Lambert-W solves the noncommutative Φ^4-model

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joint work with Erik Panzer, arXiv:1807.02945 (today)

based on long-term collaboration with Harald Grosse

Conjecture

The non-linear integral equation for the $\lambda\Phi_2^{\star 4}$ two-point function

$$
\begin{aligned}
(1+a+b)G_\lambda(a,b) &= 1 + \lambda\int_0^\infty dp \Big(\frac{G_\lambda(p,b)-G_\lambda(a,b)}{p-a} + \frac{G_\lambda(a,b)}{1+p}\Big) \\
&\quad + \lambda\int_0^\infty dq \Big(\frac{G_\lambda(a,q)-G_\lambda(a,b)}{q-b} + \frac{G_\lambda(a,b)}{1+q}\Big) \\
&\quad - \lambda^2\int_0^\infty dp\int_0^\infty dq\, \frac{G_\lambda(a,b)G_\lambda(p,q)-G_\lambda(a,q)G_\lambda(p,b)}{(p-a)(q-b)}
\end{aligned}
$$

where $a, b \in \mathbb{R}_+$, is for any $\lambda > -\frac{1}{2\log 2} \approx -0.721348$ solved by

$$
G_\lambda(a,b) = G_\lambda(b,a) = \frac{(1+a+b)\exp(N_\lambda(a,b))}{\big(b+\lambda W\big(\frac{1}{\lambda}e^{(1+a)/\lambda}\big)\big)\big(a+\lambda W\big(\frac{1}{\lambda}e^{(1+b)/\lambda}\big)\big)}
$$

where W is the Lambert function, $W(z)e^{W(z)} = z$, and

$$
N_\lambda(a,b) := \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} dt\, \log\Big(1-\frac{\lambda\log(\frac{1}{2}-\mathrm{i}t)}{b+\frac{1}{2}+\mathrm{i}t}\Big)\frac{d}{dt}\log\Big(1-\frac{\lambda\log(\frac{1}{2}+\mathrm{i}t)}{a+\frac{1}{2}-\mathrm{i}t}\Big)
$$

$N_\lambda(a,b)$ expands into Nielsen polylogarithms and Riemann ζ's.

Noncommutative quantum field theory

This formula removes the last obstacle to construct the functor

$$\mathrm{NCQFT}_{\blacksquare} : \mathfrak{Cobord}_2^2 \to \mathfrak{Vect}$$

which assigns

to any 2-manifold Σ with boundaries and boundary defects [identified with $\Sigma = (I_1^1 \times \cdots \times I_{N_1}^1) \otimes \cdots \otimes (I_1^B \times \cdots \times I_{N_B}^B)$]

a function $G_\Sigma \in C^\infty(\mathbb{R}_+^\Sigma)$,

compatible with

- gluing of defects + gluing of $\blacksquare$ $\leftrightarrow$ Dyson-Schwinger eq. for covariance $C(x,y) = \frac{1}{\mu^2+x+y}$
- $\mathrm{NCQFT}_{\blacksquare}(\Sigma_g) = 0$ for $g \geq 1$ (large-$\mathcal{N}$ limit)
- Ward identity of Disertori-Gurau-Magnen-Rivasseau

Remark: $\mathrm{NCQFT}_{\blacktriangle}$ was constructed in [Grosse-Sako-W, 2016]

Dyson-Schwinger equations for matrix model with action $S(\Phi) = V\,\mathrm{Tr}(E\Phi^2 + \frac{\lambda}{4}\Phi^4)$

1. $$G_{|ab|} = \frac{1}{E_a+E_b}\Big\{1 - \frac{\lambda}{V}\sum_{p=0}^{\mathcal{N}}\Big(G_{|ab|}G_{|ap|} - \frac{G_{|pb|}-G_{|ab|}}{E_p - E_a}\Big)\Big\} + \mathcal{O}(\tfrac{1}{V})$$
2. $$G_{|ab|}=\frac{1}{E_a+E_b}\Big\{1-\frac{\lambda}{V}\sum_{p=0}^{\mathcal{N}}G_{|ab|}(G_{|ap|}+G_{|pb|})-\frac{\lambda}{V^2}\sum_{p,q=0}^{\mathcal{N}}G_{|abpq|}\Big\} + \mathcal{O}(\tfrac{1}{V})$$
3. $$G_{|abcd|} = (-\lambda)\frac{G_{|ab|}G_{|cd|} - G_{|ad|}G_{|cb|}}{(E_a - E_c)(E_b - E_d)} + \mathcal{O}(\tfrac{1}{V})$$

1+3 were found in [Grosse-W. 2012] using the Ward identity of [Disertori-Gurau-Magnen-Rivasseau, 2006]; 2 is easy

- For 2D-Moyal space: $V = \frac{\theta}{4}$, $E = (E_m\delta_{mn})$, $E_m = \frac{\mu^2}{2} + \frac{m}{V}$
- Eliminate $\sum_p G_{|ap|}$, take limit $V, \mathcal{N} \to \infty$ with $\frac{\mathcal{N}}{V} = \Lambda^2$ fixed.
- Get Riemann integrals over continuous functions $G(a,b)$, $a, b \in [0, \Lambda^2]$. Renormalisation $\mu^2 = 1 - 2\lambda\log(1+\Lambda^2)$

A boundary value problem à la Gakhov

Initiated by Alexander Hock who showed me Gakhov's book "Boundary value problem" (1966).

Contains chapter on problems in several complex variables.

Main message: For $\Phi(z_1,\dots,z_n) := \int \frac{dp_1 \dots dp_n \; f(p_1,\dots,p_n)}{(p_1-z_1)\cdots(p_n-z_n)}$, consider all 2^n possible boundary values $z_i = a_i \pm \mathrm{i}\epsilon$:

Theorem

Consider the following function, holomorphic on $(\mathbb{C}\setminus\mathbb{R}_+)^2$:

$$\Psi(z,w) = 1 + z + w - \lambda\log(-z) - \lambda\log(-w) + \lambda^2 \int_0^\infty dp \int_0^\infty dq \, \frac{G_\lambda(p,q)}{(p-z)(q-w)}$$

Then: $\Psi(a+,b+)\Psi(a-,b-) = \Psi(a+,b-)\Psi(a-,b+)$

This simple equation completely determines NCQFT∎

Solution up to three loops

It would be great to have a complex solution theory.
For the time being we try an ansatz as formal power series in λ:

$$
\begin{aligned}
G_\lambda(a,b) &= \frac{1}{1+a+b} \\
&+ \frac{\lambda\log(1+a)+\lambda\log(1+b)}{(1+a+b)^2} \\
&+ \frac{\lambda^2}{(1+a+b)^2}\Big(-\frac{1+2a}{a(1+a)}\log(1+a)-\frac{1+2b}{b(1+b)}\log(1+b)\Big) \\
&+ \frac{\lambda^2}{(1+a+b)^3}\Big(\zeta(2)+(\log(1+a))^2+(\log(1+b))^2 \\
&\qquad +\log(1+a)\log(1+b)-\mathrm{Li}_2(-a)-\mathrm{Li}_2(-b)\Big) \\
&+ \text{[continued on next page]}
\end{aligned}
$$

Solution up to three loops, continued

$$
\begin{aligned}
&+\frac{\lambda^3}{(1+a+b)^2}\Big(\frac{\log(1+a)}{(1+a)^2}+\frac{\log(1+a)}{a(1+a)}+\frac{\log(1+b)}{(1+b)^2}+\frac{\log(1+b)}{a(1+b)}\\
&\qquad -\frac{(\log(1+a))^2}{2(1+a)^2}-\frac{(\log(1+a))^2}{2a^2}-\frac{(\log(1+b))^2}{2(1+b)^2}-\frac{(\log(1+b))^2}{2b^2}\Big)\\
&+\frac{\lambda^3}{(1+a+b)^3}\Big(\frac{1+2a}{a(1+a)}\big(\mathrm{Li}_2(-a)-\log(1+a)\log(1+b)-2(\log(1+a))^2\big)-\frac{\zeta(2)}{1+a}\\
&\qquad +\frac{1+2b}{b(1+b)}\big(\mathrm{Li}_2(-b)-\log(1+b)\log(1+a)-2(\log(1+b))^2\big)-\frac{\zeta(2)}{1+b}\Big)\\
&+\frac{\lambda^3}{(1+a+b)^4}\Big(\frac{2}{3}(\log(1+a))^3+\frac{2}{3}(\log(1+b))^3-2\mathrm{Li}_3(-a)-2\mathrm{Li}_3(-b)\\
&\qquad +2\mathrm{Li}_3(\tfrac{1}{1+a})+(\log(1+a))^2\log a+2\mathrm{Li}_3(\tfrac{1}{1+b})+(\log(1+b))^2\log b\\
&\qquad +\big(\log(1+a)\log(1+b)-\mathrm{Li}_2(-a)-\mathrm{Li}_2(-b)-5\zeta(2)\big)\times\\
&\qquad\qquad \times\big(\log(1+a)+\log(1+b)\big)\Big)+\mathcal{O}(\lambda^4)
\end{aligned}
$$

The 4-loop result is done, but too long to show here

A real approach to the solution

- recall: $\Psi(a+,b+)\Psi(a-,b-) = \Psi(a+,b-)\Psi(a-,b+)$
- write it as $|\Psi(a+,b+)| = |\Psi(a+,b-)|$
- there exists real angle function $\tau_b(a) \in [0,\pi]$ with $\Psi(a+,b+)e^{-\mathrm{i}\tau_a(b)} = \Psi(a+,b-)e^{\mathrm{i}\tau_a(b)}$
- split into Re/Im gives two singular integral equations of Carleman type which can be algebraically integrated to

1. $$G_\lambda(a,b) = \frac{\sin\tau_a(b)}{\lambda\pi} e^{\mathcal{H}_b[\tau_a(\bullet)]}$$
2. $$\lambda\pi\cot\tau_b(a) = 1 + a + b - \lambda\log a + \frac{1}{\pi}\int_0^\infty dp\left(e^{-\mathcal{H}_p[\tau_a(\bullet)]}\sin\tau_a(p) - \frac{\lambda\pi}{1+p}\right)$$

where $\mathcal{H}_a[f(\bullet)] = \frac{1}{\pi}\fint_0^\infty dp\,\frac{f(p)}{p-a}$ is one-sided Hilbert transform.

1 was already found in [Grosse-W, 2012]; 2 was missing

Hilbert transform

A perturbative approach to the solution starts with $\tau_b(a) = \frac{\lambda\pi}{1+a+b}$.
After some iteration these integrals are needed:

$$\begin{aligned}
\mathcal{H}_a\Big[\pi\frac{(\log(1+\bullet))^3}{(1+\bullet+b)^4}\Big] &= -\frac{1}{6}\frac{d^3}{db^3}\mathcal{H}_a\Big[\pi\frac{(\log(1+\bullet))^3}{(1+\bullet+b)}\Big] \\
&= -\frac{1}{6}\frac{d^3}{db^3}\mathcal{H}_a\Big[\frac{\Gamma(4)\mathrm{Im}(\mathrm{Li}_4(1+\bullet+\mathrm{i}\epsilon))}{(1+\bullet+b)}\Big] \\
&= -\frac{d^3}{db^3}\mathcal{H}_a\Big[\mathrm{Im}\Big(\frac{\mathrm{Li}_4(1+\bullet+\mathrm{i}\epsilon)-\mathrm{Li}_4(-b)}{(1+\bullet+\mathrm{i}\epsilon+b)}\Big)\Big] \\
&= -\frac{d^3}{db^3}\mathrm{Re}\Big(\frac{\mathrm{Li}_4(1+a+\mathrm{i}\epsilon)-\mathrm{Li}_4(-b)}{(1+a+\mathrm{i}\epsilon+b)}\Big)
\end{aligned}$$

The miracle

recall: $\lambda\pi \cot \tau_b(a) = \cdots + \frac{1}{\pi} \int_0^\infty dp \, \Big(e^{-\mathcal{H}_p[\tau_a(\bullet)]} \sin \tau_a(p) - \frac{\lambda\pi}{1+p}\Big)$

$$
\begin{aligned}
\overset{\text{4th order}}{=} \lambda^4 \int_0^\infty dp \Big\{ & \frac{\pi^2}{6(1+a)(1+a+p)^3} + \frac{\pi^2}{6(1+p)(1+a+p)^3} - \frac{\log(1+a)}{(1+a)^2(1+a+p)^2} \\
& - \frac{\log(1+a)}{a(1+a)(1+a+p)^2} + \frac{\pi^2 \log(1+a)}{2(1+a+p)^4} + \frac{(\log(1+a))^2}{2a^2(1+a+p)^2} + \frac{(\log(1+a))^2}{2(1+a)^2(1+a+p)^2} \\
& - \frac{(\log(1+a))^2 \log a}{(1+a+p)^4} + \frac{(\log(1+a))^3}{3(1+a+p)^4} - \frac{13\pi^2 \log p}{3(1+a+p)^4} + \frac{2\log(1+a)\log p}{(1+a)(1+a+p)^3} \\
& + \frac{2\log(1+a)\log p}{a(1+a+p)^3} - \frac{3\log(1+a)(\log p)^2}{(1+a+p)^4} + \frac{4(\log p)^3}{(1+a+p)^4} + \frac{\log(1+p)}{p(1+p)(1+a+p)^2} \\
& + \frac{\log(1+p)}{(1+p)^2(1+a+p)^2} - \frac{23\pi^2 \log(1+p)}{6(1+a+p)^4} + \frac{\log(1+a)\log(1+p)}{(1+a)(1+a+p)^3} \\
& + \frac{\log(1+a)\log(1+p)}{a(1+a+p)^3} + \frac{\log(1+a)\log(1+p)}{p(1+a+p)^3} + \frac{\log(1+a)\log(1+p)}{(1+p)(1+a+p)^3} \\
& - \frac{4(\log p)\log(1+p)}{p(1+a+p)^3} - \frac{4(\log p)\log(1+p)}{(1+p)(1+a+p)^3} - \frac{4\log(1+a)\log(1+p)\log p}{(1+a+p)^4}
\end{aligned}
$$

+ [next page]

The miracle

$$
\begin{aligned}
&+\frac{9\log(1+p)(\log p)^2}{(1+a+p)^4}-\frac{(\log(1+p))^2}{2p^2(1+a+p)^2}-\frac{(\log(1+p))^2}{2(1+p)^2(1+a+p)^2}-\frac{2(\log(1+p))^2}{p(1+a+p)^3}\\
&-\frac{2(\log(1+p))^2}{(1+p)(1+a+p)^3}-\frac{\log(1+a)(\log(1+p))^2}{(1+a+p)^4}+\frac{5(\log(1+p))^2\log p}{(1+a+p)^4}\\
&+\frac{4(\log(1+p))^3}{3(1+a+p)^4}-\frac{\mathrm{Li}_2(-a)}{a(1+a+p)^3}-\frac{\mathrm{Li}_2(-a)}{(1+a)(1+a+p)^3}-\frac{\log(1+a)\mathrm{Li}_2(-a)}{(1+a+p)^4}\\
&+\frac{2\mathrm{Li}_2(-a)\log p}{(1+a+p)^4}+\frac{\log(1+p)\mathrm{Li}_2(-a)}{(1+a+p)^4}-\frac{\mathrm{Li}_2(-p)}{p(1+a+p)^3}-\frac{\mathrm{Li}_2(-p)}{(1+p)(1+a+p)^3}\\
&-\frac{\log(1+a)\mathrm{Li}_2(-p)}{(1+a+p)^4}+\frac{2\mathrm{Li}_2(-p)\log p}{(1+a+p)^4}+\frac{\log(1+p)\mathrm{Li}_2(-p)}{(1+a+p)^4}+\frac{2\mathrm{Li}_3(-a)}{(1+a+p)^4}\\
&-\frac{2\mathrm{Li}_3(1/(1+a))}{(1+a+p)^4}+\frac{2\mathrm{Li}_3(-p)}{(1+a+p)^4}-\frac{2\mathrm{Li}_3(1/(1+p))}{(1+a+p)^4}\Big\}\\
&=\lambda\Big(\frac{\lambda}{1+a}\Big)^3\Big\{((1+a)+a)\frac{\log(1+a)}{a}-(2(1+a)^2+a(1+a)+3a^2)\frac{(\log(1+a))^2}{2a^2}\\
&\qquad+((1+a)^3+a^3)\frac{(\log(1+a))^3}{3a^3}\Big\}
\end{aligned}
$$

Why is it (and lower orders) so simple? Does it continue?

Les Houches

- It seems that behind the very complicated 2-point function $G_\lambda(a,b)$ there is a relatively simple angle function $\tau_b(a)$ whose Hilbert transform yields $G(a,b)$. Can we prove this?
- I met many polylogarithmic experts at the Les Houches summer school. Our problem is routine for them:

- Singularities confined to hyperplanes
$a=0,\ \ b=0,\ \ a+1=0,\ \ b+1=0,\ \ a+b+1=0$
- Corresponds to moduli space $\mathfrak{M}_{0,5}$ which is completely understood [Francis Brown, 2009]
- Need iterated integrals for an alphabet. Form Hopf algebra with shuffle product and deconcatenation coproduct (not unrelated to Connes-Kreimer-Moscovici Hopf algebra)
- powerful computer algebra code is available

The miracle continues

- Erik Panzer was able to compute with HyperInt over night the angle function to order 7, later to order 9(!)
- The miracle continued: only powers of $\log(1+a)$ and rational function of a
- The results are of such striking simplicity and structure that we could guess the formula to all orders, given on next page

The initial Conjecture is true if one accepts:

Hypothesis

A sequence (a_n) of rational numbers arising from a seemingly simple mathematical problem, for which $a_1, \ldots, a_{36}$ are integers of combinatorial significance, must continue as the same integer sequence.

Extrapolation to all orders

$$I_\lambda(a) := \frac{1}{\pi}\int_0^\infty dp\, \Big(e^{-\mathcal{H}_p[\tau_a(\bullet)]}\sin\tau_a(p) - \frac{\lambda\pi}{1+p}\Big)$$
$$= -\lambda\log(1+a) + \sum_{n=1}^\infty \lambda^{n+1}\Big(\frac{(\log(1+a))^n}{na^n} + \frac{(\log(1+a))^n}{n(1+a)^n}\Big)$$
$$+ \sum_{n=1}^\infty \frac{(n-1)!(-\lambda)^{n+1}}{(1+a)^n}\sum_{k=1}^{n-1}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{k!}\frac{s_{n-j,n-k}}{(n-j)!}\big(1+\big(\tfrac{1+a}{a}\big)^j\big)(\log(1+a))^k$$

- the $s_{k,l}$ are Stirling numbers of the 1st kind
- 28 rational numbers of $n \leq 9$ match
  next 8 numbers of $n = 10$ correctly predicted
- next use generating function for Stirling numbers to remove two sums

Resummation

$$I_\lambda(a) = \underbrace{\sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{d^{n-1}}{da^{n-1}} (-\log(1+a))^n}_{K(a,\lambda)} - \lambda \underbrace{\sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{d^{n-1}}{da^{n-1}} \frac{(-\log(1+a))^n}{a}}_{L(a,\lambda)}$$

three summation strategies:

1. Cauchy's formula and geometric series.
   Gives residue at $a = z + \lambda \log(1+z)$.
2. Use $\frac{(\log(1+a))^k}{k!} = \sum_{n=k}^{\infty} \frac{s_{n,k}}{n!} a^n$
   and recursion of Stirling numbers to derive PDE for K, L
3. Use Lagrange inversion theorem,
   in particular the Lagrange-Bürmann formula

Theorem (Lagrange 1770, Bürmann 1799)

Let $\phi(w)$ be analytic at $w=0$ with $\phi(0)\neq 0$ and $f(w)=\frac{w}{\phi(w)}$.
Its inverse $g(z)$, namely $z=f(g(z))$, is analytic at $z=0$ and given by

$$g(z)=\sum_{n=1}^{\infty}\frac{z^n}{n!}\frac{d^{n-1}}{dw^{n-1}}\big(\phi(w)^n\big)\Big|_{w=0}$$

Moreover, for any analytic function H with $H(0)=0$,

$$H(g(z))=\sum_{n=1}^{\infty}\frac{z^n}{n!}\frac{d^{n-1}}{dw^{n-1}}\big(H'(w)\phi(w)^n\big)\Big|_{w=0}$$

- set $z=\lambda$, $\phi(w)=-\log(1+a+w)$, $g(\lambda)=K(a,\lambda)$

$\Rightarrow$ $K(a,\lambda)=-\lambda\log(1+a+K(a,\lambda))$, $K(a,\lambda)=W(\frac{1}{\lambda}e^{\frac{1+a}{\lambda}})-1-a$

- set $H'(w)=\frac{1}{a+w}$ $\Rightarrow$ $L(a,\lambda)=H(K(a,\lambda))=\log\big(1+\frac{K(a,\lambda)}{a}\big)$

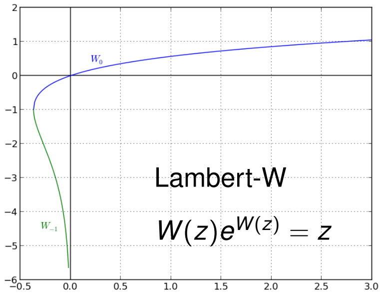
Main Lemma

The integral equation

$$\lambda\pi \cot \tau_b(a) = 1 + a + b - \lambda \log a + \frac{1}{\pi}\int_0^\infty dp \left(e^{-\mathcal{H}_p[\tau_a(\bullet)]} \sin \tau_a(p) - \frac{\lambda\pi}{1+p} \right)$$

is solved by

$$\lambda\pi \cot \tau_b(a) = b + \lambda W\Big(\frac{e^{(1+a)/\lambda}}{\lambda}\Big) - \lambda \log\Big(W\Big(\frac{e^{(1+a)/\lambda}}{\lambda}\Big) - 1\Big)$$



rediscovered again and again since [Lambert, 1752]

branch W_0 for $\lambda \geq 0$,
branch W_{-1} for $-1 < \lambda < 0$

$\tau_b(a) \in [0, \pi]$ for $\lambda \geq 0$,
$\tau_b(a) \in [-\pi, 0]$ for $\lambda \leq 0$

analytic(!) switch at $\lambda = 0$

Preliminary summary

$$G(a,b) \equiv G(b,a) = \frac{\exp\left(\frac{1}{\pi}\int_0^\infty \frac{dp}{p-a}\arctan\left(\frac{\lambda\pi}{b+\lambda W\left(\frac{e^{(1+p)/\lambda}}{\lambda}\right)-\lambda\log\left(\lambda W\left(\frac{e^{(1+p)/\lambda}}{\lambda}\right)-1\right)}\right)\right)}{\sqrt{(\lambda\pi)^2+\left(b+\lambda W\left(\frac{e^{(1+a)/\lambda}}{\lambda}\right)-\lambda\log\left(\lambda W\left(\frac{e^{(1+a)/\lambda}}{\lambda}\right)-1\right)\right)^2}}$$

- both arguments a,b arise in highly asymmetric manner, but the final result has to be symmetric:
- symmetry identified the equation for $\tau_b(a)$ which we solved (by guess) in terms of Lambert-W

Can we make the symmetry manifest?

Cleaning I

Lemma

In the previous branch conventions one has

$$\mathcal{H}_a\left[\arctan\left(\frac{\lambda\pi}{1+b+\bullet-\lambda\log(\bullet)}\right)\right] = \log\left(\frac{\sqrt{(1+a+b-\lambda\log a)^2+(\lambda\pi)^2}}{a+\lambda W\big(\frac{1}{\lambda}e^{(1+b)/\lambda}\big)}\right),$$

$$\frac{1}{2\pi\mathrm{i}}\int_{\gamma_\epsilon}\frac{dz}{z-a}\log\left(1-\frac{\lambda\log(-z)}{1+b+z}\right) = \log\left(\frac{1+a+b}{a+\lambda W\big(\frac{1}{\lambda}e^{(1+b)/\lambda}\big)}\right)$$

where $\mathcal{H}_a$ is the one-sided Hilbert transform and γ_ϵ the curve which encircles the positive real axis clockwise at distance ϵ.

Cleaning II

Proposition

Let $\tau_b(a)$ be the angle function given in the Main Lemma. Then

$$\begin{aligned}
&\mathcal{H}_a[\tau_b(\bullet)] \\
&= \log \sqrt{\big(b+\lambda W(\tfrac{1}{\lambda}e^{(1+a)/\lambda}) - \lambda\log(\lambda W(\tfrac{1}{\lambda}e^{(1+a)/\lambda})-1)\big)^2 + (\lambda\pi)^2} \\
&+ \log\left(\frac{(1+a+b)\exp(N_\lambda(a,b))}{(b+\lambda W(\frac{1}{\lambda}e^{(1+a)/\lambda}))(a+\lambda W(\frac{1}{\lambda}e^{(1+b)/\lambda}))}\right),
\end{aligned}$$

where

$$N_\lambda(a,b) = \frac{1}{2\pi\mathrm{i}}\int_{\gamma_\epsilon} dz \log\left(1-\frac{\lambda\log(-z)}{1+b+z}\right)\frac{d}{dw}\log\left(1-\frac{\lambda\log(1+z+w)}{a-(z+w)}\right)\Big|_{w=0}$$

Expansion of N into convergent power series permits to deform γ_ϵ into $-\frac{1}{2}+\mathrm{i}\mathbb{R}$. Gives symmetric integral of the initial Conjecture

The Nielsen function

Remains to understand the function $N_\lambda(a,b)$. The symmetric integral shows real analyticity for all $\lambda > -\frac{1}{2\log 2}$ and $a,b \geq 0$.

The contour integral has convergent expansion

$$N_\lambda(a,b) = \sum_{m,n=1}^{\infty} \frac{(-\lambda)^{m+n}}{m!m!n!n!} \frac{\partial^{m-1}}{\partial a^{m-1}} \frac{\partial^{n-1}}{\partial b^{n-1}} \frac{\partial^m}{\partial \alpha^m} \frac{\partial^n}{\partial \beta^m} R_{\alpha,\beta}(a,b;w)\Big|_{\alpha=\beta=w=0}$$

$$R_{\alpha,\beta}(a,b;w) := \frac{1}{\pi}\int_0^\infty dz\, \mathrm{Im}\Big(\frac{(-z-\mathrm{i}\epsilon)^\beta(1+z+w+\mathrm{i}\epsilon)^\alpha}{(1+b+z+\mathrm{i}\epsilon)(a-z-w-\mathrm{i}\epsilon)}\Big)$$

Lemma

$$R_{\alpha,\beta}(a,b;w) = \frac{1}{(1+a+b-w)} \frac{\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)} \Big\{ -(1+w)^{\alpha+\beta} + (1+w)^\alpha(1+b)^\beta {}_2F_1\Big(\begin{matrix}-\alpha,\beta\\ 1-\alpha\end{matrix}\Big|\frac{w-b}{1+w}\Big) + (1+w)^\beta(1+a)^\alpha {}_2F_1\Big(\begin{matrix}-\beta,\alpha\\ 1-\beta\end{matrix}\Big|\frac{w-a}{1+w}\Big)\Big\}.$$

Nielsen polylogarithms and Riemann zeta values

Theorem [Kölbig-Mignaco-Remiddi, 1970]

$$ {}_2F_1\left(\begin{matrix}-\alpha,\beta\\ 1-\alpha\end{matrix}\middle| z\right) = 1 - \sum_{n,p=1}^{\infty} \alpha^n\beta^p S_{n,p}(z) \quad \text{where} $$

$$ S_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{dt}{t} (\log t)^{n-1}(\log(1-zt))^p $$

is the generalised polylogarithm of [Nielsen, 1909]

e.g. $S_{n,1}(z) = \mathrm{Li}_{n+1}(z)$,
$S_{1,2}(z) = \zeta(3) - \mathrm{Li}_3(z) + \log(1-z)(\zeta(2) - \mathrm{Li}_2(z) - \frac{1}{2}(\log z)^2)$

Moreover, $\dfrac{\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)} = \exp\left(\sum_{k=2}^{\infty}((\alpha+\beta)^k - \alpha^k - \beta^k)\frac{\zeta(k)}{k}\right)$

Extracting the λ^k-contribution to $N_\lambda(a,b)$ becomes an exercise in book-keeping.

Outlook I: NC-Φ^4

- All $G(a_1,\dots,a_N)$ are polynomials in $G(a,b)$ and $\frac{1}{E_c-E_d}$ [Grosse-W, 2012]. Non-crossing cord diagrams arise.
- Functions $G(a_1^1,\dots,a_{N_1}^1|\dots|a_1^B,\dots,a_{N_B}^B)$ solve linear integral equations; they are always solvable. Need $N_\lambda(a,b)$ for explicit formulae.
- Passing from 2D to 4D amounts to change integration measure $dp \mapsto pdp$. Renormalisation more involved.
- Expect the same Gakhov boundary value problem. One equation $G(a,b) = \frac{e^{\mathcal{H}_a[\tau_b(\bullet)]-\mathcal{H}_0[\tau_0(\bullet)]}\sin\tau_b(a)}{\lambda\pi a}$ already there.
- HyperInt will easily treat the renormalised 4D equations.

Does all this help for reflection positivity?

Outlook II: QFT

Every realistic QFT evaluates to [harmonic, elliptic, Goncharov, ...] polylogarithms. Here is the most famous example from QED:

$$\frac{g}{2} = 1 + \frac{1}{2}\frac{e^2}{\pi} + \Big\{\frac{197}{144} + \Big(\frac{1}{2} - 3\log 2\Big)\zeta(2) + \frac{3}{4}\zeta(3)\Big\}\Big(\frac{e^2}{\pi}\Big)^2$$
$$+ \Big\{\frac{28259}{5184} + \Big(\frac{17101}{235} - \frac{596}{2}\cdot\log 2\Big)\zeta(2) + \frac{139}{18}\zeta(3) + \frac{100}{3}\mathrm{Li}_4\Big(\frac{1}{2}\Big)$$
$$+ \frac{25}{3}\Big(\frac{\log^4 2}{6} - \zeta(2)\log^2 2\Big) - \frac{239\zeta(4) - 166\zeta(2)\cdot\zeta(3) + 215\zeta(5)}{24}\Big\}\Big(\frac{e^2}{\pi}\Big)^3$$
$$+ \Big\{\dots\Big\}\Big(\frac{e^2}{\pi}\Big)^4$$
$$= 1.001\,159\,652\,153\,5$$

- Is there a structure behind the rational numbers?
- Could the tremendous complexity of QFT arise as integral transform of a simple function which solves a transcendental equation?

Outlook III: Integrability

Solving a non-linear problem by (generalised) radicals is only possible if a friendly algebraic structure permits it.

We have no idea what it is, but it is there: The Φ^4-matrix model is now an honest integrable model.

As such it stands on same level as the Kontsevich model Φ^3 which

- relates to an infinite-dimensional Lie algebra
- relates to the τ-function of the KdV-hierarchy
- generates intersection numbers of stable cohomology classes on the moduli space of complex curves

All that might exist for Φ^4, and its number theory is much richer!

We might have hit only the tip of an iceberg. Much more new mathematics could wait under water for discovery.

I invite you to dive