Integrability in QFT-models on noncommutative geometry

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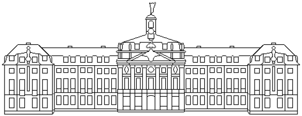


joined work with Erik Panzer

based on long-term collaboration with Harald Grosse

Overview

- Feynman graphs evaluate to (harmonic, elliptic) polylogarithms. Something of that should be non-perturbatively preserved in correlation functions and scattering amplitudes.
- As modest step to understand the secrets behind such series we study QFT toy models. They describe Euclidean scalar fields on a simple noncommutative geometry.
- Making essential use of Dyson-Schwinger equations we identify integrable structures in these models: Known for the Kontsevich model Φ^3, possibly new for Φ^4.
- The last remaining obstacle for the construction Φ_2^4 was removed during this school in joint work with Erik Panzer.
- Discussions with Johannes Blümlein, David Broadhurst and Alexander Hock, and the long-term collaboration with Harald Grosse, also contributed to this step.

Background: QFT on noncommutative geometries

1. Compactification of M-theory on noncommutative tori
   [Connes-Douglas-Schwarz 1997]
2. Limits of string theory in presence of magnetic fields
   [Schomerus, Seiberg-Witten 1999]
3. QFT-models on Moyal space (B =const) show UV/IR-mixing
   [Minwalla-van Raamsdonk-Seiberg 1999]

UV/IR cured in $\lambda\phi^{\star 4}_4$ with harmonic propagation

$$S(\phi) := \frac{1}{64\pi^2}\int_{\mathbb{R}^4} dx\Big(\frac{1}{2}\phi\star(-\Delta+4\Omega^2\|\Theta^{-1}x\|^2+\mu^2)\star\phi+\frac{\lambda}{4}\phi^{\star 4}\Big)(x)$$

- renormalisable to all orders in λ [Grosse-W 2004]
- β-function is zero to all orders
  [Disertori-Gurau-Magnen-Rivasseau 2006]

Can it be constructed?

- Yes, up to solution of a fixed point problem [Grosse-W 2012]
- Complete solution for $\phi^{\star 3}_{\{2,4,6\}}$ [Grosse-Sako-W 2016]

From NCQFT to matrix models

Noncommutative manifold is a noncommutative algebra $\mathcal{A}$ which often has finite-dimensional approximations: matrices

Example: Moyal algebra = Rieffel deformation of $C^\infty(\mathbb{R}^2)$

$$(f\star g)(x) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{dy\,dk}{(2\pi)^2} f(x+\tfrac{1}{2}\Theta k)\, g(x+y)\, e^{\mathrm{i}\langle k,y\rangle}, \quad \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

- $f_{mn}(x) = 2(-1)^m \sqrt{\frac{m!}{n!}} \Big(\sqrt{\frac{2}{\theta}}x_1 + \mathrm{i}x_2\Big)^{n-m} L_m^{n-m}\Big(\frac{2\|x\|^2}{\theta}\Big) e^{-\frac{\|x\|^2}{\theta}}$

  satisies $f_{mn}\star f_{kl} = \delta_{nk} f_{ml}$ and $\int \frac{dx}{8\pi} f_{mn}(x) = \frac{\theta}{4}\delta_{mn}$

- $$S(\phi) := \frac{1}{(8\pi)^{D/2}} \int_{\mathbb{R}^D} dx \Big(\frac{1}{2}\phi\star(-\Delta + 4\cdot 1\|\Theta^{-1}x\|^2 + \mu^2)\star\phi + \frac{\lambda}{p}\phi^{\star p}\Big)(x)$$
  $$= V\,\mathrm{Tr}(E\Phi^2 + \tfrac{\lambda}{p}\Phi^p)$$

  where $V = (\frac{\theta}{4})^{D/2}$, $E = (E_{\underline{m}}\delta_{\underline{m}\underline{n}})$, $E_{\underline{m}} = \frac{\mu^2}{2} + \frac{m_1+\dots+m_{D/2}}{V^{2/D}}$

- $p=3$ is the Kontsevich model

$\phi^{\star 3}_D$ [with Harald Grosse and Akifumi Sako]

- action $S(\Phi) = V\,\mathrm{tr}(ZE\Phi^2 + (\kappa+\nu E+\zeta E^2)\Phi + \frac{\lambda_{bare}Z^{\frac{3}{2}}}{3}\Phi^3)$
  for $E = \Big(\frac{\mu^2_{bare}}{2} + \mu^2 e\big(\frac{|n|}{\mu^2 V^{2/D}}\big)\delta_{mn}\Big)$, $\quad m,n \in \mathbb{N}^{D/2}$
- $\mu_{bare}, \lambda_{bare}, Z, \kappa, \nu, \zeta$ to be fixed by normalisation conditions
- partition function $\mathcal{Z}(J) = \int d\Phi\ \exp(-S(\Phi) + \mathrm{i}V\,\mathrm{tr}(\Phi J))$

$$\log\frac{\mathcal{Z}(J)}{\mathcal{Z}(0)} = \sum_{B=1}^{\infty}\sum_{N_B\geq\cdots\geq N_1\geq 1}\frac{V^{2-B}}{S_{N_1\dots N_B}} G_{|p_1^1\dots p_{N_1}^1|\dots|p_1^B\dots p_{N_B}^B|}\prod_{\beta=1}^{B}\Big(\prod_{j_\beta=1}^{N_\beta}\mathrm{i}J_{p^\beta_{j_\beta}p^\beta_{j_\beta+1}}\Big)_{cycl}$$

Strategy

- $\mathcal{Z}(J)$ is meaningless for $\lambda \in \mathbb{R}$!
- $\mathcal{Z}(J)$ is only used as tool to derive identities (Dyson-Schwinger equations) between $G_{|p_1^1\dots p_{N_1}^1|\dots|p_1^B\dots p_{N_B}^B|}$
- Forget $\mathcal{Z}$, declare SD-equations as exact and search for rigorous solutions $G_{\dots}$ of them!

Dyson-Schwinger equations

1-point function in dimension $D \leq 6$, $\underline{a} = (a_1, \dots, a_{D/2})$:

$$G_{|\underline{a}|} = \frac{1}{2ZE_{\underline{a}}}\Big\{-\kappa - \nu E_{\underline{a}} - \zeta E_{\underline{a}}^2 - \lambda_{bare} Z^{\frac{3}{2}}\Big(G_{|\underline{a}|}^2 + \frac{1}{V}\sum_{\underline{m}\in\mathbb{N}_{\mathcal{N}}^{D/2}} G_{|\underline{am}|} + \frac{G_{|\underline{a}|\underline{a}|}}{V^2}\Big)\Big\}$$

- typical feature: Dyson-Schwinger equation for n-point function depends on $(m > n)$-point function
- Here we are rescued:
  1. $G_{|a|a|}$ comes with $\frac{1}{V^2}$, goes away in limit $V \to \infty$
  2. $G_{|am|}$ expressable in terms of $G_{|a|}, G_{|m|}$ thanks to Ward-Takahashi identity for $U(\infty)$-group action:

Theorem (Disertori-Gurau-Magnen-Rivasseau 2006)

$$\sum_n \frac{\partial^2 \mathcal{Z}(J)}{\mathrm{i}^2 \partial J_{bn} \partial J_{na}} = \sum_n \frac{V}{Z(E_a - E_b)}\Big(J_{an}\frac{\partial}{\partial J_{bn}} - J_{nb}\frac{\partial}{\partial J_{na}}\Big)\mathcal{Z}(J) - \frac{V}{Z}(\nu + \zeta(E_a + E_b))\frac{\partial \mathcal{Z}(J)}{\mathrm{i}\partial J_{ba}}$$

(for $a \neq b$)

Closed equation for 1-point function

- WTI gives:

$$G_{|\underline{a}\underline{b}|} = \frac{1}{Z(E_{\underline{a}}+E_{\underline{b}})}\Big(1+\lambda_{bare}Z^{\frac{1}{2}}\frac{(G_{|\underline{a}|}-G_{|\underline{b}|})}{E_{\underline{a}}-E_{\underline{b}}} + \lambda_{bare}Z^{\frac{1}{2}}(\nu+\zeta(E_{\underline{a}}+E_{\underline{b}}))G_{|\underline{a}\underline{b}|}\Big)$$

- scaling limit $\mathcal{N}, V \to \infty$ with $\frac{\mathcal{N}}{V^{2/D}} = \mu^2\Lambda^2$ fixed gives non-linear integral equation for $G(a) = \lim \mu^{1-\frac{D}{2}} G_{|\underline{a}|}\Big|_{|\underline{a}|=V^{2/D}\mu^2 a}$

similar to equation from Virasoro constraint in Kontsevich model:

Theorem [Makeenko-Semenoff 1991]

$$W^2(A) + \int_\alpha^\beta dT\rho(T)\frac{W(A)-W(T)}{A-T} = A + \text{const}$$

is solved by $W(A) = \sqrt{A+c} + \frac{1}{2}\int_\alpha^\beta \frac{dT\ \rho(T)}{(\sqrt{A+c}+\sqrt{T+c})\sqrt{T+c}}$

together with a consistency condition on c.

$A = (2e(a)+1)^2$, $\rho(T) = \dfrac{2\lambda^2\big(e^{-1}(\frac{\sqrt{T}-1}{2})\big)^{D/2-1}}{\Gamma(D/2)\sqrt{T}e'(e^{-1}(\frac{\sqrt{T}-1}{2}))}$, Moyal: $e(a)=a$

Explicit solution of Φ^3_6

Theorem [Grosse-Sako-W, 2016]

Given coupling constant λ, eigenvalue function $e(x)$ of 6D degeneracy. Then the renormalised 1-point function of $\lambda\Phi^3_6$ is

$$G(a) = \frac{\sqrt{(A+c)(1+c)} - c - \sqrt{A}}{2\lambda} + \frac{\lambda}{4}\int_1^\infty \frac{dT\,\big(e^{-1}(\frac{\sqrt{T}-1}{2})\big)^2\,(\sqrt{A+c}-\sqrt{1+c})^2}{\sqrt{T}e'(e^{-1}(\frac{\sqrt{T}-1}{2}))(\sqrt{A+c}+\sqrt{T+c})(\sqrt{1+c}+\sqrt{T+c})^2\sqrt{T+c}}$$

where $\sqrt{A} := 2e(a)+1$ and $c(\lambda)$ is implicit solution of

$$-c = \lambda^2 \int_1^\infty \frac{dT\,\big(e^{-1}(\frac{\sqrt{T}-1}{2})\big)^2}{\sqrt{T}e'(e^{-1}(\frac{\sqrt{T}-1}{2}))(\sqrt{1+c}+\sqrt{T+c})^3\sqrt{T+c}}$$

- explicit integrals for Moyal space with $e(x) = x$ (later)
- matches perfectly renormalised Feynman graph calculation

Higher correlation functions

…satisfy linear integral equations, easily reduced to $(1+\ldots+1)$:

$$G(a^1_1,\ldots,a^1_{N_1}|\ldots|a^B_1,\ldots,a^B_{N_B}|)$$
$$=\frac{1}{\lambda^B}\sum_{k_1=1}^{N_1}\cdots\sum_{k_B=1}^{N_B} G(a^1_{k_1}|\ldots|a^B_{k_B}|)\prod_{\beta=1}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{4\lambda}{A^\beta_{k_\beta}-A^\beta_{l_\beta}}$$

Theorem [Grosse-Sako-W 2016]

$$G(a|b)=\frac{4\lambda^2}{\sqrt{A+c}\cdot\sqrt{B+c}\cdot(\sqrt{A+c}+\sqrt{B+c})^2}$$
where $A=(2(e(a)+1))^2$

$$G(a^1|\ldots|a^B)=\frac{d^{B-3}}{dt^{B-3}}\left(\frac{(-2\lambda)^{3B-4}}{(R(t))^{B-2}}\frac{1}{\sqrt{A^1+c-2t}^3}\cdots\frac{1}{\sqrt{A^B+c-2t}^3}\right)\Bigg|_{t=0}$$

$$R(T)=\lim_{\tilde\Lambda\to\infty}\left(\frac{1}{\sqrt{Z(\tilde\Lambda)}}-\int_1^{\tilde\Lambda}\frac{dT\rho(T)}{\sqrt{T+c}}\frac{1}{(\sqrt{T+c}+\sqrt{T+c-2t})\sqrt{T+c-2t}}\right)$$

Proof: ansatz for recursion and experience with Bell polynomials

Remarks

1. $\lambda\Phi^3_6$ has positive β-function for $\lambda \in \mathbb{R}$
2. λ-expansion of exact solution for Moyal $e(a) = a$:

$$G(a) = \frac{\lambda}{4(2a+1)}\big(2(1+a)^2\log(1+a) - a(2+3a)\big) + \frac{\lambda^3}{16(2a+1)^3}\big(a^3(2+3a)(2\log 2 - 1)^2\big) + \mathcal{O}(\lambda^5)$$

   only $\log(1+a)$ arises and only at one-loop, nothing more
3. matches Feynman ribbon graphs (vertices, edges & faces)

   label faces (not loops!) by positive reals $x \in \mathbb{R}_+$,
   edges by $\frac{1}{1+x+y}$ if they separate faces x, y,
   vertices by $(-\lambda)$,

   integrate over inner faces with weight $\rho(x)$. [$= \frac{x^2}{2}$ for $\lambda\Phi^3_6$],
   renormalise with BPHZ
4. individual ribbon graphs produce polylogarithms and renormalons, but they all cancel

New development: Φ^4_2 and polylogarithms

Dyson-Schwinger equations for $S(\Phi) = V\,\mathrm{Tr}(E\Phi^2 + \frac{\lambda}{4}\Phi^4)$

1. $$G_{|ab|} = \frac{1}{E_a+E_b}\Big\{1 - \frac{\lambda}{V}\sum_{p=0}^{\mathcal{N}}\Big(G_{|ab|}G_{|ap|} - \frac{G_{|pb|}-G_{|ab|}}{E_p - E_a}\Big)\Big\} + \mathcal{O}(\tfrac{1}{V})$$
2. $$G_{|ab|} = \frac{1}{E_a+E_b}\Big\{1 - \frac{\lambda}{V}\sum_{p=0}^{\mathcal{N}}G_{|ab|}(G_{|ap|}+G_{|pb|}) - \frac{\lambda}{V^2}\sum_{p,q=0}^{\mathcal{N}}G_{|abpq|}\Big\}$$
3. $$G_{|abcd|} = (-\lambda)\frac{G_{|ab|}G_{|cd|} - G_{|ad|}G_{|cb|}}{(E_a - E_c)(E_b - E_d)} + \mathcal{O}(\tfrac{1}{V})$$

eliminate $\sum_p G_{|ap|}$, take limit:

$$(a+b+\mu^2)G(a,b) = 1 + \lambda\int_0^{\Lambda^2} dq\,\frac{G(a,q)-G(a,b)}{q-b} + \lambda\int_0^{\Lambda^2} dp\,\frac{G(p,b)-G(a,b)}{p-a}$$
$$-\lambda^2\int_0^{\Lambda^2} dp\int_0^{\Lambda^2} dq\,\frac{G(a,b)G(p,q) - G(a,q)G(p,b)}{(p-a)(q-b)}$$

Gakhov's method

A few months ago, Alexander Hock showed me Gakhov's book "Boundary value problem" which treats them in several variables.

Main message: For $\Phi(z_1,\dots,z_n) := \int \frac{dp_1\dots dp_n\, f(p_1,\dots,p_n)}{(p_1-z_1)\cdots(p_n-z_n)}$, consider all 2^n possible boundary values $z_i = a_i \pm \mathrm{i}\epsilon$.

Two weeks before this school I applied Gakhov to our equation (with $\mu^2 = 1 - 2\lambda\log(1+\Lambda^2)$):

Theorem

Consider the following function, holomorphic on $(\mathbb{C}\setminus\mathbb{R}_+)^2$:

$$\Psi(z,w) = 1 + z + w - \lambda\log(-z) - \lambda\log(-w) + \lambda^2 \int_0^\infty dp \int_0^\infty dq\, \frac{G(p,q)}{(p-z)(q-w)}$$

Then: $\Psi(a+,b+)\Psi(a-,b-) = \Psi(a+,b-)\Psi(a-,b+)$

This simple equation captures everything of the $\lambda\Phi^4_2$ matrix model

Solution up to three loops

It would be great to have a complex solution theory. For the time being we try perturbation theory:

$$G(a,b) = \frac{1}{1+a+b}$$
$$+ \frac{\lambda \log(1+a) + \lambda \log(1+b)}{(1+a+b)^2}$$
$$+ \frac{\lambda^2}{(1+a+b)^2}\Big(- \frac{1+2a}{a(1+a)} \log(1+a) - \frac{1+2b}{b(1+b)} \log(1+b)\Big)$$
$$+ \frac{\lambda^2}{(1+a+b)^3}\Big(\zeta(2) + (\log(1+a))^2 + (\log(1+b))^2 + \log(1+a)\log(1+b) - \mathrm{Li}_2(-a) - \mathrm{Li}_2(-b)\Big)$$

$+$ [continued on next page]

Solution up to three loops, continued

$$
\begin{aligned}
&+\frac{\lambda^3}{(1+a+b)^2}\Big(\frac{\log(1+a)}{(1+a)^2}+\frac{\log(1+a)}{a(1+a)}+\frac{\log(1+b)}{(1+b)^2}+\frac{\log(1+b)}{a(1+b)}\\
&\qquad-\frac{(\log(1+a))^2}{2(1+a)^2}-\frac{(\log(1+a))^2}{2a^2}-\frac{(\log(1+b))^2}{2(1+b)^2}-\frac{(\log(1+b))^2}{2b^2}\Big)\\
&+\frac{\lambda^3}{(1+a+b)^3}\Big(\frac{1+2a}{a(1+a)}\big(\mathrm{Li}_2(-a)-\log(1+a)\log(1+b)-2(\log(1+a))^2\big)-\frac{\zeta(2)}{1+a}\\
&\qquad+\frac{1+2b}{b(1+b)}\big(\mathrm{Li}_2(-b)-\log(1+b)\log(1+a)-2(\log(1+b))^2\big)-\frac{\zeta(2)}{1+b}\Big)\\
&+\frac{\lambda^3}{(1+a+b)^4}\Big(\frac{2}{3}(\log(1+a))^3+\frac{2}{3}(\log(1+b))^3-2\mathrm{Li}_3(-a)-2\mathrm{Li}_3(-b)\\
&\qquad+2\mathrm{Li}_3(\tfrac{1}{1+a})+(\log(1+a))^2\log a+2\mathrm{Li}_3(\tfrac{1}{1+b})+(\log(1+b))^2\log b\\
&\qquad+\big(\log(1+a)\log(1+b)-\mathrm{Li}_2(-a)-\mathrm{Li}_2(-b)-5\zeta(2)\big)\times\\
&\qquad\qquad\times\big(\log(1+a)+\log(1+b)\big)\Big)+\mathcal{O}(\lambda^4)
\end{aligned}
$$

The 4-loop result is done, but too long to show here

A real approach to the solution

- recall: $\Psi(a+,b+)\Psi(a-,b-)=\Psi(a+,b-)\Psi(a-,b+)$
- write it as $|\Psi(a+,b+)|=|\Psi(a+,b-)|$
- there exists real angle function $\tau_b(a)\in[0,\pi]$ with $\Psi(a+,b+)e^{-\mathrm{i}\tau_a(b)}=\Psi(a+,b-)e^{\mathrm{i}\tau_a(b)}$
- split into Re/Im gives two singular integral equations of Carleman type which can be algebraically integrated to

1. $$G(a,b)=\frac{\sin\tau_a(b)}{\lambda\pi}e^{\mathcal{H}_b[\tau_a(\bullet)]}$$
2. $$\lambda\pi\cot\tau_b(a)=1+a+b-\lambda\log a +\frac{1}{\pi}\int_0^\infty dp\,\Big(e^{-\mathcal{H}_p[\tau_a(\bullet)]}\sin\tau_a(p)-\frac{\lambda\pi}{1+p}\Big)$$

where $\mathcal{H}_a[f(\bullet)]=\frac{1}{\pi}\fint_0^\infty dp\,\frac{f(p)}{p-a}$ is one-sided Hilbert transform

Hilbert transform

A perturbative approach to the solution starts with $\tau_b(a) = \frac{\lambda\pi}{1+a+b}$.
After some iteration these integrals are needed:

$$\begin{aligned}
\mathcal{H}_a\Big[\pi\frac{(\log(1+\bullet))^3}{(1+\bullet+b)^4}\Big] &= -\frac{1}{6}\frac{d^3}{db^3}\mathcal{H}_a\Big[\pi\frac{(\log(1+\bullet))^3}{(1+\bullet+b)}\Big] \\
&= -\frac{1}{6}\frac{d^3}{db^3}\mathcal{H}_a\Big[\frac{\Gamma(4)\mathrm{Im}(\mathrm{Li}_4(1+\bullet+\mathrm{i}\epsilon))}{(1+\bullet+b)}\Big] \\
&= -\frac{d^3}{db^3}\mathcal{H}_a\Big[\mathrm{Im}\Big(\frac{\mathrm{Li}_4(1+\bullet+\mathrm{i}\epsilon)-\mathrm{Li}_4(-b)}{(1+\bullet+\mathrm{i}\epsilon+b)}\Big)\Big] \\
&= -\frac{d^3}{db^3}\mathrm{Re}\Big(\frac{\mathrm{Li}_4(1+a+\mathrm{i}\epsilon)-\mathrm{Li}_4(-b)}{(1+a+\mathrm{i}\epsilon+b)}\Big)
\end{aligned}$$

What happens at next order?

- $\mathrm{Im}((\log(-a-\mathrm{i}\epsilon)^4) \sim \pi(\log a)^3$,
- $\mathrm{Im}((\log(-a))^3\log(1+a) - \mathrm{Li}_4(-a)) \sim \pi\log(1+a)(\log a)^2$,
- $\mathrm{Im}(\mathrm{Li}_4(1+\frac{1}{a}) \sim \pi(\log(1+a) - \log a)^3$

give all weight-3 Hilbert transforms, hence $G(a,b)$ to 4 loops.

Question (before Les Houches)

Which function F, holomorphic on $\mathbb{C}\setminus\mathbb{R}_+$, satisfies
$\mathrm{Im}(F(a+\mathrm{i}\epsilon)) = (\log(1+a))^2(\log a)^2$?

Thanks to experts here, this and much more is answered:

- Erik Panzer's HyperInt computes $\mathcal{H}_a[\frac{(\log(1+\bullet))^2(\log\bullet)^2}{1+\bullet+b}]$ instantly
- Johannes Blümlein explained me the theory behind: shuffles, harmonic polylogarithms, iterated integrals for an alphabet [here of two letters 0 and -1, the Nielsen class]

  known cuts let us do $\mathcal{H}_a[\frac{(\log(1+\bullet))^2(\log\bullet)^2}{1+\bullet+b}]$ by hand

The miracle

recall: $\lambda\pi\cot\tau_b(a) = \cdots + \frac{1}{\pi}\int_0^\infty dp\,\Big(e^{-\mathcal{H}_p[\tau_a(\bullet)]}\sin\tau_a(p) - \frac{\lambda\pi}{1+p}\Big)$

$$
\begin{aligned}
&\overset{\text{4th order}}{=} \lambda^4\int_0^\infty dp\Big\{\frac{\pi^2}{6(1+a)(1+a+p)^3} + \frac{\pi^2}{6(1+p)(1+a+p)^3} - \frac{\log(1+a)}{(1+a)^2(1+a+p)^2}\\
&- \frac{\log(1+a)}{a(1+a)(1+a+p)^2} + \frac{\pi^2\log(1+a)}{2(1+a+p)^4} + \frac{(\log(1+a))^2}{2a^2(1+a+p)^2} + \frac{(\log(1+a))^2}{2(1+a)^2(1+a+p)^2}\\
&- \frac{(\log(1+a))^2\log a}{(1+a+p)^4} + \frac{(\log(1+a))^3}{3(1+a+p)^4} - \frac{13\pi^2\log p}{3(1+a+p)^4} + \frac{2\log(1+a)\log p}{(1+a)(1+a+p)^3}\\
&+ \frac{2\log(1+a)\log p}{a(1+a+p)^3} - \frac{3\log(1+a)(\log p)^2}{(1+a+p)^4} + \frac{4(\log p)^3}{(1+a+p)^4} + \frac{\log(1+p)}{p(1+p)(1+a+p)^2}\\
&+ \frac{\log(1+p)}{(1+p)^2(1+a+p)^2} - \frac{23\pi^2\log(1+p)}{6(1+a+p)^4} + \frac{\log(1+a)\log(1+p)}{(1+a)(1+a+p)^3}\\
&+ \frac{\log(1+a)\log(1+p)}{a(1+a+p)^3} + \frac{\log(1+a)\log(1+p)}{p(1+a+p)^3} + \frac{\log(1+a)\log(1+p)}{(1+p)(1+a+p)^3}\\
&- \frac{4(\log p)\log(1+p)}{p(1+a+p)^3} - \frac{4(\log p)\log(1+p)}{(1+p)(1+a+p)^3} - \frac{4\log(1+a)\log(1+p)\log p}{(1+a+p)^4}
\end{aligned}
$$

+ [next page]

The miracle

$$
\begin{aligned}
&+\frac{9\log(1+p)(\log p)^2}{(1+a+p)^4}-\frac{(\log(1+p))^2}{2p^2(1+a+p)^2}-\frac{(\log(1+p))^2}{2(1+p)^2(1+a+p)^2}-\frac{2(\log(1+p))^2}{p(1+a+p)^3}\\
&-\frac{2(\log(1+p))^2}{(1+p)(1+a+p)^3}-\frac{\log(1+a)(\log(1+p))^2}{(1+a+p)^4}+\frac{5(\log(1+p))^2\log p}{(1+a+p)^4}\\
&+\frac{4(\log(1+p))^3}{3(1+a+p)^4}-\frac{\mathrm{Li}_2(-a)}{a(1+a+p)^3}-\frac{\mathrm{Li}_2(-a)}{(1+a)(1+a+p)^3}-\frac{\log(1+a)\mathrm{Li}_2(-a)}{(1+a+p)^4}\\
&+\frac{2\mathrm{Li}_2(-a)\log p}{(1+a+p)^4}+\frac{\log(1+p)\mathrm{Li}_2(-a)}{(1+a+p)^4}-\frac{\mathrm{Li}_2(-p)}{p(1+a+p)^3}-\frac{\mathrm{Li}_2(-p)}{(1+p)(1+a+p)^3}\\
&-\frac{\log(1+a)\mathrm{Li}_2(-p)}{(1+a+p)^4}+\frac{2\mathrm{Li}_2(-p)\log p}{(1+a+p)^4}+\frac{\log(1+p)\mathrm{Li}_2(-p)}{(1+a+p)^4}+\frac{2\mathrm{Li}_3(-a)}{(1+a+p)^4}\\
&-\frac{2\mathrm{Li}_3(1/(1+a))}{(1+a+p)^4}+\frac{2\mathrm{Li}_3(-p)}{(1+a+p)^4}-\frac{2\mathrm{Li}_3(1/(1+p))}{(1+a+p)^4}\Big\}\\
&=\lambda\Big(\frac{\lambda}{1+a}\Big)^3\Big\{((1+a)+a)\frac{\log(1+a)}{a}-(2(1+a)^2+a(1+a)+3a^2)\frac{(\log(1+a))^2}{2a^2}\\
&\qquad+((1+a)^3+a^3)\frac{(\log(1+a))^3}{3a^3}\Big\}
\end{aligned}
$$

Why is it (and lower orders) so simple? Does it continue?

Les Houches

- It seems that behind the very complicated 2-point function $G(a,b)$ there is a relatively simple angle function $\tau_b(a)$ whose Hilbert transform yields $G(a,b)$.
- With help by experts, and a lot of luck, it might even be possible to guess the angle function
- We had luck: Erik Panzer was able to compute with HyperInt over night the angle function to order 7, later to order 9 (!)
- The miracle continued: only powers of $\log(1+a)$ and rational function of a
- The arising fractions were so characteristic that Erik found a recursion that is solved by Stirling numbers of the 1st kind

Extrapolation to all orders

HyperInt gives:

$$I_\lambda(a) := \frac{1}{\pi}\int_0^\infty dp\,\Big(e^{-\mathcal{H}_p[\tau_a(\bullet)]}\sin\tau_a(p) - \frac{\lambda\pi}{1+p}\Big)$$

$$= -\lambda\log(1+a) + \sum_{n=2}^\infty \lambda^n\Big(\frac{(\log(1+a))^{n-1}}{(n-1)a^{n-1}} + \frac{(\log(1+a))^{n-1}}{(n-1)(1+a)^{n-1}}\Big)$$

$$+ \sum_{n=2}^\infty \frac{(n-2)!(-\lambda)^n}{(1+a)^{n-1}}\sum_{k=1}^{n-2}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{k!}\frac{s_{n-1-j,n-1-k}}{(n-1-j)!}\big(1+\big(\tfrac{1+a}{a}\big)^j\big)(\log(1+a))^k$$

next use generating function for Stirling numbers to remove two sums

Resummation

$$I_\lambda(a) = -\lambda\log(1{+}a) + \sum_{n=2}^{\infty}\frac{(-\lambda)^n}{(n{-}1)!}\frac{d^{n-2}}{da^{n-2}}\Big(\frac{(1{+}2a)\log(1{+}a))^{n-1}}{a(1+a)}\Big)$$

$$\stackrel{|a|<1}{=} \underbrace{\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} s_{m+n-1,n}(-\lambda)^n\frac{a^m}{m!}}_{F(a,\lambda)} - \lambda\underbrace{\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{s_{m-n,n}}{m+n}(-\lambda)^n\frac{a^m}{m!}}_{G(a,\lambda)}$$

- Either write the 1st line via Cauchy's formula and resum. Gives residue at $a = z + \lambda\log(1+z)$.
- Better use recursion of Stirling numbers to get PDE for F, G:

$$\Big[(1+a+\lambda)\frac{\partial}{\partial a} + \lambda\frac{\partial}{\partial\lambda}\Big]F(a,\lambda) = F(a,\lambda) - \lambda$$

$$\Big[a\frac{\partial}{\partial a} + \lambda\frac{\partial}{\partial\lambda}\Big]G(a,\lambda) = \frac{\partial F(a,\lambda)}{\partial a}$$

Solution

with boundary conditions $F(0,\lambda) = 0$ and $F(a,0) = 0$:

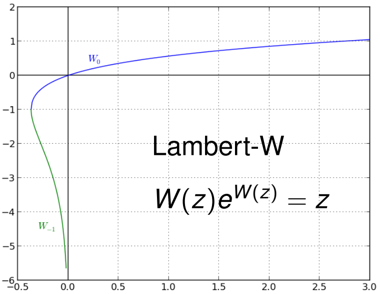
$$F(a,\lambda) = \lambda W\big(\tfrac{1}{\lambda} e^{(1+a)/\lambda}\big) - 1 - a$$

$$G(a,t) = -\int_0^1 \frac{dt}{t} \frac{1}{1 + W\big(\frac{e^{(1+at)/(\lambda t)}}{\lambda t}\big)}$$

real-analytic solution for $\lambda \in]-1,0[\cup]0,\infty[$

branch W_0 for $\lambda > 0$,
branch W_{-1} for $-1 < \lambda < 0$

extends continuously,
but not analytically, to $\lambda = 0$
(perturbation series
does not converge)



Summary

Theorem [Erik Panzer+RW, Les Houches, 11 June 2018]

The non-linear integral equation for the Φ^4_2 two-point function

$$
\begin{aligned}
&(a+b+1-2\lambda\log(1+\Lambda^2))G(a,b)\\
&= 1+\lambda\int_0^{\Lambda^2} dq\,\frac{G(a,q)-G(a,b)}{q-b}+\lambda\int_0^{\Lambda^2} dp\,\frac{G(p,b)-G(a,b)}{p-a}\\
&-\lambda^2\int_0^{\Lambda^2} dp\int_0^{\Lambda^2} dq\,\frac{G(a,b)G(p,q)-G(a,q)G(p,b)}{(p-a)(q-b)}
\end{aligned}
$$

is for $\Lambda\to\infty$ solved by $G(a,b)=\dfrac{\sin\tau_a(b)}{\lambda\pi}e^{\mathcal{H}_b[\tau_a(\bullet)]}$, where

$$\lambda\pi\cot\tau_b(a)=b-\lambda\log a+\lambda W\Big(\frac{e^{(1+a)/\lambda}}{\lambda}\Big)+\int_0^1\frac{dt}{t}\,\frac{1}{1+W\Big(\frac{e^{(1+at)/(\lambda t)}}{\lambda t}\Big)}$$

Remarks

- I was hunting such a formula, with Harald Grosse, for 10 years. Now with Erik Panzer we solved the problem within 5 days.
- All N-point functions of Φ^4_2 depend algebraically on $G(a,b)$. Linear integral eqs. for $(N_1+\ldots+N_B)$-point fn. with $N_i \leq 2$.
- Φ^4_4 should be similar: $G(a,b) = \frac{\sin\tau_b(a)}{\lambda\pi a} e^{\mathcal{H}_a[\tau_b(\bullet)]-\mathcal{H}_0[\tau_0(\bullet)]}$ is there since 2012, only the eq. for $\tau_b(a)$ was missing.
- Lambert-W is asymptotic to

$$W\Big(\frac{e^{(1+a)/\lambda}}{\lambda}\Big) = \Big(\frac{1+a}{\lambda} - \log\lambda\Big) - \log\Big(\frac{1+a}{\lambda} - \log\lambda\Big) - \sum_{\ell=0}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^m s_{\ell+m,m+1}}{m!} \frac{(\log(\frac{1+a}{\lambda} - \log\lambda))^m}{(\frac{1+a}{\lambda} - \log\lambda)^{\ell+m}}$$

  Transseries and resurgence anticipated $\log\log$ and $e^{\frac{1}{\lambda}}$.
- Everything you expect is there!

Dreams

The Φ^4-matrix model is now an honest integrable model.
As such it stands on same level as the Kontsevich model Φ^3.
But its number theory is much richer!

The Kontsevich model

- relates to an infinite-dimensional Lie algebra
- relates to the τ-function of the KdV-hierarchy
- generates intersection numbers of stable cohomology classes on the moduli space of complex curves

All that might exist for Φ^4.

We might have hit only the tip of an iceberg. Much more new mathematics could wait under water for discovery.

I invite you to dive