Solvable quantum field theories on noncommutative geometry

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What is QFT? – physics point of view

- Quantum field theory (QFT) is the theory that describes Nature at very high energy density.
- One famous such experiment measures the magnetic moment g of the electron: $\frac{g_{experiment}}{2} = 1.001\,159\,652\,180\,7$
- QFT predicts that number in terms of the electron charge e measured to $e^{-2} = 137.035\,999\,084$:

$$\frac{g_{QFT}}{2} = 1 + \frac{1}{2}\frac{e^2}{\pi} + \Big\{\frac{197}{144} + \Big(\frac{1}{2} - 3\log 2\Big)\zeta(2) + \frac{3}{4}\zeta(3)\Big\}\Big(\frac{e^2}{\pi}\Big)^2$$
$$+ \Big\{\frac{28259}{5184} + \Big(\frac{17101}{235} - \frac{596}{2}\cdot\log 2\Big)\zeta(2) + \frac{139}{18}\zeta(3) + \frac{100}{3}\mathrm{Li}_4\Big(\frac{1}{2}\Big)$$
$$+ \frac{25}{3}\Big(\frac{\log^4 2}{6} - \zeta(2)\log^2 2\Big) - \frac{239\zeta(4) - 166\zeta(2)\cdot\zeta(3) + 215\zeta(5)}{24}\Big\}\Big(\frac{e^2}{\pi}\Big)^3$$
$$+ \Big\{\dots\Big\}\Big(\frac{e^2}{\pi}\Big)^4$$
$$= 1.001\,159\,652\,153\,5$$

What is QFT? – mathematics point of view

A quantum field is an unbounded operator-valued distribution $\mathcal{S}^D \ni f \mapsto \Phi(f) : \mathcal{D} \to \mathcal{D} \subset \mathcal{H}$ satisfying Wightman's axioms

1. covariance: $U(a,L)^*\Phi(f)U(a,L) = \Phi(f^{a,L})$, $f^{a,L}(x) = f(L^{-1}(x-a))$
2. vacuum: 1-dim subspace of invariant vectors $U(a,L)\Omega = \Omega$
3. positive spectrum: if $U(a,1) = \exp(\mathrm{i}\sum_\mu a_\mu P_\mu)$, then (joint spectrum of P_μ) $\subseteq V_+$ (forward lightcone)
4. locality: for f, g causally independent, $[\Phi(f), \Phi(g)] = 0$

Unbounded operators are difficult

1. Haag-Kastler: $\mathbb{R}^D \supset \mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ – type-III von Neumann alg.
2. via Wightman distributions to Schwinger functions

The Euclidean approach

Wightman distributions $W(x_1,\dots,x_N) := \langle\Omega, \Phi(x_1)\cdots\Phi(x_N)\Omega\rangle$
...are boundary values of holomorphic functions

(initially in $x_j - x_{j+1} = \xi_j - \mathrm{i}\eta_j$, $\eta_j \in V_+$
but extended by Poincaré$_{\mathbb{C}}$ and locality)

- restriction of $W(z_1,\dots,z_N)$ to real subspace of Euclidean points (minus diagonals) defines Schwinger functions
- Schwinger functions inherit real analyticity, Euclidean invariance, complete symmetry and reflection positivity

Theorem [Osterwalder-Schrader, 1974]

These properties are sufficient to reconstruct Wightman theory!

Fruitful exchange with statistical physics and probability theory
...but so far no non-trivial QFT model in 4 dimensions

Why is it difficult?

Schwinger 2-point function in momentum space

$$\hat{S}_2(p) \propto \frac{1}{(p^2+m^2)^{1-\eta/2}}, \qquad \eta \text{ – anomalous dimension}$$

- reflection positivity requires $\eta \geq 0$
- ...but convergence in 4D needs $\eta \leq 0$

proposed way out: require decay of effective vertex functions with $\|p_i\| \to \infty$ (asymptotic freedom)

- Yang-Mills theory expected to have asymptotic freedom
- tendency to produce infrared problems ($\to$ confinement), no solution so far, Millenium Prize Problem for Yang-Mills

Circumvent convergence via exactly solvable models

A new class of solvable models arises from noncommutative geometry. Reflection positivity is on the agenda (not yet done).

The free Euclidean quantum field

$\mathcal{A}$ – Fréchet $*$-algebra, $\mathcal{A}_*$ its subspace of self-adjoint elements.

Later: $\mathcal{A} = \bigcup_{\mathcal{N}=1}^{\infty} \mathcal{A}^{\mathcal{N}}$ with $\mathcal{A}^{\mathcal{N}+1} = \begin{pmatrix} \mathcal{A}^{\mathcal{N}} & 0 \\ 0 & 0 \end{pmatrix}$ finite-dimensional

Theorem [Bochner-Minlos]

Let $\mathcal{F} : \mathcal{A}_* \to \mathbb{R}$ with $\mathcal{F}(0) = 1$ be continuous and of positive type: $\sum_{i,j=1}^{K} c_i \bar{c}_j \mathcal{F}(a_i - a_j) \geq 0$ for any $a_i \in \mathcal{A}_*$, $c_i \in \mathbb{C}$.
Then $\exists!$ Borel measure $d\mathcal{M}$ on the dual space $\mathcal{A}'_*$ with

$$\mathcal{F}(a) = \int_{\mathcal{A}'_*} e^{\mathrm{i}\Phi(a)} d\mathcal{M}(\Phi)$$

For any inner product $C : \mathcal{A}_* \times \mathcal{A}_* \to \mathbb{R}$, called covariance, $\mathcal{F}(a) := \exp(-\frac{1}{2}C(a,a))$ is of positive type and defines $d\mathcal{M}_C(\Phi)$

A choice

- $\mathcal{A}^{\mathcal{N}} = \mathrm{span}(e_{kl} \;:\; k, l \leq \mathcal{N})$, $\quad e_{kl}e_{mn} = \delta_{lm}e_{kn}$, $(e_{kl})^* = e_{lk}$
- $C_E(e_{kl}, e_{mn}) = \dfrac{\delta_{kn}\delta_{lm}}{V\tilde{Z}(\tilde{E}_k + \tilde{E}_l)}$ for $(\tilde{E}_k)$ positive & increasing

Formal approach to interacting quantum fields

For functional S_{int} on $\mathcal{A}'_*$, define moments of perturbed measure

$$\langle a_1 \otimes \cdots \otimes a_N \rangle := \frac{\int_{\mathcal{A}'_*} d\mathcal{M}_C(\Phi)\, \Phi(a_1)\cdots\Phi(a_N)\exp(-S_{int}(\Phi))}{\int_{\mathcal{A}'_*} d\mathcal{M}_C(\Phi)\, \exp(-S_{int}(\Phi))}$$

- OK for finite $\mathcal{N}$, but limit $\mathcal{N} \to \infty$ enormous challenge
- existence for commutative $\mathcal{A}$ only established in a few (super-renormalisable, fermionic) cases
- exceptional new tools if $\mathcal{A}$ is limit of matrix algebras!

Strategy

1. determine sequences $\tilde{Z}, \tilde{\lambda}, \ldots$ parametrising S_{int} such that some moments constant under $\mathcal{A}^{\mathcal{N}} \hookrightarrow \mathcal{A}^{\mathcal{N}+1}$ (renormalisation)
2. find equations between moments, well-defined for $\mathcal{N} \to \infty$
3. show that all can be solved recursively, starting from an initial solution of a non-linear problem

Two solvable models

define $\Phi_{kl} := \Phi(e_{kl})$, recall $C_E(e_{kl}, e_{mn}) = \frac{1}{V\tilde{Z}} \frac{\delta_{kn}\delta_{lm}}{\tilde{E}_k+\tilde{E}_l}$,
choose $\tilde{E}_k = \frac{\tilde{\mu}^2}{2} + \mu^2 e(\frac{k}{\mu^2 V})$ with $e(0) = 0$

3. The Φ^3 model, closely related to [Kontsevich, 1991]-model

$$S_{int}(\Phi) = \frac{V\tilde{Z}^{3/2}\tilde{\lambda}}{3} \sum_{k,l,m} \Phi_{kl}\Phi_{lm}\Phi_{mk} + \sum_k (\tilde{\kappa} + \tilde{\nu}\tilde{E}_k + \tilde{\zeta}\tilde{E}_k^2)\Phi_{kk}$$

Remark: $\int_{\mathcal{A}'_*} d\mathcal{M}_C(\Phi) \exp(-S_{int}(\Phi))$ is a formal power series in $t_i = (2i-1)!! \sum_k \tilde{E}_k^{-2i-1}$ which generates intersection numbers on moduli space $\overline{\mathcal{M}}_{g,s}$ of complex curves

4. The Φ^4 model

$$S_{int}(\Phi) = \frac{V\tilde{Z}^2\lambda}{4} \sum_{k,l,m,n} \Phi_{kl}\Phi_{lm}\Phi_{mn}\Phi_{nk}$$

suggested the method [Grosse-W.], but more complicated

The partition function and its derivatives

$\mathcal{Z}_C(J) = \int_{\mathcal{A}'_*} d\mathcal{M}_C(\Phi)\; e^{-S_{int}(\Phi)} e^{\mathrm{i}\Phi(J)}$

- generates N-point functions by

$$\langle a_1 \otimes \cdots \otimes a_N \rangle = \frac{(-\mathrm{i})^N}{\mathcal{Z}_C(0)} \frac{\partial^N \mathcal{Z}_C(t_1 a_1 + \cdots + t_N a_N)}{\partial t_1 \cdots \partial t_N}\Big|_{t_i=0}$$

- multiple J-derivatives are zero unless organised in cycles:

$$\sum_{g=0}^{\infty} V^{2-2g-B} G^{(g)}_{|k_1^1 \dots k_{N_1}^1|\dots|k_1^B \dots k_{N_B}^B|} = \frac{\partial^{N_1+\dots+N_B} \log \mathcal{Z}_C(J)}{\partial \mathbb{J}_{k_{N_B}^B \dots k_1^B} \cdots \partial \mathbb{J}_{k_{N_1}^1 \dots k_1^1}}\Big|_{J\equiv 0},$$

where $\frac{\partial^N}{\partial \mathbb{J}_{k_N \dots k_1}} = (-\mathrm{i})^N \frac{\partial^N}{\partial J_{k_N k_{N-1}} \cdots \partial J_{k_2 k_1} \partial J_{k_1 k_N}}$

These J-derivatives combine with $e^{-S_{int}(\frac{\partial}{\partial J_{kl}})}$ to Dyson-Schwinger equations between the $G^{(g)}_{|k_1^1 \dots k_{N_1}^1|\dots|k_1^B \dots k_{N_B}^B|}$

The partition function and its derivatives

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- generates N-point functions by

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Dyson-Schwinger equations for Φ^3-model: Part I.

A closed non-linear equation for $W^{(0)}_{|k|} := 2\lambda G^{(0)}_{|k|} + 2E_K$:

$$(W^{(0)}_{|k|})^2 + 2\tilde{\nu}\lambda W^{(0)}_{|k|} + \frac{2\lambda^2}{V}\sum_n \frac{W^{(0)}_{|k|} - W^{(0)}_{|n|}}{E_k^2 - E_n^2} = \frac{4E_k^2}{\tilde{Z}} - \tilde{\kappa}'$$

- with $E_k = \tilde{E}_k - \frac{\tilde{\mu}^2}{2} + \frac{\mu^2}{2}$, $\quad \tilde{\kappa}' = \tilde{\nu}^2\lambda^2(1+\tilde{Z}^{-1}) + 4\tilde{\kappa}'\lambda\tilde{Z}^{-1}$
- first renormalisation: $\lambda = \tilde{\lambda}\tilde{Z}^{\frac{1}{2}}$, $\tilde{\mu}^2 = \mu^2 + \lambda\tilde{\nu}$, $\lambda\tilde{\zeta} = \tilde{Z} - 1$

Key observation

$W^{(0)}_{|k|}$ extends to a meromorphic function $W(z)$

where $z^2 = 4E_k^2 + c$ and $W^{(0)}_{|k|} = W((4E_k^2 + c)^{1/2})$

solvable by complex analysis [Makeenko-Semenoff, 1992]

$$W^{(0)}(z) = \frac{z}{\tilde{Z}^{1/2}} - \lambda\tilde{\nu} + \frac{4\lambda^2}{V}\sum_n \frac{1}{\big(z + (4E_n^2+c)^{1/2}\big)(4E_n^2+c)^{1/2}}$$

where $-\tilde{\kappa}' = \frac{c}{\tilde{Z}} + \frac{8\lambda^2}{V\tilde{Z}^{1/2}}\sum_n \frac{1}{(4E_n^2+c)^{1/2}}$ $\qquad \to \tilde{\nu}, \mathcal{Z}, c$

Dyson-Schwinger equations for Φ^3-model: Part II.

All other functions satisfy affine equations which are iteratively solvable. First, everything relates to $1+\dots+1$-point function:

$$G^{(g)}(z_1^1,\dots,z_{N_1}^1|\dots|z_1^B,\dots,z_{N_B}^B)$$
$$=\sum_{i_1=1}^{N_1}\dots\sum_{i_B=1}^{N_B} G^{(g)}(z_{i_1}^1|\dots|z_{i_B}^B)\Big(\prod_{i_1\neq j_1=1}^{N_1}\frac{\lambda}{z_{i_1}^1-z_{j_1}^1}\Big)\cdots\Big(\prod_{i_B\neq j_B}^{N_B}\frac{\lambda}{z_{i_B}^B-z_{j_B}^B}\Big)$$

For $g=0$ these are obtained in [Grosse-Sako-W., 2017]

$$G^{(0)}(z_1|z_2)=\frac{4\lambda^2}{z_1 z_2(z_1+z_2)^2}$$

$$G^{(0)}(z_1|\dots|z_B)=\frac{d^{B-3}}{dt^{B-3}}\Big(\frac{(-2\lambda)^{3B-4}}{(R(t))^{B-2}\prod_{\beta=1}^B(z_\beta^2-2t)^{\frac{3}{2}}}\Big)\Big|_{t=0},$$

$$R(t):=\frac{1}{\tilde{Z}^{\frac{1}{2}}}-\frac{4\lambda^2}{V}\sum_n\frac{1}{(4E_n^2+c)^{\frac{1}{2}}\big((4E_n^2+c)^{\frac{1}{2}}+(4E_n^2+c-2t)^{\frac{1}{2}}\big)(4E_n^2+c-2t)^{\frac{1}{2}}}$$

Relation to Topological recursion?

(asked by Roland Speicher, confirmed by Alexander Hock)

Topological recursion [Eynard, Orantin, Chekhov,. . .]

consists of a spectral curve $y(x)$ (varies from case to case) together with a universal recursive definition of sequences $\omega_{g,B}$ of meromorphic B-forms, starting with $\omega_{0,2}(z_1, z_2) = \frac{\text{const}\cdot dz_1 dz_2}{(z_1 - z_2)^2}$

- matrix models
- Kontsevich model, hyperbolic volumes of moduli spaces
- Gromov-Witten invariants

- $y(z) = \lim_{\mathcal{N}\to\infty}\Big(\frac{z}{2\lambda\tilde{Z}^{\frac{1}{2}}} - \frac{\tilde{\nu}}{2} + \frac{2\lambda}{V}\sum_n \frac{1}{(z+(4E_n^2+c)^{\frac{1}{2}})(4E_n^2+c)^{\frac{1}{2}}}\Big)$
  with $z^2 = x$ (branch point at $z = 0$, involution $\sigma_0(z) = -z$)
- $\omega_{g,B}(z_1, \dots, z_B) = z_1\cdots z_B G^{(g)}(z_1|\dots|z_B)dz_1\cdots dz_B$, $2g+B\geq 3$
- Dyson-Schwinger eqs. = recursion [Grosse-Hock-W., soon]

Dyson-Schwinger equations for Φ^4-model

Theorem [Grosse-W., 2011-15]

1. $\exists$ closed non-linear equation for $G^{(0)}_{|kl|}$:

$$G^{(0)}_{|kl|} = \frac{1}{\tilde{Z}(\tilde{E}_k+\tilde{E}_l)} - \frac{\tilde{Z}\lambda}{V(\tilde{E}_k+\tilde{E}_l)} \sum_n \Big(G^{(0)}_{|kl|} G^{(0)}_{|kn|} - \frac{G^{(0)}_{|nl|} - G^{(0)}_{|kl|}}{\tilde{Z}(\tilde{E}_n - \tilde{E}_k)} \Big)$$

2. extension to sectionally-holomorphic function $G(z_1, z_2)$
3. in limit $\frac{1}{V}\sum_n \delta(t-\mu^2 e(\frac{n}{\mu^2 V})) \to \rho(t) \in C([0,\Lambda^2])$

   partially solved by $G(a,b) = \dfrac{e^{\mathcal{H}^\Lambda_a[\tau_b(\bullet)]} \sin \tau_b(a)}{\tilde{Z}\lambda\pi\rho(a)}$,

   where $\tau_b(a) := \arctan_{[0,\pi]} \left(\dfrac{\lambda\pi\rho(a)}{b + \frac{\tilde{Z}^{-1} + \lambda\pi\mathcal{H}^\Lambda_a[\rho(\bullet)G(\bullet,0)]}{G(a,0)}} \right)$

   and $\mathcal{H}^\Lambda_a[f(\bullet)]$ – Hilbert transform over $[0,\Lambda^2]$
4. solution exists by Schauder fixed point theorem

Solution for $\rho(t) = 1$ [joint work with Erik Panzer, 2018]

Key Lemma

For any Hölder-continuous function f on $(0, \Lambda^2)$ one has

$$\int_0^{\Lambda^2} dp\, e^{\pm \mathcal{H}^\Lambda_p[f(\bullet)]} \sin(f(p)) = \int_0^{\Lambda^2} dp\, f(p)$$

gives new (and closed!) equation for $\tau_b(a)$

$$\tau_b(a) = \arctan_{[0,\pi]}\left(\frac{\lambda\pi}{1+a+b-\lambda\log a + \frac{1}{\pi}\int_0^\infty dp\,\left(\tau_a(p) - \frac{\lambda\pi}{1+p}\right)}\right)$$

Theorem (for $\lambda > 0$)

$$\tau_a(p) = \arctan_{[0,\pi]}\left(\frac{\lambda\pi}{a + \lambda W_0\left(\frac{1}{\lambda}e^{\frac{1+p}{\lambda}}\right) - \lambda\log\left(\lambda W_0\left(\frac{1}{\lambda}e^{\frac{1+p}{\lambda}}\right) - 1\right)}\right)$$

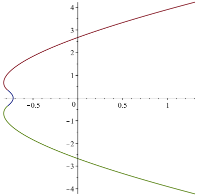
- W_0 – principal branch of Lambert function, $W(z)e^{W(z)} = z$
- found by series expansion, computer algebra and Lagrange-Bürmann resummation

Solution for $\rho(t) = 1$ [joint work with Erik Panzer, 2018]

$$G(a,b) = \frac{(1+a+b)\exp(N_\lambda(a,b))}{\big(a+\lambda W\big(\frac{1}{\lambda}e^{(1+b)/\lambda}\big)\big)\big(b+\lambda W\big(\frac{1}{\lambda}e^{(1+a)/\lambda}\big)\big)} \quad \text{where}$$

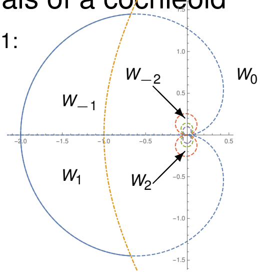
$$N_\lambda(a,b) := \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} dt\; \log\Big(1-\frac{\lambda\log(\frac{1}{2}-\mathrm{i}t)}{a+\frac{1}{2}+\mathrm{i}t}\Big)\frac{\partial}{\partial t}\log\Big(1-\frac{\lambda\log(\frac{1}{2}+\mathrm{i}t)}{b+\frac{1}{2}-\mathrm{i}t}\Big)$$

- holomorphic in complex λ-domain $\supset (-\frac{1}{\log 4}, \infty)$



- Lambert-W branches changed at spirals of a cochleoid

for $a = 1$:



- $N_\lambda(a,b)$ has series expansion in Nielsen's polylogarithms $S_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 dt\,\frac{\log^{n-1}(t)\log^p(1-zt)}{t}$ and Riemann's $\zeta(n)$

Outlook

What about higher functions? For $B = 1$, $g = 0$ we know that

$$G^{(0)}_{|k_0k_1\dots k_{N-1}|} = (-\lambda)\sum_{l=1}^{\frac{N-2}{2}} \frac{G^{(0)}_{|k_0k_1\dots k_{2l-1}|}G^{(0)}_{|k_{2l}k_{2l+1}\dots k_{N-1}|} - G^{(0)}_{|k_{2l}k_1\dots k_{2l-1}|}G^{(0)}_{|k_0k_{2l+1}\dots k_{N-1}|}}{(E_{k_0}-E_{k_{2l}})(E_{k_1}-E_{k_{N-1}})}$$

is organised into Catalan tables [de Jong-Hock-W., soon]

Open problems

- Is there topological recursion for the $(2-2g-B \leq 0)$-sector of the Φ^4-model? What is its spectral curve?
- Is the Φ^4-equation also solvable for $\rho(t) \neq 1$?
- What classes of functions are involved?
- Does the Φ^4-model compute any intersection numbers on some moduli space?
- What can be said about Osterwalder-Schrader positivity?