# From QFT on noncommutative spaces studied with Manfred to integrable QFT models

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# Personal prehistory

In January 2000 I joined Harald Grosse at Universität Wien. This was my 2nd postdoc after CPT Marseille.

My plan was to resolve a puzzle discovered by [Minwalla, van Raamsdonk, Seiberg, 1999]:

**Ultraviolet/Infrared-mixing**

Certain conditionally UV-convergent (but absolutely divergent) subgraphs in QFT-models on noncommutative $\mathbb{R}^D$ produce untreatable IR-divergences.

I had some hope that gauge models are better. With Harald and Thomas Krajewski we quickly proved that the ghost sector is fine, but in the end this did not help.

Next hope was SUSY, where quadratic divergences cancel. We did not have much experience with SUSY.

# First contact with Manfred

It was great luck that Manfred contacted Harald precisely during this time of uncertainty.

Manfred, with a group of young students, was already working on SUSY on noncommutative $\mathbb{R}^D$.

He invited us to join – the beginning of a very successful collaboration.

In fact the results were ready, but Harald and I wrote the introduction with an extended overview of the field:

"*The superfield formalism applied to the noncommutative Wess-Zumino model*," hep-th/0007050.

# Manfred's work with students

Manfred always had very many students.

He had a remarkable method to guide them:
Manfred regularly scanned arXiv. When he found an interesting hot topic, he gave it to a student, or a group of students.

He had an exceptional intuition what topic is fruitful.

# Students

I had interaction with 3 generations of students:

1. The closest with Andreas Bichl, Jesper Grimstrup and Lukas Popp on the Seiberg-Witten map (more below).
2. Volkmar Putz joined me in Leipzig; we applied the Seiberg-Witten map to super-Yang-Mills.
   Further coauthors: A. Gerholt, M. Ertl, M. Wickenhauser
3. Another generation around Peter Fischer worked on deformed Minkowki space.
   Further coauthors: H. Bozkaya, M. Pitschmann.

Later generations include Michael Wohlgenannt and Daniel Blaschke, and probably others in the audience

# Moyal space = noncommutative $\mathbb{R}^D$

vector space of Schwartz functions, equipped with non-local, associative, noncommutative product

$$(f \star g)(x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{d\eta\, dk}{(2\pi)^2} f(x+\tfrac{1}{2}\Theta k)\, g(x+\eta)\, e^{\mathrm{i}\langle k,\eta\rangle} \qquad \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

- strict deformation by $\mathbb{R}^D$-group action (Rieffel)
- compared with formal deformation quantisation (Kontsevich)
- restricting formal deformation to finite order gives local theory

For gauge theory, [Seiberg-Witten, 1999] found a map to higher-derivative gauge theory on usual $\mathbb{R}^D$.

I was sceptical that this is interesting. The young students, after discussion with the group of Julius Wess, convinced me that we should try. It turned out to be a gold mine.

# Seiberg-Witten map

Thanks to Mathematica programming skills available at TU Wien we were several months ahead of all other groups.

Our strongest result is:

**Theorem [Bichl-Grimstrup-Grosse-Popp-Schweda-W, 2001]**

After field redefinition, the Seiberg-Witten mapped photon selfenergy is renormalisable to all orders in $\theta$.

In the very end this did not help: as soon as there are fermions, the method breaks down.

I abandoned the subject in summer 2002, also paused the interaction with Manfred

# Final attempt: matrix basis and RG

UV/IR-mixing resisted all treatments. Now being at MPI Leipzig, I started with Harald a final attack based on:

- Use matrix basis of Moyal space. No positions, no oscillating phase factors, direct control of convergence (inspired by José Gracia-Bondía)
- Control large-$N$ matrix limit by renormalisation group à la Polchinski (inspired by Thomas Krajewski)

## Matrix basis (in 2D)

$\phi(x) = \sum_{m,n=0}^{\infty} \Phi_{mn} f_{mn}(x)$

$$f_{mn}(x) = 2(-1)^m \sqrt{\frac{m!}{n!}} \Big(\sqrt{\tfrac{2}{\theta}} x_1 + \mathrm{i} x_2\Big)^{n-m} L_m^{n-m}\Big(\frac{2|x|^2}{\theta}\Big) e^{-\frac{|x|^2}{\theta}}$$

satisfies $f_{mn} \star f_{kl} = \delta_{nk} f_{ml}$ and $\int dx\, f_{mn}(x) = 2\pi\theta\delta_{mn}$

This alone did not help, but it pointed out a natural modification of the noncommutative $\phi^{\star 4}$-action

# Harmonic propagation

Adding a 4th marginal coupling makes everything well-defined:

$$S[\phi] = \int d^4x \Big( \frac{Z}{2} \phi \star \big( -\Delta + \Omega^2 (2\Theta^{-1} x)^2 + \mu_0^2 \big) \phi + \frac{\lambda_0 Z^2}{4} \phi \star \phi \star \phi \star \phi \Big)(x)$$

- renormalisable as formal power series in $\lambda_0$ [Grosse-W., 2004] (renormalisation of $\mu_0^2, \lambda_0, Z \in \mathbb{R}_+$ and $\Omega \in [0,1]$) means: well-defined perturbative quantum field theory
- Langmann-Szabo duality: theories at $\Omega$ and $\Omega^* = \frac{1}{\Omega}$ are the same; self-dual case $\Omega = 1$ is matrix model
- one-loop RG-flow of running coupling is bounded, no Landau ghost [Grosse-W., 2004]
- later: $\beta$-function vanishes to all orders in $\lambda_0$ for $\Omega = 1$ [Disertori-Gurau-Magnen-Rivasseau, 2006]

# Manfred's help with my habilitation

- I tried to assemble renormalisation proof and one-loop vanishing of $\beta_\lambda$ into a habilition thesis.
- University of Leipzig rejected my request (but permitted lectures).
- Manfred rescued me and offered to do the habilitation at TU Wien (submitted 08/2004, defended 05/2005).
- Manfred also supported applications 2004+2005 for a Marie Curie Chair at TU Wien
(was not realised; I got a job in Münster).

# Main results since then (with Harald)

$\beta = 0$ should permit rigorous construction of $\lambda\phi^{\star 4}$

- action in matrix basis $S(\Phi) = V\,\mathrm{tr}(ZE\Phi^2 + \frac{Z^2\lambda}{4}\Phi^4)$ for $E = \left(\left(\frac{\mu^2_{bare}}{2} + \frac{4|\underline{n}|}{\theta}\right)\delta_{\underline{n}\underline{m}}\right)$, $\underline{n} = (n_1, n_2)$, $V = (\frac{\theta}{4})^2$
- use partition function $\mathcal{Z}(J) := \frac{1}{\mathcal{Z}(0)}\int D\Phi\ \exp(-S(\Phi) + \sum \Phi_{mn}J_{mn})$ as tool to derive equations of motion (Schwinger-Dyson eq.)
- covariance under $\Phi \mapsto U^*\Phi U$ yields Ward-Takahashi identity:

**Theorem [Disertori-Gurau-Magnen-Rivasseau, 2006]**

$$0 = \sum_{n\in I}\left(\frac{(E_a - E_p)}{V}\frac{\partial^2 \mathcal{Z}}{\partial J_{an}\partial J_{np}} + J_{pn}\frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}}{\partial J_{np}}\right)$$

formula for $\sum_{n\in I}\frac{\partial^2 \mathcal{Z}}{\partial J_{an}\partial J_{np}}$ after controling the kernel

# Schwinger-Dyson equations

In a scaling limit $V \to \infty$ and $\frac{1}{V}\sum_{p\in I}$ finite, we have:

## 1. A closed non-linear equation for $G_{|ab|}$ (2009/2011)

$$G_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a + E_b)} \frac{1}{V} \sum_{p\in I} \Big( G_{|ab|} G_{|ap|} - \frac{G_{|pb|} - G_{|ab|}}{E_p - E_a} \Big)$$

## 2. For $N \geq 4$ a universal algebraic recursion formula (2012)

$$G_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|} G_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G_{|b_{2l} b_1 \dots b_{2l-1}|} G_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

- similar formulae for other topological sectors
- no index summation in $G_{|abcd|}$ $\Rightarrow$ $\beta$-function zero!

# Solution of the closed equation for $G_{|ab|}$

All QFT-difficulties (divergences, non-linearity, summation of perturbation series) contained in a single equation for $G_{|ab|}$

- becomes integral equation in limit $V \to \infty$ and $\frac{1}{V}\sum_{p \in I}$ finite
- renormalisation of all Feynman graphs at once achieved
- techniques from complex analysis permit reduction to fixed point equation for $G(x,0)$:

$$G(x,0) = \frac{1}{1+x} \exp\left( -\lambda \int_0^x dt \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}_p[G(\bullet,0)]}{G(p,0)}\right)^2} \right)$$

- proved existence of solution with $G(x,0) \stackrel{x\to\infty}{\longrightarrow} \frac{1}{(1+x)^{1-\eta}}$, conjecture $\eta = -\frac{\arcsin(\lambda\pi)}{\pi}$
- reflection positivity requires $\eta > 0$, i.e. negative $\lambda$

# State of the art

- [Grosse-Sako-W, 2016] derives explicit and exact formulae, analytic in $\lambda$, for all matrix correlation functions of $\lambda\Phi^3$.
- Perfect match with perturbation series
- $\lambda\phi^4$ currently on hold because exact solution of fixed point equation not yet available (but a lot is known).
- The topic inspired Razvan Gurau and Vincent Rivasseau to develop coloured tensor models as one approach to quantum gravity.
- We currently work on translating matricial correlation functions back to Schwinger functions in position space; main question concerns Osterwalder-Schrader axioms.

Exchange with Manfred was indispensible foundation for these achievements.