Matricial quantum field theory: Integrability. Positivity?

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based on arXiv:1610.00526 & 1612.07584 with Harald Grosse and Akifumi Sako

and arXiv: 1205.0465, 1306.2816, 1402.1041, 1406.7755 & 1505.05161
with Harald Grosse

Goal: Quantum Field Theory satisfying axioms

- 1932: axioms for quantum mechanics [von Neumann]
- 1950's: unique extension to quantum fields [Wightman]
  = unbounded op.-valued distributions $f \mapsto \Phi(f) : \mathcal{D} \to \mathcal{D} \subset \mathcal{H}$

  Theorem: vacuum expectation values $\langle \Omega, \Phi(x_1) \cdots \Phi(x_N)\Omega \rangle$ are boundary values of holomorphic functions
- their restriction to real subspace of Euclidean points (minus diagonals) defines Schwinger functions
- Schwinger functions inherit real analyticity, Euclidean invariance, complete symmetry and reflection positivity

Theorem [Osterwalder-Schrader 1974]

These properties are sufficient to reconstruct Wightman theory!

So far no non-trivial QFT model in 4 dimensions ...

State of the art

Constructed models (selection)

1. exactly solvable 2D-models (e.g. Thirring, Schwinger)
2. ϕ^4_2 by hard work [Glimm-Jaffe 1968–72]
3. $P[\phi]_2$: candidate Schwinger functions as moments of Feynman-Kac perturbation of Gaußian measure [Glimm, Jaffe, Simon, . . . 1974]
4. ϕ^4_3 similar techniques [Feldman-Osterwalder 1976]
5. Gross-Neveu$_2$: Fermionic summation techniques [Gawędzki-Kupiainen 1985, Feldman-Magnen-Rivasseau-Sénéor 1986]

- probable non-example: ϕ^4_4 [Aizenman 1981, Fröhlich 1982]
- Clay Math Institute Millenium Prize Problem: Prove existence of Yang-Mills$_4$ with mass gap

Standard model of particle physics

- confimed by all experiments of last 40 years
- It is a NONCOMMUTATIVE GEOMETRY
  [Alain Connes 1986–now, with collaborators:
  Lott, Chamseddine, Marcolli, van Suijlekom, Mukhanov, ...]
- No rigorous mathematical treatment available; instead:
  1. Feynman graphs and pertubative renormalisation theory
     [Bogoliubov-Parasiuk-Hepp-Zimmermann-Lowenstein]
  2. Monte Carlo simulation on supercomputers
- Nonetheless mathematically interesting:
  1. zero-locus of Symanzik polynomials is algebraic variety
  2. amplitudes evaluate to special numbers
     (polylogarithms, multiple zeta values)
  3. renormalisation is Birkhoff-decomposition of a loop in the group of characters of the Connes-Kreimer Hopf algebra

Matricial quantum field theory

. . . is the marriage of

1. matrix models for 2D quantum gravity
2. QFT on noncommutative spaces

1. Kontsevich model (1992)
   designed to prove Witten's conjecture that hermitean one-matrix model computes intersection numbers of stable cohomology classes on the moduli space of complex curves
2. Space-time should become a noncommutative manifold at short distances.
   - Euclidean scalar field $\phi \in \mathcal{A}$ (noncommutative algebra)
   - $\mathcal{A}$ often has finite-dimensional approximations
   - For converse convergence of matrices to $\mathcal{A}$, see Marc Rieffel's talk

The Kontsevich model

defined by partition function

$$\mathcal{Z}(E) := \frac{\int d\Phi \ \exp\Big(-\mathrm{Tr}\big(E\Phi^2 + \frac{\mathrm{i}}{6}\Phi^3\big)\Big)}{\int d\Phi \ \exp\Big(-\mathrm{Tr}\big(E\Phi^2\big)\Big)}$$

- Asymptotic expansion in 'coupling constant' $\frac{\mathrm{i}}{6}$ gives rational function of eigenvalues e_i of E. This rational function generates the intersection numbers.
- Related to Hermitean one-matrix model
  $$\mathcal{Z}(E)[[t_n]] = \int DM \ \exp(-\mathcal{N}\sum_n t_n \,\mathrm{tr}(M^n))$$
  where $t_n := (2n-1)!!\mathrm{tr}(E^{-(2n-1)})$
- Large-$\mathcal{N}$ limit gives KdV evolution equation. Exact solution related to Virasoro algebra.

QFT on noncommutative geometries

Example: Moyal algebra = Rieffel deformation of $C^\infty(\mathbb{R}^2)$

$$(f\star g)(\xi)=\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{d\eta\, dk}{(2\pi)^2} f(x+\tfrac{1}{2}\Theta k)\, g(\xi+\eta)\, e^{\mathrm{i}\langle k,\eta\rangle} \qquad \Theta=\begin{pmatrix}0 & \theta\\ -\theta & 0\end{pmatrix}$$

- matrix basis $\phi(\xi)=\sum_{m,n=0}^{\infty}\Phi_{mn}f_{mn}(\xi)$

$$f_{mn}(\xi)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\tfrac{2}{\theta}}\xi_1+\mathrm{i}\xi_2\Big)^{n-m}L_m^{n-m}\Big(\frac{2\|\xi\|^2}{\theta}\Big)e^{-\frac{\|\xi\|^2}{\theta}}$$

- satisies $f_{mn}\star f_{kl}=\delta_{nk}f_{ml}$ and $\int\frac{d\xi}{8\pi}f_{mn}(\xi)=\frac{\theta}{4}\delta_{mn}$

- Consider scalar field theories on Moyal space

$$S(\phi):=\frac{1}{(8\pi)^{D/2}}\int_{\mathbb{R}^D}d\xi\Big(\frac{1}{2}\phi\star(-\Delta+4\Omega^2\|\Theta^{-1}\xi\|^2)\star\phi+\mathrm{tr}(\mathrm{pol}(\phi))\Big)$$

- f_{mn}-expansion at $\Omega=1$ yields Kontsevich-type matrix model

$$S(\Phi)=V\,\mathrm{tr}(E\Phi^2+\mathrm{pol}(\Phi)),\qquad E=\big((\tfrac{\mu^2}{2}+\tfrac{n}{V})\delta_{mn}\big),\quad V=(\tfrac{\theta}{4})^{D/2}$$

Two independent dimensions

1. Topological dimension 2 from expansion of matrix models into ribbon graphs, i.e. simplicial 2-complexes.
   - dual to triangulations (Φ^3) or quadrangulations (Φ^4) of 2D-surfaces
   - partition function counts them = 2D quantum gravity
   - non-planar ribbon graphs suppressed in large-$\mathcal{N}$ limit
2. Dynamical dimension D encoded in spectrum of the unbounded positive operator E,

$$D = \inf\{p \in \mathbb{R}_+ \;:\; \mathrm{tr}((1+E)^{-\frac{p}{2}}) < \infty\}$$

   - ignored in 2D quantum gravity
   - highly relevant for renormalisation of matricial QFT

polynomial	finite	super-ren	just ren.	not ren.
Φ^3	$D < 2$	$2[\frac{D}{2}] \in \{2,4\}$	$2[\frac{D}{2}] = 6$	$2[\frac{D}{2}] > 6$
Φ^4	$D < 2$	$2[\frac{D}{2}] = 2$	$2[\frac{D}{2}] = 4$	$2[\frac{D}{2}] > 4$

Φ^3_6 matricial QFT

- action $S(\Phi) = V\,\mathrm{tr}(ZE\Phi^2 + (\kappa + \nu E + \zeta E^2)\Phi + \frac{\lambda_{bare} Z^{\frac{3}{2}}}{3}\Phi^3)$
  for $E = \Big(\frac{\mu^2_{bare}}{2} + \mu^2 e\big(\frac{|n|}{\mu^2 V^{2/D}}\big)\delta_{mn}\Big), \quad m, n \in \mathbb{N}^{D/2}$
- $\mu_{bare}, \lambda_{bare}, Z, \kappa, \nu, \zeta$ to be fixed by normalisation conditions
- partition function $\mathcal{Z}(J) = \int d\Phi \;\exp(-S(\Phi) + V\,\mathrm{tr}(\Phi J))$

$$\log \frac{\mathcal{Z}(J)}{\mathcal{Z}(0)} = \sum_{B=1}^{\infty} \sum_{N_B \geq \cdots \geq N_1 \geq 1} \frac{V^{2-B}}{S_{N_1 \ldots N_B}} G_{|p_1^1 \ldots p_{N_1}^1| \ldots |p_1^B \ldots p_{N_B}^B|} \prod_{\beta=1}^{B} \Big(\prod_{j_\beta=1}^{N_\beta} J_{p^\beta_{j_\beta} p^\beta_{j_\beta+1}} \Big)_{cycl}$$

Strategy

- $\mathcal{Z}(J)$ is meaningless for $\lambda \in \mathbb{R}$!
- $\mathcal{Z}(J)$ is only used as tool to derive identities (Schwinger-Dyson equations) between $G_{|p_1^1 \ldots p_{N_1}^1| \ldots |p_1^B \ldots p_{N_B}^B|}$
- Forget $\mathcal{Z}$, declare SD-equations as exact and search for rigorous solutions $G_{\ldots}$ of them!

Schwinger-Dyson equations

Inserting $\mathcal{Z}(J) = \exp\Big(-\frac{Z^{3/2}\lambda_{bare}}{3V^2}\sum \frac{\partial^3}{\partial J_{kl}\partial J_{lm}\partial J_{mk}}\Big)\mathcal{Z}_{\leq 2}(J)$ into

$G_{|a|} \equiv \frac{1}{V}\frac{\partial \log \mathcal{Z}[J]}{\partial J_{aa}}\Big|_{J=0}$ gives equation quadratic in $G_{|a|}$, linear in $\sum_m G_{|am|}$ and $G_{|a|a|}$

- typical feature: SD-equation for n-point function depends on $(m > n)$-point function
- Here we are rescued:
  1. $G_{|a|a|}$ comes with $\frac{1}{V^2}$, goes away in limit $V^{2/D} \sim \theta \to \infty$
  2. $G_{|am|}$ expressable in term of $G_{|a|}, G_{|m|}$ thanks to Ward-Takahashi identity for $U(\infty)$-group action:

Theorem (Disertori-Gurau-Magnen-Rivasseau 2006)

$$\sum_n \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{bn}\partial J_{na}} = \sum_n \frac{V}{Z(E_a - E_b)}\Big(J_{an}\frac{\partial}{\partial J_{bn}} - J_{nb}\frac{\partial}{\partial J_{na}}\Big)\mathcal{Z}[J] - \frac{V}{Z}(\nu + \zeta(E_a + E_b))\frac{\partial \mathcal{Z}[J]}{\partial J_{ba}} \qquad \text{(for } a \neq b\text{)}$$

Scaling limit $\mathcal{N}, V \to \infty$ with $\frac{\mathcal{N}}{V^{2/D}} = \mu^2\Lambda^2$ fixed

Non-linear integral equation for $\tilde{G}(x) = \mu^{1-D/2} G_{|a|}\Big|_{|a|=V^{2/D}\mu^2 x}$ similar to the string equation:

Theorem [Makeenko-Semenoff 1991]

$$W^2(X) + \int_a^b dY \rho(Y) \frac{W(X)-W(Y)}{X-Y} = X + \text{const}$$

is solved by $W(X) = \sqrt{X+c} + \frac{1}{2}\int_a^b \frac{dY\ \rho(Y)}{(\sqrt{X+c}+\sqrt{Y+c})\sqrt{Y+c}}$

together with a consistency condition on c.

Identification $X = (2e(x)+1)^2$, $\rho(Y) = \dfrac{2\lambda^2 (e^{-1}(\frac{\sqrt{Y}-1}{2}))^{D/2-1}}{\Gamma(D/2)\sqrt{Y} e'(e^{-1}(\frac{\sqrt{Y}-1}{2}))}$

Solution of renormalised equation for $D = 6$

$$W(X) = \sqrt{X+c}\sqrt{1+c} - c + \frac{1}{2}\int_1^\infty \frac{dT\ \rho(T)\ (\sqrt{X+c}-\sqrt{1+c})^2}{(\sqrt{X+c}+\sqrt{T+c})(\sqrt{1+c}+\sqrt{T+c})^2\sqrt{T+c}},$$

$$-c = \int_1^\infty \frac{dT\ \rho(T)}{(\sqrt{1+c}+\sqrt{T+c})^3\sqrt{T+c}}$$

gives $\tilde{G}(x) = \frac{1}{2\lambda}(W(X) - \sqrt{X})$

Higher correlation functions

…satisfy linear integral equations, easily reduced to $(1+\ldots+1)$:

$$G_{|a^1_1\ldots a^1_{N_1}|\ldots|a^B_1\ldots a^B_{N_B}|} = \lambda^{N_1+\cdots+N_B-B}\sum_{k_1=1}^{N_1}\cdots\sum_{k_B=1}^{N_B} G_{|a^1_{k_1}|\ldots|a^B_{k_B}|}\prod_{\beta=1}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{1}{E^2_{a^1_{k_1}}-E^2_{a^1_{l_1}}}$$

Proposition

$$G(X|Y) = \frac{4\lambda^2}{\sqrt{X+c}\cdot\sqrt{Y+c}\cdot(\sqrt{X+c}+\sqrt{Y+c})^2}$$

$$G(X^1|\ldots|X^B) = \frac{d^{B-3}}{dt^{B-3}}\left(\frac{(-2\lambda)^{3B-4}}{(R(t))^{B-2}}\frac{1}{\sqrt{X^1+c-2t}^3}\cdots\frac{1}{\sqrt{X^B+c-2t}^3}\right)\Bigg|_{t=0}$$

$$R(T) \overset{6D}{=} \sqrt{1+c} - \int_1^\infty dT\,\rho(T)\,\frac{\{\sqrt{1+c}(2\sqrt{T+c}+\sqrt{1+c})(\sqrt{T+c-2t}+\sqrt{T+c}) + t(\sqrt{T+c-2t}+2\sqrt{T+c})\}}{\sqrt{T+c}(\sqrt{1+c}+\sqrt{T+c})^2(\sqrt{T+c}+\sqrt{T+c-2t})^2\sqrt{T+c-2t}}$$

$$\overset{D\leq 6}{=} 1 - \int_1^\infty \frac{dT\rho(T)}{\sqrt{T+c}}\,\frac{1}{(\sqrt{T+c}+\sqrt{T+c-2t})\sqrt{T+c-2t}}$$

Proof: ansatz for recursion and experience with Bell polynomials

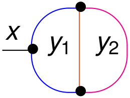
Consistency: (new?) identity for Bell polynomials

For any $l, n_0, \dots, n_p \in \mathbb{N}$, the Bell polynomials satisfy

$$\begin{aligned}
&\frac{(2l{+}5)!!}{(l+2)!} \sum_{K\geq 0} (N{-}2{+}K)! \frac{B_{N-M-l-4,K}(\{x_r\})}{(N{-}M{-}l{-}4)!} \\
&- \sum_{K\geq 0} (N{-}3{+}K)! \frac{B_{N-M-l-4,K}(\{x_r\})}{(N{-}M{-}l{-}4)!} \sum_{i=0}^{p} n_i \frac{(2l+2i+3)!!(2i+1)i!}{(2i+1)!!(l+i+1)!} \\
&= \sum_{j\geq 1} \sum_{K\geq 0} (N{-}2{+}K)! \frac{(2j{+}2l{+}5)!!(j{+}1)!}{(2j{+}1)!!(j{+}l{+}2)!} \cdot \frac{x_j}{j!} \cdot \frac{B_{N-M-l-j-4,K}(\{x_r\})}{(N{-}M{-}l{-}j{-}4)!} \\
&+ \frac{1}{2} \sum_{l'=0}^{l} \sum_{n_0'=0}^{n_0} \cdots \sum_{n_p'=0}^{n_p} \sum_{K',K''\geq 0} \frac{(2l'{+}1)!!(2l''{+}1)!!}{l'!\, l''!} \binom{n_0}{n_0'} \cdots \binom{n_p}{n_p'} \\
&\times (N'{-}2{+}K')! \frac{B_{N'-M'-l'-2,K'}(\{x_r\})}{(N'{-}M'{-}l'{-}2)!} (N''{-}2{+}K'')! \frac{B_{N''-M''-l''-2,K''}(\{x_r\})}{(N''{-}M''{-}l''{-}2)!}
\end{aligned}$$

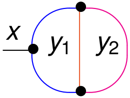
where $l'' := l - l'$, $N'' := N - N'$ and $M'' := M - M'$ and for $? = \{\varnothing, ', ''\}$: $n_0^? + \cdots + n_p^? = N^?$ and $0n_0^? + 1n_1^? + \cdots + pn_p^? = M^?$

Simplest 6D-ribbon graph with overlapping divergence



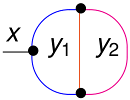
$$= \frac{(-\lambda)^3}{(2x+1)} \int_0^\infty \frac{y_1^2 dy_1}{2} \int_0^\infty \frac{y_2^2 dy_2}{2} \left\{ \frac{1}{(x+y_1+1)^2(y_1+y_2+1)(x+y_2+1)} \right\}$$

Simplest 6D-ribbon graph with overlapping divergence



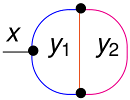
$$
\begin{aligned}
= \frac{(-\lambda)^3}{(2x+1)} \int_0^\infty \frac{y_1^2\, dy_1}{2} \int_0^\infty \frac{y_2^2\, dy_2}{2} \Bigg\{ & \frac{1}{(x+y_1+1)^2(y_1+y_2+1)(x+y_2+1)} \\
& + \frac{1}{x+y_2+1}\Big(-\frac{1}{(y_1+1)^3}\Big) + \frac{1}{(x+y_1+1)^2}\Big(-\frac{1}{(y_2+1)^2}+\frac{y_1+x}{(y_2+1)^3}\Big) \\
& + \frac{1}{(y_1+y_2+1)}\Big(-\frac{1}{(y_1+1)^2(y_2+1)} + \frac{2x}{(y_1+1)^3(y_2+1)} + \frac{x}{(y_1+1)^2(y_2+1)^2} \\
& \qquad - \frac{3x^2}{(y_1+1)^4(y_2+1)} - \frac{x^2}{(y_1+1)^2(y_2+1)^3} - \frac{2x^2}{(y_1+1)^3(y_2+1)^2}\Big) \\
& + \Big(-\frac{1}{(y_1+1)^3}\Big)\Big(-\frac{1}{y_2+1}+\frac{x}{(y_2+1)^2}-\frac{x^2}{(y_2+1)^3}\Big) \\
& + \Big(\Big(-\frac{1}{(y_1+1)^2}+\frac{2x}{(y_1+1)^3}-\frac{3x^2}{(y_1+1)^4}\Big)\Big(-\frac{1}{(y_2+1)^2}+\frac{y_1}{(y_2+1)^3}\Big) \\
& \qquad + \Big(-\frac{1}{(y_1+1)^2}+\frac{2x}{(y_1+1)^3}\Big)\Big(\frac{x}{(y_2+1)^3}\Big)\Big) \Bigg\}
\end{aligned}
$$

Simplest 6D-ribbon graph with overlapping divergence



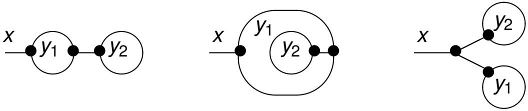
$$= \frac{-\lambda^3}{4(2x+1)^3}\Big\{(x+1)(2x+1)(3x+2)\log(1+x) + (x+1)^3(3x+1)(\log(1+x))^2$$
$$+ x(1+x)(1+3x+3x^2)\Big((\log(1+x))^2 - 2\log(1+x)\log x + 2\mathrm{Li}_2\Big(\frac{1}{1+x}\Big)\Big)$$
$$-3x^3(2+3x)\zeta(2)\Big\} + \frac{\lambda^3 x}{2(2x+1)}\Big(\zeta(2) + 1 - \frac{x}{2}\Big)$$

Simplest 6D-ribbon graph with overlapping divergence



$$= \frac{-\lambda^3}{4(2x+1)^3}\Big\{(x+1)(2x+1)(3x+2)\log(1+x) + (x+1)^3(3x+1)(\log(1+x))^2$$
$$+ x(1+x)(1+3x+3x^2)\Big((\log(1+x))^2 - 2\log(1+x)\log x + 2\mathrm{Li}_2\Big(\frac{1}{1+x}\Big)\Big)$$
$$- 3x^3(2+3x)\zeta(2)\Big\} + \frac{\lambda^3 x}{2(2x+1)}\Big(\zeta(2) + 1 - \frac{x}{2}\Big)$$

adding:



gives the λ^3-order of the exact formula for $\tilde{G}(x)$!

Schwinger functions

undo the passage to the f_{mn}-matrix basis of Moyal space:

Theorem [HG+RW, 2013]: *connected* **Schwinger functions**

$$S_N^c(\mu\xi_1,\dots,\mu\xi_N) := \lim_{V\mu^2\to\infty} \sum_{m_i,n_i=0}^{\infty} f_{m_1n_1}(\xi_1)\cdots f_{m_Nn_N}(\xi_N) \frac{(V\mu^2)^{-2}\mu^{3N}\partial^N \log\mathcal{Z}(J)}{\partial J_{m_1n_1}\dots\partial J_{m_Nn_N}}\Big|_{J=0}$$

$$= \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \Big(\prod_{\beta=1}^{B} \frac{2^{\frac{DN_\beta}{2}}}{N_\beta} \int_{\mathbb{R}^D} \frac{dp_\beta}{(2\pi\mu^2)^{\frac{D}{2}}} e^{\mathrm{i}\langle p_\beta, \sum_{i=1}^{N_\beta}(-1)^{i-1}\xi_{\sigma(N_1+\dots+N_{\beta-1}+i)}\rangle}\Big)$$

$$\times \frac{1}{(8\pi)^{\frac{D}{2}} S_{N_1\dots N_B}} \tilde{G}\Big(\underbrace{\frac{\|p_1\|^2}{2\mu^2},\dots,\frac{\|p_1\|^2}{2\mu^2}}_{N_1}\Big|\dots\Big|\underbrace{\frac{\|p_B\|^2}{2\mu^2},\dots,\frac{\|p_B\|^2}{2\mu^2}}_{N_B}\Big)$$

Confinement of noncommutativity: have internal interaction of matrices; commutative subsector propagates to outside world

- Schwinger functions are symmetric and invariant under full Euclidean group (completely unexpected for NCQFT!)
- remains: reflection positivity (…and non-triviality)

Reflection positivity $S(\bar{f}^r \otimes f) \geq 0$

- f stands for sequences of test functions of complicated support
- $f_1^r(\tau, \vec{\xi}) = f_1(-\tau, \vec{\xi})$ is time reflection

Implies for very special f:
The temporal Fourier transform of $\tilde{S}$ (in all independent energies) is, for any spatial momenta, a positive definite function.

Theorem (Hausdorff-Bernstein-Widder, 1921-1912/28-1941)

For a [smooth] function F on $(\mathbb{R}_+)^N \ni t = (t^1, \dots, t^N)$ are equivalent:

1. F is positive definite, i.e. $\sum_{i,j=1}^K \overline{c_i} c_j F(t_i + t_j) \geq 0$
2. F is the joint Laplace transform of a positive measure*
3. F is completely monotonic, $(-1)^{k_1+\dots+k_N} \partial_{t^1}^{k_1} \dots \partial_{t^N}^{k_N} F(t) \geq 0$

*This is 80% of the proof of the Osterwalder-Schrader theorem.

Stieltjes functions

Prototype for $N = 1$

$$\int_{-\infty}^{\infty} \frac{e^{ip^0 t}}{(p^0)+\vec{p}^2+m^2} = \Big(\frac{2\pi t}{\sqrt{\vec{p}^2+m^2}}\Big)^{\frac{1}{2}} K_{\frac{1}{2}}\big(t\sqrt{\vec{p}^2+m^2}\big) = \frac{\pi e^{-t\sqrt{\vec{p}^2+m^2}}}{\sqrt{\vec{p}^2+m^2}}$$

Theorem

Up to integration in m^2 with positive measure, $\frac{1}{(p^0)+\vec{p}^2+m^2}$ is the only function with positive definite Fourier transform for $N = 1$.

- $p^2 \mapsto \int_0^\infty \frac{\varrho(m^2)dm^2}{p^2+m^2}$ forms the class of Stieltjes functions
- in QFT, $\varrho(m^2)$ is the Källén-Lehmann spectral measure

Is $\tilde{G}(\frac{\|p\|^2}{2\mu^2}, \frac{\|p\|^2}{2\mu^2})$ Stieltjes?

- We work on this for Φ^4_4 since 2013. Have some analytic evidence, confirmed by computer, but no complete proof.
- For Φ^3_D we have the answer:

Reflection positivity of the 2-point function

Theorem (Grosse-Sako-W 2016)

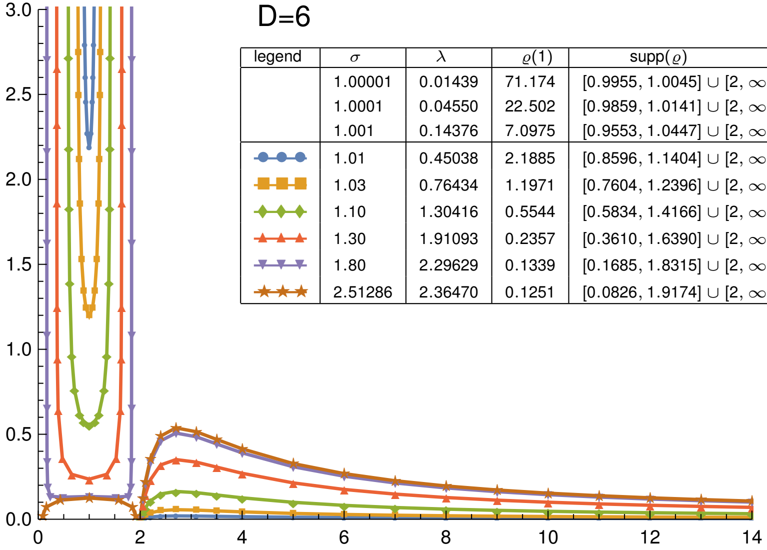
1. The Φ^3_D-matricial QFT is not reflection positive for $\lambda \in \mathrm{i}\mathbb{R}$.
2. The Φ^3_D two-point function is reflection positive for $D \in \{4,6\}$ and some range of $\lambda \in \mathbb{R}$, but not in $D = 2$.

measure supported on fuzzy mass shell plus scattering part:

$$\tilde{G}\Big(\frac{\|p\|^2}{2\mu^2}, \frac{\|p\|^2}{2\mu^2}\Big) \overset{6D}{=} \frac{\lambda^2}{4\pi(\sigma^2-1)} \int_0^\pi d\phi \, \frac{\begin{array}{c}\{2\frac{\log(1+\sigma)}{\sigma}-1+\sigma(\sigma-1)\tan^2\phi \\ -\tan\phi(1+\sigma^2\tan^2\phi)\big(\arctan_{[0,\pi]}(\sigma\tan\phi)-\phi\big)\}\end{array}}{1-\frac{\sqrt{\sigma^2-1}}{\sigma}\cos\phi + \frac{\|p\|^2}{\mu^2}} + \frac{\lambda^2}{4}\int_2^\infty dt \, \frac{t(t-2)/(t-1)^3}{t+\frac{\|p\|^2}{\mu^2}},$$

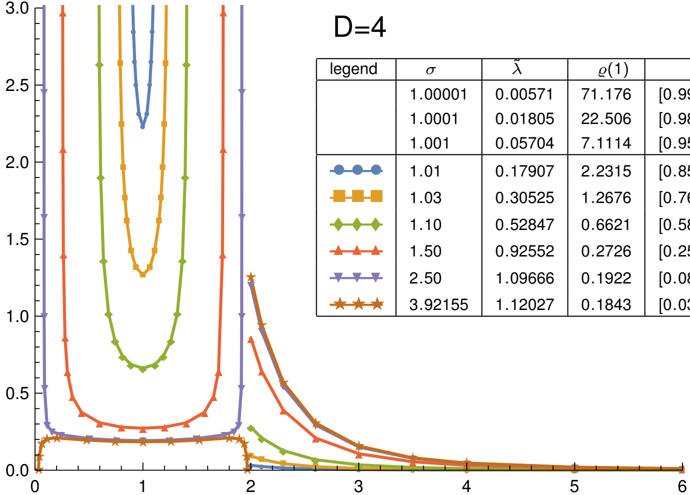
where $\sigma := \frac{1}{\sqrt{1+c}} \in [1, -2W_{-1}(-\frac{1}{2\sqrt{e}})-1]$ is the inverse solution of $\lambda^2 = \frac{4(\sigma^2-1)}{\sigma^2-2\sigma+2\log(1+\sigma)} \in [1, \frac{8W_{-1}(-\frac{1}{2\sqrt{e}})}{1+2W_{-1}(-\frac{1}{2\sqrt{e}})}]$

Källén-Lehmann measure: plots



legend	σ	λ	$\varrho(1)$	supp(ϱ)
	1.00001	0.01439	71.174	$[0.9955, 1.0045] \cup [2, \infty[$
	1.0001	0.04550	22.502	$[0.9859, 1.0141] \cup [2, \infty[$
	1.001	0.14376	7.0975	$[0.9553, 1.0447] \cup [2, \infty[$
●—●—●	1.01	0.45038	2.1885	$[0.8596, 1.1404] \cup [2, \infty[$
■—■—■	1.03	0.76434	1.1971	$[0.7604, 1.2396] \cup [2, \infty[$
◆—◆—◆	1.10	1.30416	0.5544	$[0.5834, 1.4166] \cup [2, \infty[$
▲—▲—▲	1.30	1.91093	0.2357	$[0.3610, 1.6390] \cup [2, \infty[$
▼—▼—▼	1.80	2.29629	0.1339	$[0.1685, 1.8315] \cup [2, \infty[$
★—★—★	2.51286	2.36470	0.1251	$[0.0826, 1.9174] \cup [2, \infty[$

Källén-Lehmann measure: plots



D=4

legend	σ	$\tilde{\lambda}$	$\varrho(1)$	supp(ϱ)
	1.00001	0.00571	71.176	$[0.9955, 1.0045] \cup [2, \infty[$
	1.0001	0.01805	22.506	$[0.9859, 1.0141] \cup [2, \infty[$
	1.001	0.05704	7.1114	$[0.9553, 1.0447] \cup [2, \infty[$
●—●—●	1.01	0.17907	2.2315	$[0.8596, 1.1404] \cup [2, \infty[$
■—■—■	1.03	0.30525	1.2676	$[0.7604, 1.2396] \cup [2, \infty[$
◆—◆—◆	1.10	0.52847	0.6621	$[0.5834, 1.4166] \cup [2, \infty[$
▲—▲—▲	1.50	0.92552	0.2726	$[0.2546, 1.7454] \cup [2, \infty[$
▼—▼—▼	2.50	1.09666	0.1922	$[0.0835, 1.9165] \cup [2, \infty[$
★—★—★	3.92155	1.12027	0.1843	$[0.0331, 1.9669] \cup [2, \infty[$

Reflection positivity of higher Schwinger functions?

- Connected Schwinger functions $S^c_{N\geq 4}$ are not positive!
- Anyway too much, one needs positivity of FT of full functions

e.g. $$\tilde{G}\big(\tfrac{\|p\|^2}{2\mu^2},\tfrac{\|p\|^2}{2\mu^2}\big)\tilde{G}\big(\tfrac{\|q\|^2}{2\mu^2},\tfrac{\|q\|^2}{2\mu^2}\big)+\tilde{G}\big(\tfrac{\|p\|^2}{2\mu^2},\tfrac{\|p\|^2}{2\mu^2}\big|\tfrac{\|q\|^2}{2\mu^2},\tfrac{\|q\|^2}{2\mu^2}\big)$$

- Difficult for $N=4$,
  but $G(2|2|2)+G(2)G(2)G(2)$ is not positive.

Very probable conclusion

The Φ^3_D matricial QFT does not satisfy Osterwalder-Schrader.

- Reason: Higher functions too much localised in p-space!
  already $\tilde{G}\big(\frac{\|p\|^2}{2\mu^2},\frac{\|p\|^2}{2\mu^2}\big)\propto\frac{C_1\log(\|p\|^2+\mu^2)+C_2}{\|p\|^2+\mu^2}$ almost fails
- For Φ^4_4 we expect $\tilde{G}\big(\frac{\|p\|^2}{2\mu^2},\frac{\|p\|^2}{2\mu^2}\big)\propto\frac{C}{(\|p\|^2+\mu^2)^{1-\frac{1}{\pi}\arcsin(|\lambda|\pi)}}$ (hope!)
- Keeps us busy for the next time!


Bon anniversaire
Alain!