R(t)}\Bigg],
\end{aligned}
$$

where $\{0,\pm\alpha_1,..\pm\alpha_d\}$ are all solutions of $R(z)-R(-z)=0$.

$\Omega^{(g)}(u_1, \ldots, u_m)$ for $2 - 2g - m < 0$

Proposition ($g = 0$) / Conjecture ($g \geq 1$)

$$
\begin{aligned}
R'(z)\Omega^{(g)}_{m+1}(u_1, \ldots, u_m, z) = \operatorname*{Res}_{q \to 0, -u_l, \beta_i} \frac{dq}{(q - z)\mathfrak{G}_0(q)} \Bigg[\\
\sum_{\substack{I_1 \uplus I_2 = \{u_1, \ldots, u_m\} \\ g_1 + g_2 = g \\ (I_1, g_1) \neq (\emptyset, 0) \neq (I_2, g_2)}} \Omega^{(g_1)}_{|I_1|+1}(I_1, q) \frac{\lambda}{N} \sum_{n=1}^{d} \frac{r_n \mathcal{T}^{(g_2)}(I_2 \| q, \varepsilon_n |)}{R(\varepsilon_n) - R(-q)} \\
+ \sum_{i=1}^{m} \frac{\partial}{\partial R(u_i)} \mathcal{T}^{(g)}(u_1, ..\check{u}_i.., u_m \| u_i, q |) \\
+ \frac{\lambda}{N} \sum_{n=1}^{d} \frac{r_n \mathcal{T}^{(g-1)}(u_1, \ldots, u_m, q \| q, \varepsilon_n |)}{R(\varepsilon_n) - R(-q)} \\
+ \frac{\lambda}{N} \sum_{n=1}^{d} \frac{r_n \mathcal{T}^{(g-1)}(u_1, \ldots, u_m \| q | \varepsilon_n |)}{(R(\varepsilon_n) - R(q))(R(\varepsilon_n) - R(-q))} - \mathcal{T}^{(g-1)}(u_1, \ldots, u_m \| q | q |) \Bigg]
\end{aligned}
$$

$\beta_1, \ldots \beta_{2d}$ – ramification points, $R'(\beta_i) = 0$

Differential forms $\omega_{g,m}(z_1,\dots,z_m) = \lambda^{2-2g-m}\Omega_m^{(g)}(z_1,\dots,z_m)\prod_{k=1}^m dR(z_i)$

$$\omega_{0,3}(u_1,u_2,z) = -\sum_{i=1}^{2d} \frac{\Big(\frac{1}{(u_1-\beta_i)^2}+\frac{1}{(u_1+\beta_i)^2}\Big)\Big(\frac{1}{(u_2-\beta_i)^2}+\frac{1}{(u_2+\beta_i)^2}\Big)du_1\,du_2\,dz}{R'(-\beta_i)R''(\beta_i)(z-\beta_i)^2}$$
$$+\Big[d_{u_1}\Big(\frac{\omega_{0,2}(u_2,u_1)}{(dR)(u_1)}\frac{dz}{R'(-u_1)(z+u_1)^2}\Big)+u_1\leftrightarrow u_2\Big]$$

$$\omega_{1,1}(z) = \sum_{i=1}^{2d}\frac{dz}{R'(-\beta_i)R''(\beta_i)}\Big\{-\frac{1}{8(z-\beta_i)^4}+\frac{R'''(\beta_i)}{24R''(\beta_i)(z-\beta_i)^3}$$
$$+\frac{\frac{R''''(\beta_i)}{48R''(\beta_i)}-\frac{(R'''(\beta_i))^2}{48(R''(\beta_i))^2}+\frac{R''(-\beta_i)R'''(\beta_i)}{48R'(-\beta_i)R''(\beta_i)}+\frac{(R''(-\beta_i))^2}{48(R'(-\beta_i))^2}-\frac{1}{8\beta_i^2}}{(z-\beta_i)^2}\Big\}$$
$$-\frac{dz}{8(R'(0))^2z^3}+\frac{R''(0)dz}{16(R'(0))^3z^2}$$

- We also have $\omega_{0,4}$, $\omega_{0,5}$. All are simple and structured. We are confident that poles at $z=-u_k$ can soon be understood.
- Of particular relevance will be the Laurent expansion about the fixed point $z=0$ of the involution $z\leftrightarrow -z$. This is new.

Blobbed topological recursion [Borot-Shadrin 15]

Proposition $(g,m) \in \{(0,3),(0,4),(0,5),(1,1)\}$ / Conjecture

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be ramified cover identified in $\mathcal{G}^{(0)}(z,w)$ with ramification points $\beta_1, \dots, \beta_{2d}$. Define $\omega_{0,1}(z) = -R(-z)R'(z)dz$ and for $2-2g-m \leq 0$ the $\omega_{g,m}$ as before.
Then parts $\mathcal{P}\omega_{g,m}$ containing the poles at ramification points:

$$\begin{aligned}
&\mathcal{P}_z\omega_{g,m}(u_1,\dots,u_{m-1},z)\\
&= \sum_{i=1}^{2d} \operatorname*{Res}_{q\to\beta_i} \frac{\frac{1}{2}\int_{q'=\sigma(q)}^{q'=q} B(z,q')}{\omega_{0,1}(q)-\omega_{0,1}(\sigma_i(q))} \Bigg(\omega_{g-1,m+1}(u_1,\dots,u_{m-1},q,\sigma_i(q)) \\
&\qquad + \sum_{\substack{I_1 \uplus I_2 = \{u_1,\dots,u_{m-1}\}\\ g_1+g_2=g \\ (I_1,g_1)\neq(\emptyset,0)\neq(I_2,g_2)}} \omega_{g_1,|I_1|+1}(I_1,q)\,\omega_{g_2,|I_2|+1}(I_2,\sigma_i(q)) \Bigg)
\end{aligned}$$

where σ_i = local Galois involution near β_i, i.e. $R(z) = R(\sigma_i(z))$, $\sigma_i(\beta_i) = \beta_i$, $\sigma_i \neq \mathrm{id}$ and $B(u,z) = \frac{du\,dz}{(u-z)^2}$ Bergman kernel.

$\Omega_n^{(g)}$ in terms of correlation functions and ribbon graphs

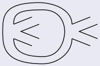
$$\Omega_2^{(0)}(R^{-1}(\zeta), R^{-1}(\eta)) = \frac{(R^{-1})'(\zeta)(R^{-1})'(\eta)}{\big(R^{-1}(\zeta)-R^{-1}(\eta)\big)^2} + \frac{(R^{-1})'(\zeta)(R^{-1})'(\eta)}{\big(R^{-1}(\zeta)+R^{-1}(\eta)\big)^2}$$

$$= \frac{1}{(\zeta-\eta)^2} + \big(G^{(0)}(\zeta,\eta)\big)^2 + \frac{1}{N^2}\sum_{k,l=1}^{d} r_k r_l G^{(0)}(\zeta, e_k|\eta, e_l)$$

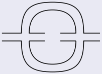
$$+ \frac{1}{N}\sum_{k=1}^{d} r_k\big(G^{(0)}(\zeta,e_k,\eta,e_k) + G^{(0)}(\zeta,e_k,\zeta,\eta) + G^{(0)}(\eta,e_k,\eta,\zeta)\big)$$

Example: Ribbon graph expansion of $G^{(0)}(\zeta, e_k|\eta, e_l)$ to $\mathcal{O}(\lambda^2)$

$$= \frac{(-\lambda)^2}{(\zeta+e_k)^2(\eta+e_l)^2}\Big(\frac{1}{(\zeta+\eta)^2}+\frac{1}{(\zeta+e_l)^2}+\frac{1}{(e_k+\eta)^2}+\frac{1}{(e_k+e_l)^2}$$



$$+\frac{1}{(\zeta+\eta)(e_k+e_l)}+\frac{1}{(\zeta+e_l)(e_k+\eta)}\Big)$$



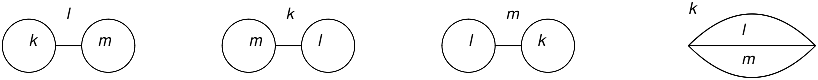
order	λ^0	λ^1	λ^2	λ^3	λ^4	λ^5
# ribbon graphs in $\Omega_2^{(0)}$	1	7	58	522	4941	48411

Interpretation

There exists a (g, n)-indexed family of distinguished polynomials in correlation functions, themselves series of ribbon graphs, which resum to expressions which

- are much simpler when transformed with R^{-1} (Taylor approximation to implicit function theorem)
- are of topological significance (intersection numbers on $\overline{\mathcal{M}}_{g,n}$, see [Borot-Shadrin 15])

Remark: Structurally analogous to Kontsevich model



$$\frac{(-\lambda)^2}{(e_k+e_l)(2e_l)(e_m+e_l)}+\frac{(-\lambda)^2}{(e_m+e_k)(2e_k)(e_l+e_k)}+\frac{(-\lambda)^2}{(e_l+e_m)(2e_m)(e_k+e_m)}+\frac{(-\lambda)^2}{(e_k+e_l)(e_l+e_m)(e_m+e_k)}$$

$$=\frac{(-\lambda)^2}{2e_k e_l e_m}\underbrace{\langle\tau_0\tau_0\tau_0\rangle}_{=1} \sim \Omega_3^{(0)}(z_1, z_2, z_3)$$

Which structure does the quartic Kontsevich model generate?

Summary

Quartic Kontsevich model offers exceptional possibilities to study structures in QFT:

- Formulated as system of Dyson-Schwinger equations (Feynman ribbon graph expansion possible).
- Renormalisation directly in Dyson-Schwinger equations.
- Dyson-Schwinger equations exactly and explicitly solvable (at least for finite-dimensional approximation).
- Connections to complex algebraic geometry (blobbed TR).
- Unexpected simplification in distinguished series of graphs.

Could there be something similar in realistic QFT?

1. Is enormous complexity in QFT partly due to change of variables via complicated inverse of a simple function R?
2. Are there polynomials in correlation functions which are simple (in terms of R^{-1}) and of topological significance?