Matrix models

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Overview

Matrix models are a common topic e.g. in

- enumerative combinatorics,
- quantum gravity in two dimensions,
- complex algebraic geometry,
- quantum fields on noncommutative geometry.

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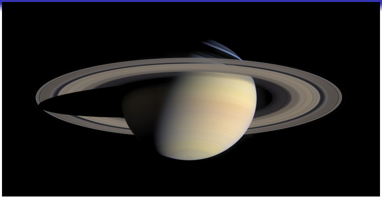
In form of random matrix theory, they are important in

- stochastics,
- free probability.

Enumerative combinatorics

Consider a planet of genus g on which all countries are (possibly degenerate) polygons neighbouring each other.



We are interested in world maps of

- n_3 triangle countries,
- n_4 quadrangle countries,
- n_5 pentagon countries, etc

We admit a fixed number of oceans:

- l_3 of them triangles,
- l_4 of them quadrangles, etc.

How many different world maps are there?

Enumerative combinatorics

William Tutte (1963) counted these numbers in the case of a spherical planet (genus 0) with one ocean (rooted planar maps):

- All countries and the ocean form $n = n_4 + 1$ quadrangles. Then here are $\frac{2 \cdot 3^n}{(n+2)} C_n$ different world maps, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.
- All counties and the ocean from $n_3 + 1$ triangles. Then $n_3 + 1 = 2n$ is even, and there are $\frac{2^{3n+1}\Gamma(\frac{3n}{2}+1)}{(n+2)!\Gamma(\frac{n}{2}+1)}$ different world maps.

Generating functions

Let $M_s^{(g)}(v)$ be the set of world maps on a planet of genus g, with s oceans and in total v vertices.

It gives rise to a generating function

$$W_s^{(g)}(x_1, \dots, x_s, t_1, \dots, t_d; t) = \sum_{v=1}^{\infty} t^v \sum_{m \in M_s^{(g)}(v)} \frac{t_3^{n_3(m)} \cdots t_d^{n_d(m)}}{x_1^{1+l_1(m)} \cdots x_s^{1+l_s(m)}} \frac{1}{\#\mathrm{Aut}(m)}$$

where

- $n_3(m), n_4(m), \dots, n_d(m)$ are the numbers of triangles, quadrangles, ..., d-gons of the world map m
- the j-th ocean in m is a $l_j(m)$-gon, $j = 1, \dots, s$,
- $\#\mathrm{Aut}(m)$ is the order of the automorphism group of the map.

Tutte equations

To count the number of maps, study what happens if one edge at the ocean is removed:

- either both sides are ocean; the map disconnects,
- or one side is land; removing the dyke, that country sinks into the sea.

If $s=1$ and $T_l^{(0)}$ is the x_1^{-1-l} coefficient in $W^{(0)}$ ($\rightarrow$ residue), then

$$T_{l+1}^{(0)} = \sum_{j=0}^{l-1} T_j^{(0)} T_{l-1-j}^{(0)} + \sum_{j=3}^{d} t_j T_{l+j-1}^{(0)}$$

- 1st term: two disconnected continents in the ocean, for which the ocean is a j-gon and a $(l-1-j)$-gon
- 2nd term: one j-gon country sinks into the ocean

These equations can be solved!

Matrix models

[Brézin, Itzykson, Parisi, Zuber 78] gave an interpretation of the generating function as a formal matrix integral

$$\mathcal{Z}(t_3,\dots,t_d;t)$$

$$:= \int_{M_N^*} dM \exp\Big(-N\,\mathrm{Tr}\Big(\frac{M^2}{2t}\Big)\Big)\exp\Big(\frac{N}{t}\,\mathrm{Tr}\Big(\frac{t_3}{3}M^3+\dots+\frac{t_d}{d}M^d\Big)\Big)$$

$$= \sum_{\Sigma\in \substack{\text{disconn. maps}\\ \text{no ocean}}} \Big(\frac{N}{t}\Big)^{\chi(\Sigma)} t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)}\cdots t_d^{n_d(\Sigma)}\cdot\frac{t^{v(\Sigma)}}{\#\mathrm{Aut}(\Sigma)}$$

where

- the integral is over self-adjoint $N\times N$-matrices M, with dM the (normalised) Lebesgue measure.
- $\chi(\Sigma)$ is the Euler characteristics of Σ
- Σ has $n_3(\Sigma)$ triangles, …, $n_d(\Sigma)$ d-gons and in total $v(\Sigma)$ vertices

Background: Gaußian matrix integral

For $dM = 2^{-N/2}(\frac{N}{\pi t})^{N^2/2} \prod_i dM_{ii} \prod_{i<j} d\mathrm{Re}(M_{ij}) d\mathrm{Im}(M_{ij})$:

$$\int_{M_N^*} dM \exp\left(-N\,\mathrm{Tr}\left(\frac{M^2}{2t}\right)\right) = 1$$

$$\int_{M_N^*} dM\; M_{kl} M_{mn} \exp\left(-N\,\mathrm{Tr}\left(\frac{M^2}{2t}\right)\right) = \frac{t}{N}\delta_{kn}\delta_{lm} =: \langle M_{kl} M_{mn}\rangle$$

$$\int_{M_N^*} dM \prod_{i=1}^{v} M_{k_i l_i} \exp\left(-N\,\mathrm{Tr}\left(\frac{M^2}{2t}\right)\right) = \begin{cases} \displaystyle\sum_{\text{pairings}} \prod_{\text{pairs (i,j)}} \langle M_{k_i l_i} M_{k_j l_j}\rangle \\ 0 \text{ if } v \text{ is odd} \end{cases}$$

A pairing is a permutation in which all cycles are of length 2.

In the physics literature, the last property is known as Wick's theorem

Expansion

$$\exp\Big(\frac{N}{t}\mathrm{Tr}\Big(\frac{t_3}{3}M^3+\frac{t_4}{4}M^4+\cdots+\frac{t_d}{d}M^d\Big)\Big)$$
$$=\sum_{n_3,\dots,n_d=0}^{\infty}\frac{1}{n_3!\cdots n_d!}\Big(\frac{N}{t}\Big)^{n_3+\cdots+n_d}\Big(\frac{t_3}{3}\mathrm{Tr}(M^3)\Big)^{n_3}\cdots\Big(\frac{t_d}{d}\mathrm{Tr}(M^d)\Big)^{n_d}$$

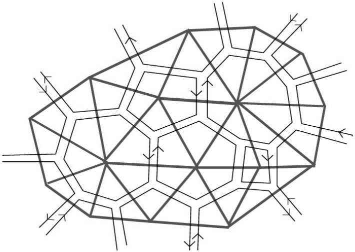
- Under the Gaußian integral, it gives a sum over closed ribbon graphs with n_3 trivalent vertices, ..., n_d d-valent vertices.
- Factorials and $(\frac{1}{k})^{n_k}$ combine to $\frac{1}{\#\mathrm{Aut}(\Sigma)}$
- Each such ribbon graphs Γ comes with prefactor $(\frac{N}{t})^{\chi(\Gamma)}$, where $\chi(\Gamma)=v(\Gamma)-e(\Gamma)+f(\Gamma)$ is the Euler characteristics.

Duality

Ribbon graphs Γ and maps Σ are dual to each other:

- Starting from a map, choose one inner point in every polygon.
- Draw a line between these inner points of two polygons whenever the polygons are neighbours in the map.

The duality sends k-valent vertices into k-gons. It preserves $\chi(\Gamma) = \chi(\Sigma)$



Including oceans

Oceans which are $l_1, \dots, l_s$ gones are generated by

$$\frac{\displaystyle\int_{M_N^*} dM\, \mathrm{Tr}(M^{l_1}) \cdots \mathrm{Tr}(M^{l_s}) e^{-N\,\mathrm{Tr}(\frac{M^2}{2t})} e^{\frac{N}{t}\,\mathrm{Tr}(\frac{t_3}{3}M^3 + \frac{t_4}{4}M^4 + \cdots + \frac{t_d}{d}M^d)}}{\displaystyle\int_{M_N^*} dM\; e^{-N\,\mathrm{Tr}(\frac{M^2}{2t})} e^{\frac{N}{t}\,\mathrm{Tr}(\frac{t_3}{3}M^3 + \frac{t_4}{4}M^4 + \cdots + \frac{t_d}{d}M^d)}}$$

They can be formally collected into resolvents
$W(x) = \mathrm{Tr}((x - M)^{-1})$, for $x \notin \mathbb{R}$.

- The family of these integrals constitute the Hermitean 1-matrix model.
- Generalisations to several matrices M_i exist where potentials $V_j(M_j)$ are connected by terms $\exp(M_i M_j)$.

Tutte equations = loop equations

Derivatives of generating function $\mathcal{Z}(t_1 = 0, t_2 = -1, t_3, \dots, t_d)$ of ribbon graphs are not independent:

$$0 = \Big(\sum_{j=1}^{\infty}(k+j)t_j\frac{\partial}{\partial t_{k+j}} + \frac{t^2}{N^2}\sum_{l=1}^{k-1} l(k-l)\frac{\partial^2}{\partial t_l \partial t_{k-l}} + 2tk\frac{\partial}{\partial t_k}\Big)\mathcal{Z}$$

- Up to conjugation, these differential ops become generators L_k of the Witt (Virasoro?) algebra, $[L_k, L_l] = (k-l)L_{k+l}$.
- Recursive solution of constraints gives first terms of $\mathcal{Z}$.
- Provides connections to other structures with Virasoro constraints.
- KdV-hierarchy of PDE is the integrable structure of Hermitean 1-matrix model.

Quantum gravity I

Aim: make sense of $\displaystyle\sum_{\text{topologies}} \int_{\text{metrics}} dg\; e^{-\int_{M_g} \frac{\kappa}{2}(\text{scal}(g)-2\Lambda)d\text{vol}(g)}$

- In $D = 2$ dimensions, by Gauß-Bonnet this reduces to Euler characteristics and total volume.
- A substitute for the g-integral is the sum over world maps where each country has unit weight.
- Thus, $\mathcal{Z}(t_3, \dots, t_d; t)$ serves as partition function of 2D quantum gravity.

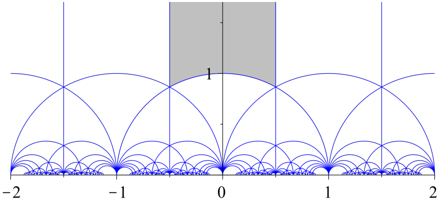
Quantum gravity II

Another proposal

- Integrate, for each genus-g planet with s oceans, over the space of diffeomorphism classes of (g, s)-planets. These form the moduli space $\mathcal{M}_{g,s}$ of complex curves.
- Compactification is necessary to have a meaningful integral.
- Being of topological nature, 2D-quantum gravity should be expressed in terms of topological invariants of $\overline{\mathcal{M}}_{g,s}$.

The moduli space of complex curves

- Any two tori (of genus 1) are homotopic, but not necessarily (complex-) diffeomorphic to each other.
- The equivalence classes of tori with marked point 0 are parametrised by points in the fundamental domain



- Compactified by adding the unique sphere with three marked points, of which two are glued to a pinched torus.

In general, the moduli space of genus-g curves with s marked points is a space of complex dimension $3g + s - 3$.

It is an orbifold, similar to a manifold, but with corners.

Intersection numbers

Consider on $\overline{\mathcal{M}}_{g,s}$ a family $\{\mathcal{L}_1, \dots, \mathcal{L}_s)\}$ of line bundles:

- Fibre of $\mathcal{L}_i$ over $x \in \overline{\mathcal{M}}_{g,s}$, which is a (nodal) curve $x = \mathcal{C}$, is the cotangent space of $\mathcal{C}$ at the i-th marked point.
- These bundles are classified by their first Chern class $\psi = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,s}, \mathbb{Q})$.
- Intersection numbers of Chern classes over $\overline{\mathcal{M}}_{g,s}$:

$$\langle \tau_{d_1} \cdots \tau_{d_s} \rangle := \int_{\overline{\mathcal{M}}_{g,s}} \prod_{j=1}^{s} \left(c_1(\mathcal{L}_j)\right)^{d_j} \in \mathbb{Q}$$

Collect them into a generating function

$$\mathcal{F}(t_0, t_1, \dots) = \sum_{k_0, k_1, \dots = 0}^{\infty} \langle \tau_0^{k_0} \tau_1^{k_1} \dots \rangle \prod_{i=0}^{\infty} \frac{t_i^{k_i}}{k_i!}$$

Witten's conjecture

Uniqueness of 2D-quantum gravity implies that generating functions of intersection numbers on $\overline{\mathcal{M}}_{g,s}$ and of genus-g world maps with s oceans are the same:

Conjecture [Witten 91]

1. The generating function $\mathcal{F}$ obeys the string equation
$$\frac{\partial \mathcal{F}}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial \mathcal{F}}{\partial t_i}$$
2. $U(\{t\}) := \frac{\partial^2}{\partial t_0^2} \mathcal{F}(\{t\})$ satisfies the KdV equations
$$\frac{\partial U}{\partial t_n} = \frac{\partial}{\partial t_0} R_{n+1}(U, \partial_{t_0} U, \partial_{t_0}^2 U, \dots)$$
for polynomials R_n recursively defined by $R_1(U) = U$ and
$$\frac{\partial}{\partial t_0} R_{n+1} = \frac{1}{2n+1} \Big(R_n \frac{\partial U}{\partial t_0} + 2U \frac{\partial R_n}{\partial t_0} + \frac{1}{4} \frac{\partial^3 R_n}{\partial t_0^3} \Big)$$

Kontsevich's proof of the Witten conjecture

Theorem [Kontsevich 91]

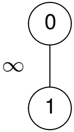
$$\sum_{d_1,\dots,d_s=0}^{\infty} \langle \tau_{d_1} \cdots \tau_{d_s} \rangle \prod_{i=1}^{s} \frac{(2d_i-1)!!}{\lambda_i^{2d_i+1}} = \sum_{\Gamma \in \mathcal{RG}_s^3} \frac{2^{-|\mathcal{V}(\Gamma)|}}{\#\mathrm{Aut}(\Gamma)} \prod_{e \in \mathcal{E}(\Gamma)} \frac{2}{\lambda'(e)+\lambda''(e)}$$

where

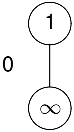
- The sum is over ribbon graphs with s faces. All vertices are 3-valent.
- $\mathcal{V}(\Gamma)$, $\mathcal{E}(\Gamma)$ are the sets of vertices and edges.
- The faces are labelled by formal variables $\lambda_1, \dots, \lambda_s$.
- $\lambda'(e), \lambda''(e)$ label the two faces separated by edge e.

The proof relies on quadratic differentials [Strebel 67].

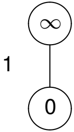
Example: $\mathcal{M}_{0,3}$



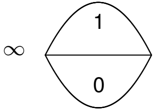
$$\frac{1}{4}\cdot\frac{2}{\lambda_0+\lambda_\infty}\cdot\frac{2}{\lambda_1+\lambda_\infty}\cdot\frac{2}{\lambda_\infty+\lambda_\infty}$$



$$+\frac{1}{4}\cdot\frac{2}{\lambda_0+\lambda_1}\cdot\frac{2}{\lambda_0+\lambda_\infty}\cdot\frac{2}{\lambda_0+\lambda_0}$$



$$+\frac{1}{4}\cdot\frac{2}{\lambda_0+\lambda_1}\cdot\frac{2}{\lambda_1+\lambda_\infty}\cdot\frac{2}{\lambda_1+\lambda_1}$$



$$+\frac{1}{4}\cdot\frac{2}{\lambda_0+\lambda_1}\cdot\frac{2}{\lambda_1+\lambda_\infty}\cdot\frac{2}{\lambda_0+\lambda_\infty} = \frac{1}{\lambda_0\lambda_1\lambda_\infty}$$

$$\langle\tau_0\tau_0\tau_0\rangle = 1$$

The Kontsevich model

- We have seen that ribbon graphs arise in matrix integrals
- Here we need to assign weight $\frac{1}{\lambda(e)'+\lambda(e)''}$ to every edge and admit only 3-valent vertices:

Theorem [Kontsevich]

$$\mathcal{F}(t_0, t_1, t_2, t_3, \dots)$$

$$:= \log\left(\frac{\displaystyle\int_{M_N^*} dM\, e^{-\frac{1}{2}\mathrm{Tr}(EM^2)+\frac{\mathrm{i}}{6}\mathrm{Tr}(M^3)}}{\displaystyle\int_{M_N^*} dM\, e^{-\frac{1}{2}\mathrm{Tr}(EM^2)}}\right) = \sum_{k_0,k_1,\dots=0}^{\infty} \langle \tau_0^{k_0}\tau_1^{k_1}\dots\rangle \prod_{i=0}^{\infty}\frac{t_i^{k_i}}{k_i!}$$

where $E = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$ and $t_i := -(2i-1)!!\mathrm{Tr}(E^{-2i-1})$.

From this representation one shows that $\mathcal{Z} := \exp(\mathcal{F})$ satisfies the same Virasoro constraints as the $\mathcal{Z}$ of the 1-matrix model.

Topological recursion

1. Tutte equations,
2. loop equations,
3. Virasoro constraints,
4. Dyson-Schwinger equations, etc,

are manifestations of a universal structure called topological recursion [Eynard, Orantin 07].

Topological recursion

- Main ingredient is a polynomial equation $\mathcal{E}(x, y) = 0$ for a plane algebraic curve: the (classical) spectral curve.
- Solution parametrised by meromorphic functions $x(z), y(z)$.
- In the Kontsevich model, $y(z)$ is the expectation value of a single resolvent $W(z) = \mathrm{Tr}((z - M)^{-1})$, and $x(z) = z^2 - c$.
- Starting from a universal 2-form $\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$, a family $\omega_{g,s}$ of s-forms on $\overline{\mathbb{C}}^s$ is constructed which satisfy universal recursive equations.

These equations can be solved by residue operations!

Quantum field theory on noncommutative geometries

- In the Euclidean picture, a quantum field is a noncommutative random variable.
- Take a Fréchet $*$-algebra $\mathcal{A}$ for which $\mathcal{A}_* = \{a = a^*\}$ is a real nuclear vector space.
  Take a positive bilinear form $C(a, b)$ on $\mathcal{A}_*$.
- Then $\mathcal{F}(a) := \exp(-\frac{1}{2}C(a, a))$ is of positive type and defines by Bochner-Minlos theorem a unique measure $d\mu_C(\Phi)$ on $\mathcal{A}'_*$ with $\mathcal{F}(a) = \int_{\mathcal{A}'_*} e^{\mathrm{i}\Phi(a)} d\mu(\Phi)$.

We would like to define moments

$$\frac{\int d\mu_C(\Phi)\ \Phi(a_1)\cdots\Phi(a_k)e^{-V(\Phi)}}{\int d\mu_C(\Phi)\ e^{-V(\Phi)}}$$

of a 'deformed' measure, for some non-linear functional V on $\mathcal{A}'_*$.

Quantum field theory on noncommutative geometries

- As in all QFTs, this deformation fails because of singularities.
- We need finite-dimensional approximations.
  Then redefine parameters to ensure a limit.
- Finite-dimensional approximations to noncommutative algebras are matrix algebras.

Consequently, matrix models arise in all QFTs on noncommutative geometry.

The inner product $C(a, b) = \sum_{k,l} \frac{a_{kl} b_{lk}}{e_a + e_b}$ gives the Kontsevich class.

There are a few differences:

- We consider all matrix sizes N and admit additional N-dependences $t_i \mapsto -\lambda_i(N)$ so that limit $N \to \infty$ exists.
- Much more general moments can be considered, not only those of topological significance.

The Φ^3 matricial QFT model

Fact: Moments in matrix models have topological expansion

$$\log\left(\frac{\int_{M_N^*} d\Phi\ \Phi_{k_1l_1}\cdots\Phi_{k_nl_n} e^{-N\,S(\Phi)}}{\int_{M_N^*} d\Phi\ e^{-N\,S(\Phi)}}\right) = \sum_{s=1}^{n}\sum_{g=0}^{\infty} N^{2-2g-s} G^{(g)}_{k_1^1\ldots k_{n_1}^1|\ldots|k_1^s\ldots k_{n_s}^s}$$

(the l's are a permutation of k's which has s cycles)

Theorem [Grosse, Sako, W 16; Grosse, Hock, W 19]

Given any increasing (e_k) with $\sum_k e_k^{-4} < \infty$. Then there exist functions $Z(N), \mu(N), \lambda(N), \kappa(N), \nu(N), \xi(N)$ such that for

$$S_N(\Phi) = \mathrm{Tr}_N((ZE-\mu)\Phi^2 + (\kappa+\nu E+\xi E^2)\Phi + Z^{\frac{3}{2}}\lambda\Phi^3)\ ,$$

any (g,s)-homogeneous contribution $G^{(g)}_{k_1^1\ldots k_{N_1}^1|\ldots|k_1^s\ldots k_{N_s}^s}$ to any moment not only has a limit for $N\to\infty$, but this limit can be written down explicitly.

Recent highlight: Exact solution of the Φ^4-model

Theorem [Grosse, Hock, W 19]

Consider quartic measure $e^{-N\,\mathrm{Tr}(E\Phi^2+(\lambda/4)\Phi^4)}d\Phi$, where E has eigenvalues $0 < e_1 < \cdots < e_d$ of multiplicities $r_1, \ldots, r_d$.
Take solutions $\{\varepsilon_k, \varrho_k\}$ with $\lim_{\lambda\to 0}\varepsilon_k = e_k$, $\lim_{\lambda\to 0}\varrho_k = r_k$ of

$$e_l = \varepsilon_l - \frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{\varepsilon_k+\varepsilon_l}\,, \qquad 1 = \frac{r_l}{\varrho_l} - \frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{(\varepsilon_k+\varepsilon_l)^2}$$

Then:

$$G^{(0)}_{ab} = \frac{1}{\varepsilon_a+\varepsilon_b}\cdot\frac{\prod_{k,l=1}^{d}\Big(1+\frac{\sigma_k(e_a)+\sigma_l(e_b)}{\varepsilon_k+\varepsilon_l}\Big)}{\prod_{k,l=1}^{d}\Big(1+\frac{\sigma_k(e_a)}{\varepsilon_k+\varepsilon_l}\Big)\prod_{k,l=1}^{d}\Big(1+\frac{\sigma_l(e_b)}{\varepsilon_k+\varepsilon_l}\Big)}$$

where $\{\varepsilon_a, -\varepsilon_1-\sigma_1(a), \ldots, -\varepsilon_d-\sigma_d(a)\}$ are all inverse solutions z of $e_a = z - \frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{\varepsilon_k+z}$