Reflection positivity in large-deformation limits of noncommutative field theories

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based on arXiv:1610.00526 & 1612.07584 with Harald Grosse and Akifumi Sako

and arXiv: 1205.0465, 1306.2816, 1402.1041, 1406.7755 & 1505.05161
with Harald Grosse

What makes reflection positivity in 4D QFT difficult?

Euclidean QFT and axioms for Schwinger functions need no introduction. They are natural. Problem is to give examples.

Schwinger 2-point function in 4D momentum space

$$\hat{S}_2(p) \propto \frac{1}{(p^2+m^2)^{1-\eta/2}}, \qquad \eta \text{ – anomalous dimension}$$

- reflection positivity requires $\eta \geq 0$
- ...but convergence in 4D needs $\eta \leq 0$

proposed way out: require decay of effective vertex functions with $p_i \to \infty$ (asymptotic freedom)

- Yang-Mills theory expected to have asymptotic freedom
- tendency to produce infrared problems ($\to$ confinement), no solution so far, Millenium Prize Problem for Yang-Mills

Question

Can one circumvent convergence via exactly solvable models?

Plan

- We will produce candidate Schwinger functions via detour into noncommutative geometry.
- Action functional essentially defines matrix models with $U(\infty)$-group action.
- Derive exact explicit formulae for any correlation function, analytic in coupling constant (hope to do it for other models).
- With some ansatz (to be discussed), get projection to Schwinger functions.
- Explicit formulae permit direct verification of reflection positivity.
  It holds in some sector, but (for chosen ansatz) not for everything.
- Future: understand freedom which goes into ansatz

Matricial quantum field theory

. . . is the marriage of

1. matrix models for 2D quantum gravity
2. QFT on noncommutative spaces

1. Kontsevich model (1992)
   designed to prove Witten's conjecture that hermitean one-matrix model computes intersection numbers of stable cohomology classes on the moduli space of complex curves

2. Space-time should become a noncommutative manifold at short distances.
   - Euclidean scalar field $\phi \in \mathcal{A}$ (noncommutative algebra)
   - $\mathcal{A}$ often has finite-dimensional approximations (matrices)
   - infinitely many constraints from $U(\infty)$-group action make the models almost solvable

Two models

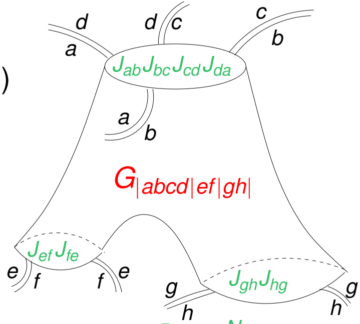
4. $S(\Phi) = V\,\mathrm{tr}(ZE\Phi^2 + \frac{Z^2\lambda}{4}\Phi^4)$
   for $E = \Big(\frac{\mu^2_{bare}}{2} + \mu^2 e\big(\frac{|n|}{\mu^2 V^{2/D}}\big)\delta_{mn}\Big), \quad m,n \in \mathbb{N}^{D/2}$
   suggested the method, solved up to fixed-point problem
3. $S(\Phi) = V\,\mathrm{tr}(ZE\Phi^2 + (\kappa+\nu E+\zeta E^2)\Phi + \frac{\lambda_{bare}Z^{\frac{3}{2}}}{3}\Phi^3)$
   completely solved thanks to relation to Kontsevich model

Strategy

- partition function meaningless for 3 and $\lambda_{bare} \in \mathbb{R}$
- use (Fourier transform of) partition function $\hat{\mathcal{Z}}(J) := \frac{1}{\mathcal{Z}} \int D\Phi \; \exp(-S(\Phi) + \mathrm{i} \sum \Phi_{mn} J_{mn})$ only as a tool to derive equations of motion (Schwinger-Dyson eq.)
- declare SD-equations as exact, try to solve them

limit $V \to \infty$ is (in examples) a large-deformation limit

Topological expansion



- Riemann surfaces with B boundary components (avoid genus expansion)
- k^{th} boundary component carries a cycle $\boldsymbol{J}^{N_k}_{p_1\dots p_{N_k}} := \mathrm{i}^{N_k}\prod_{j=1}^{N_k} J_{p_jp_{j+1}}$ of N_k external sources, $N_k+1\equiv 1$
- expand $\log\hat{\mathcal{Z}}(J) = \sum \frac{1}{S} V^{2-B} G_{|p^1_1\dots p^1_{N_1}|\dots|p^B_1\dots p^B_{N_B}|}\prod_{\beta=1}^{B}\boldsymbol{J}^{N_\beta}_{p^\beta_1\dots p^\beta_{N_\beta}}$ according to the cycle structure
- QFT of matrix models determines the weights of Riemann surfaces with decorated boundary components compatible with
  1. gluing (of fringes, not boundaries!)
  2. covariance (under $\Phi\mapsto U^*\Phi U$, which is not a symmetry!)

Schwinger-Dyson equations

1-point function in dimension $D \leq 6$, $\underline{a} = (a_1, \dots, a_{D/2})$:

$$G_{|\underline{a}|} = \frac{1}{2ZE_{\underline{a}}}\Big\{-\kappa-\nu E_{\underline{a}}-\zeta E_{\underline{a}}^2 - \lambda_{bare}Z^{\frac{3}{2}}\Big(G_{|\underline{a}|}^2 + \frac{1}{V}\sum_{\underline{m}\in\mathbb{N}_{\mathcal{N}}^{D/2}} G_{|\underline{am}|} + \frac{G_{|\underline{a}|\underline{a}|}}{V^2}\Big)\Big\}$$

- typical feature: SD-equation for n-point function depends on $(m > n)$-point function
- Here we are rescued:
  1. $G_{|a|a|}$ comes with $\frac{1}{V^2}$, goes away in limit $V \to \infty$
  2. $G_{|am|}$ expressable in terms of $G_{|a|}, G_{|m|}$ thanks to Ward-Takahashi identity for $U(\infty)$-group action:

Theorem (Disertori-Gurau-Magnen-Rivasseau 2006)

$$\sum_n \frac{\partial^2 \hat{\mathcal{Z}}(J)}{\mathrm{i}^2\partial J_{bn}\partial J_{na}} = \sum_n \frac{V}{Z(E_a - E_b)}\Big(J_{an}\frac{\partial}{\partial J_{bn}} - J_{nb}\frac{\partial}{\partial J_{na}}\Big)\hat{\mathcal{Z}}(J) - \frac{V}{Z}(\nu + \zeta(E_a + E_b))\frac{\partial \hat{\mathcal{Z}}(J)}{\mathrm{i}\partial J_{ba}} \qquad \text{(for } a \neq b\text{)}$$

Limit to integral equations

SD-equations resulting from $\hat{\mathcal{Z}}(J)$ involve sums over matrix indices

- take a limit $\mathcal{N} \to \infty$, $V \to \infty$ with $\frac{\mathcal{N}}{V^{2/D}}$ fixed in which these converge to integral equations for functions of 'continuous matrix indices'
- standard in matrix models where $V \equiv \mathcal{N}$, large-deformation limit on certain noncommutative geometries
- usual QFT-divercenges arise, but better tools to cure them:

  have exact equations for renormalisation constants $Z, \kappa, \nu, \zeta, \mu_{bare}, \lambda_{bare}$: renormalisation without expansion
- additionally the limit decouples equations into:
  - (a) one non-linear (difficult) equation for simplest function
  - (b) a hierarchy of affine equations for all other functions (iteratively solvable)

The non-linear integral equations for $\lambda\Phi^{3,4}$

4 $f(x,y)f(x,z) + \frac{f(x,y)-f(x,z)}{y-z} = \lambda \int dt\, \rho(t)\, \frac{f(x,y)f(t,z)-f(x,z)f(t,y)}{(x-t)(y-z)}$

(with $f(x,y) = f(y,x)$, $f(0,0) = 1$)

- via singular integral equations reducible to fixed point equation (for which solution exists!)

$$f(x) = \frac{1}{1+x} \exp\left(-\lambda \int_0^x dt \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}_p[f(\bullet)]}{f(p)}\right)^2}\right)$$

3 $f^2(x) + \lambda^2 \int_a^b dt \rho(t) \frac{f(x)-f(t)}{x-t} = x$

solved by $f(x) = \sqrt{x+c} + \frac{\lambda^2}{2} \int_a^b \frac{dt\ \rho(t)}{(\sqrt{x+c}+\sqrt{t+c})\sqrt{t+c}}$ plus consistency condition on c [Makeenko-Semenoff, 1991]

Remark: both equations viewed as describing boundary values of holomorphic functions

The beta-function

4. $\lambda\Phi^4_4$ has, for any dynamical matrix E, vanishing β-function

   more precisely for $\mathcal{N}, V \to \infty$:

   $$G(x,y,z,w) = (-\lambda_{bare}) \frac{G(x,y)G(z,w) - G(x,w)G(z,y)}{(e(x)-e(z))(e(y)-e(w))}$$

   (perturbative proof by [Disertori et al] initiated this work)

3. for Φ^3_6 (at cut-off Λ):

$$\beta_\lambda := \Lambda^2 \frac{d\lambda_{bare}(\Lambda)}{d\Lambda^2} = \frac{2\lambda_r^3\Lambda^6}{\left(\sqrt{1+c}+\sqrt{(2e(\Lambda^2)+1)^2+c}\right)^2 \sqrt{(2e(\Lambda^2)+1)^2+c}}$$

- positive for $\lambda_r \in \mathbb{R}$, negative for $\lambda_r \in \mathrm{i}\mathbb{R}$
- nonetheless for $\lambda_r \in \mathbb{R}$ no Landau ghost or triviality:
  $\lambda_{bare} = \frac{\lambda_r}{\sqrt{Z}}$, $Z \in [0,1]$ for $\lambda_r \in \mathbb{R}$

Explicit solution of Φ^3_6

Theorem [Grosse-Sako-W, 2016]

Given coupling constant λ, eigenvalue function $e(x)$ of 6D degeneracy. Then the renormalised 1-point function of $\lambda\Phi^3_6$ is

$$G(x) = \frac{\sqrt{(X+c)(1+c)} - c - \sqrt{X}}{2\lambda} + \frac{\lambda}{4}\int_1^\infty \frac{dT\,\big(e^{-1}(\frac{\sqrt{T}-1}{2})\big)^2\,\big(\sqrt{X+c}-\sqrt{1+c}\big)^2}{\sqrt{T}e'(e^{-1}(\frac{\sqrt{T}-1}{2}))(\sqrt{X+c}+\sqrt{T+c})(\sqrt{1+c}+\sqrt{T+c})^2\sqrt{T+c}}$$

where $\sqrt{X} := 2e(x)+1$ and $c(\lambda)$ is implicit solution of

$$-c = \lambda^2 \int_1^\infty \frac{dT\,\big(e^{-1}(\frac{\sqrt{T}-1}{2})\big)^2}{\sqrt{T}e'(e^{-1}(\frac{\sqrt{T}-1}{2}))(\sqrt{1+c}+\sqrt{T+c})^3\sqrt{T+c}}$$

- explicit integrals for Moyal space with $e(x) = x$ (later)
- matches perfectly renormalised Feynman graph calculation

Higher correlation functions

…satisfy linear integral equations, easily reduced to $(1+\ldots+1)$:

$$G(x^1_1,\ldots,x^1_{N_1}|\ldots|x^B_1,\ldots,x^B_{N_B}|) = \frac{1}{\lambda^B}\sum_{k_1=1}^{N_1}\cdots\sum_{k_B=1}^{N_B} G(x^1_{k_1}|\ldots|x^B_{k_B}|)\prod_{\beta=1}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{4\lambda}{X^\beta_{k_\beta}-X^\beta_{l_\beta}}$$

$(\lambda G(x^1_k)+\sqrt{X^1_k}/2)$ if $B=1$

$X^\beta_k = (2e(x^\beta_k)+1)^2$

Higher correlation functions

...satisfy linear integral equations, easily reduced to $(1+\ldots+1)$:

$$G(x_1^1,\ldots,x_{N_1}^1|\ldots|x_1^B,\ldots,x_{N_B}^B|)$$
$$=\frac{1}{\lambda^B}\sum_{k_1=1}^{N_1}\cdots\sum_{k_B=1}^{N_B}G(x_{k_1}^1|\ldots|x_{k_B}^B|)\prod_{\beta=1}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{4\lambda}{X_{k_\beta}^\beta-X_{l_\beta}^\beta}$$

Theorem [Grosse-Sako-W 2016]

$$G(x|y)=\frac{4\lambda^2}{\sqrt{X+c}\cdot\sqrt{Y+c}\cdot(\sqrt{X+c}+\sqrt{Y+c})^2}$$

where $X=(2(e(x)+1))^2$

$$G(x^1|\ldots|x^B)=\frac{d^{B-3}}{dt^{B-3}}\Big(\frac{(-2\lambda)^{3B-4}}{(R(t))^{B-2}}\frac{1}{\sqrt{X^1+c-2t}^3}\cdots\frac{1}{\sqrt{X^B+c-2t}^3}\Big)\Big|_{t=0}$$

$$R(T)=\lim_{\tilde{\Lambda}\to\infty}\Big(\frac{1}{\sqrt{Z(\tilde{\Lambda})}}-\int_1^{\tilde{\Lambda}}\frac{dT\rho(T)}{\sqrt{T+c}}\frac{1}{(\sqrt{T+c}+\sqrt{T+c-2t})\sqrt{T+c-2t}}\Big)$$

Proof: ansatz for recursion and experience with Bell polynomials

QFT on noncommutative geometries

Example: Moyal algebra = Rieffel deformation of $C^\infty(\mathbb{R}^2)$

$$(f \star g)(\xi) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{d\eta\, dk}{(2\pi)^2} f(\xi+\tfrac{1}{2}\Theta k)\, g(\xi+\eta)\, e^{\mathrm{i}\langle k,\eta\rangle} \qquad \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

- matrix basis $\phi(\xi) = \sum_{\underline{m},\underline{n}\in\mathbb{N}^{D/2}} \Phi_{\underline{m}\underline{n}} \prod_{i=1}^{D/2} f_{m_i n_i}(\xi_{2i-1},\xi_{2i})$

$$f_{mn}(\xi) = 2(-1)^m \sqrt{\frac{m!}{n!}} \Big(\sqrt{\tfrac{2}{\theta}}\xi_1 + \mathrm{i}\xi_2\Big)^{n-m} L_m^{n-m}\Big(\frac{2\|\xi\|^2}{\theta}\Big) e^{-\frac{\|\xi\|^2}{\theta}}$$

- satisfies $f_{\underline{m}\underline{n}} \star f_{\underline{k}\underline{l}} = \delta_{\underline{n}\underline{k}} f_{\underline{m}\underline{l}}$ and $\int \frac{d\xi}{(8\pi)^{D/2}} f_{\underline{m}\underline{n}}(\xi) = (\frac{\theta}{4})^{D/2}\delta_{\underline{m}\underline{n}}$

- consider scalar field theories on Moyal space

$$S(\phi) := \frac{1}{(8\pi)^{D/2}} \int_{\mathbb{R}^D} d\xi \Big(\frac{1}{2}\phi\star(-\Delta + 4\Omega^2\|\Theta^{-1}\xi\|^2)\star\phi + \mathrm{tr}(\mathrm{pol}(\phi))\Big)$$

- f_{mn}-expansion at $\Omega = 1$ yields matrix model

$$S(\Phi) = V\,\mathrm{tr}(E\Phi^2 + \mathrm{pol}(\Phi)), \quad E = \Big((\tfrac{\mu^2}{2} + \tfrac{n}{V^{\frac{2}{D}}})\delta_{mn}\Big), \quad V = (\tfrac{\theta}{4})^{D/2}$$

Schwinger functions on standard $\mathbb{R}^D$

from matricial moments $G(x_1^1 \dots x_{N_1}^1 | \dots | x_1^B \dots x_{N_B}^B)$ via f_{mn} to $\mathbb{R}^D$:

Definition/Theorem: *connected* Schwinger functions

$$S_N^c(\mu\xi_1, \dots, \mu\xi_N)$$

$$:= \lim_{V\mu^D \to \infty} \sum_{m_i, n_i = 0}^{\infty} f_{m_1 n_1}(\xi_1) \cdots f_{m_N n_N}(\xi_N) \frac{(V\mu^D)^{-2} \mu^{\frac{N(D+2)}{2}} \partial^N \log \hat{\mathcal{Z}}(J)}{\mathrm{i}^N \partial J_{m_1 n_1} \dots \partial J_{m_N n_N}} \Big|_{J=0}$$

$$= \sum_{\substack{N_1 + \dots + N_B = N \\ N_\beta \text{ even}}} \sum_{\sigma \in \mathcal{S}_N} \Big(\prod_{\beta=1}^{B} \frac{2^{\frac{DN_\beta}{2}}}{N_\beta} \int_{\mathbb{R}^D} \frac{dp_\beta}{(2\pi\mu^2)^{\frac{D}{2}}} e^{\mathrm{i}\langle p_\beta, \sum_{i=1}^{N_\beta} (-1)^{i-1} \xi_{\sigma(N_1 + \dots + N_{\beta-1} + i)} \rangle} \Big)$$

$$\times \frac{1}{(8\pi)^{\frac{D}{2}} S_{N_1 \dots N_B}} G\Big(\underbrace{\frac{\|p_1\|^2}{2\mu^2}, \dots, \frac{\|p_1\|^2}{2\mu^2}}_{N_1} \Big| \dots \Big| \underbrace{\frac{\|p_B\|^2}{2\mu^2}, \dots, \frac{\|p_B\|^2}{2\mu^2}}_{N_B} \Big)$$

Confinement of noncommutativity: have internal interaction of matrices; commutative subsector propagates to outside world

- Schwinger functions are symmetric and invariant under full Euclidean group (completely unexpected for NCQFT!)
- remains: reflection positivity (... and non-triviality)

Remarks

- momenta of Euclidean particles individually preserved (no production, not even momentum transfer, much less than factorising S-matrices)
- known from some classical integrable models [Moser 1975], extended to quantum fields by Kulish [1976]
- clustering is always violated
- expect very many vacua, S-matrix a constant phase in each of them

All this relies on the preliminary definition of Schwinger functions, which might change in future.

On the other hand, we can answer reflection positivity:

Reflection positivity $S(\bar{f}^r \otimes f) \geq 0$

- f stands for sequences of test functions of special support
- $f_1^r(\tau, \vec{\xi}) = f_1(-\tau, \vec{\xi})$ is time reflection

Implies for very special f:
The temporal Fourier transform of $\hat{S}$ (in all independent energies) is, for any spatial momenta, a positive definite function.

Theorem [Hausdorff-Bernstein-Widder, 1921-1912/28-1941]

For continuous/smooth function F on $(\mathbb{R}_+)^N \ni t = (t^1, \ldots, t^N)$ are equivalent:

1. F is positive definite, i.e. $\sum_{i,j=1}^K \overline{c_i} c_j F(t_i + t_j) \geq 0$
2. F is the joint Laplace transform of a positive measure
3. F is completely monotonic, $(-1)^{k_1+\cdots+k_N} \partial_{t^1}^{k_1} \ldots \partial_{t^N}^{k_N} F(t) \geq 0$

property 2 needed for Osterwalder-Schrader, 3 is what we check

Stieltjes functions

Prototype for $N = 1$

$$\int_{-\infty}^{\infty} \frac{e^{ip^0 t}}{(p^0)+\vec{p}^2+m^2} = \Big(\frac{2\pi t}{\sqrt{\vec{p}^2+m^2}}\Big)^{\frac{1}{2}} K_{\frac{1}{2}}\big(t\sqrt{\vec{p}^2+m^2}\big) = \frac{\pi e^{-t\sqrt{\vec{p}^2+m^2}}}{\sqrt{\vec{p}^2+m^2}}$$

Lemma

Up to integration in m^2 with positive measure, $\frac{1}{(p^0)+\vec{p}^2+m^2}$ is the only function with positive definite Fourier transform for $N = 1$.

- $p^2 \mapsto \int_0^\infty \frac{\varrho(m^2)dm^2}{p^2+m^2}$ forms the class of Stieltjes functions
- in QFT, $\varrho(m^2)$ is the Källén-Lehmann spectral measure

Is $G(\frac{\|p\|^2}{2\mu^2}, \frac{\|p\|^2}{2\mu^2})$ Stieltjes?

- We work on this for Φ^4_4 since 2013. Have some analytic evidence, confirmed by computer, but no complete proof.
- For Φ^3_D we have the answer:

Reflection positivity of the 2-point function

Theorem (Grosse-Sako-W 2016)

1. The Φ^3_D-matricial QFT is not reflection positive for $\lambda \in \mathrm{i}\mathbb{R}$.
2. The Φ^3_D two-point function is reflection positive for $D \in \{4,6\}$ and some range of $\lambda \in \mathbb{R}$, but not in $D = 2$.

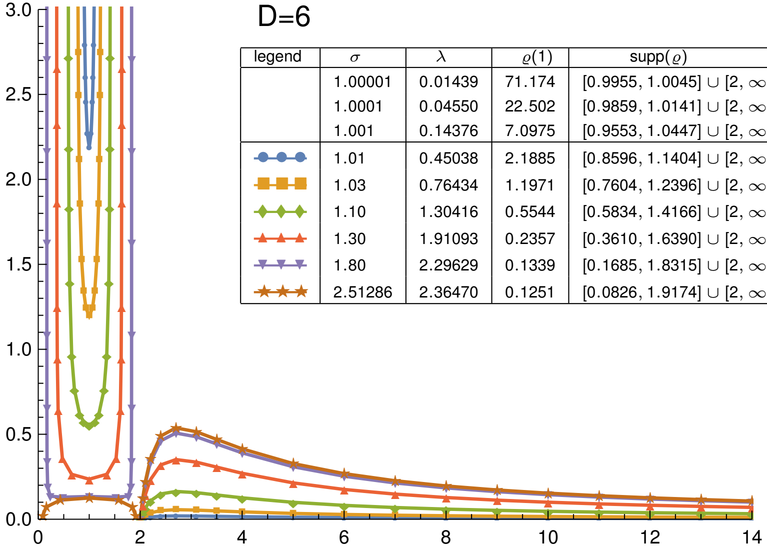
measure supported on broadened mass shell plus scattering part:

$$G\Big(\frac{\|p\|^2}{2\mu^2}, \frac{\|p\|^2}{2\mu^2}\Big) \stackrel{6D}{=} \frac{\lambda^2}{4\pi(\sigma^2-1)} \int_0^\pi d\phi \, \frac{\begin{array}{c}\{2\frac{\log(1+\sigma)}{\sigma} - 1 + \sigma(\sigma-1)\tan^2\phi \\ -\tan\phi(1+\sigma^2\tan^2\phi)\big(\arctan_{[0,\pi]}(\sigma\tan\phi)-\phi\big)\}\end{array}}{1 - \frac{\sqrt{\sigma^2-1}}{\sigma}\cos\phi + \frac{\|p\|^2}{\mu^2}}$$

$$+ \frac{\lambda^2}{4}\int_2^\infty dt \, \frac{t(t-2)/(t-1)^3}{t + \frac{\|p\|^2}{\mu^2}},$$

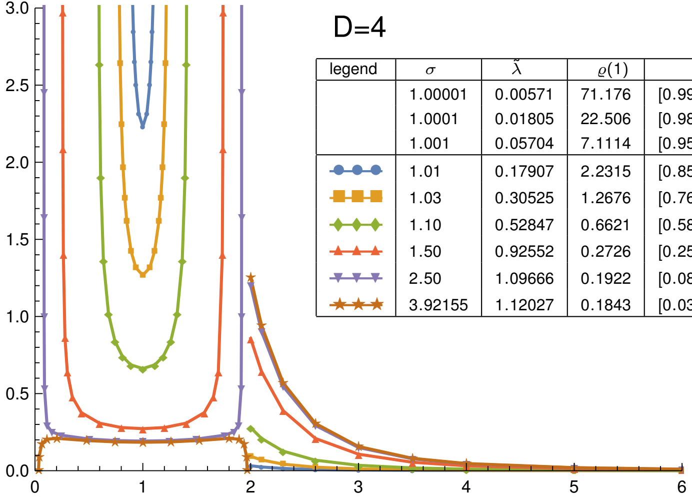
where $\sigma := \frac{1}{\sqrt{1+c}} \in [1, -2W_{-1}(-\frac{1}{2\sqrt{e}}) - 1]$ is the inverse solution of $\lambda^2 = \frac{4(\sigma^2-1)}{\sigma^2-2\sigma+2\log(1+\sigma)} \in [0, \frac{8W_{-1}(-\frac{1}{2\sqrt{e}})}{1+2W_{-1}(-\frac{1}{2\sqrt{e}})}]$

Källén-Lehmann measure: plots



legend	σ	λ	$\varrho(1)$	supp(ϱ)
	1.00001	0.01439	71.174	$[0.9955, 1.0045] \cup [2, \infty[$
	1.0001	0.04550	22.502	$[0.9859, 1.0141] \cup [2, \infty[$
	1.001	0.14376	7.0975	$[0.9553, 1.0447] \cup [2, \infty[$
circles (blue)	1.01	0.45038	2.1885	$[0.8596, 1.1404] \cup [2, \infty[$
squares (orange)	1.03	0.76434	1.1971	$[0.7604, 1.2396] \cup [2, \infty[$
diamonds (green)	1.10	1.30416	0.5544	$[0.5834, 1.4166] \cup [2, \infty[$
triangles (red)	1.30	1.91093	0.2357	$[0.3610, 1.6390] \cup [2, \infty[$
inverted triangles (purple)	1.80	2.29629	0.1339	$[0.1685, 1.8315] \cup [2, \infty[$
stars (brown)	2.51286	2.36470	0.1251	$[0.0826, 1.9174] \cup [2, \infty[$

Källén-Lehmann measure: plots



D=4

legend	σ	$\tilde{\lambda}$	$\varrho(1)$	$\text{supp}(\varrho)$
	1.00001	0.00571	71.176	$[0.9955, 1.0045] \cup [2, \infty[$
	1.0001	0.01805	22.506	$[0.9859, 1.0141] \cup [2, \infty[$
	1.001	0.05704	7.1114	$[0.9553, 1.0447] \cup [2, \infty[$
	1.01	0.17907	2.2315	$[0.8596, 1.1404] \cup [2, \infty[$
	1.03	0.30525	1.2676	$[0.7604, 1.2396] \cup [2, \infty[$
	1.10	0.52847	0.6621	$[0.5834, 1.4166] \cup [2, \infty[$
	1.50	0.92552	0.2726	$[0.2546, 1.7454] \cup [2, \infty[$
	2.50	1.09666	0.1922	$[0.0835, 1.9165] \cup [2, \infty[$
	3.92155	1.12027	0.1843	$[0.0331, 1.9669] \cup [2, \infty[$

Reflection positivity of higher Schwinger functions?

No, they are too much localised in *p*-space!

Φ^4_4 slightly better $G(\frac{\|p\|^2}{2\mu^2}, \frac{\|p\|^2}{2\mu^2}) \propto \frac{C}{(\|p\|^2+\mu^2)^{1-\frac{1}{\pi}\arcsin(|\lambda|\pi)}}$
but will not solve the problem

Way out?

- We have a nice exactly solvable matricial theory with explicit formulae for everything.
- To obtain a QFT on commutative $\mathbb{R}^D$ we take a limit which brutally projects to diagonals, loosing most information
- Can we build reasonable/natural states on matrices
$$\omega_{\xi_1,\dots,\xi_N}(e_{m_1 n_1}\otimes\dots\otimes e_{m_N n_N})$$
which keep the non-diagonal information?
- In other words: it is the state which translates matrices to space-time functions

States

More general definition of connected Schwinger functions

$$S^c_N(\xi_1,\dots,\xi_N) := \lim_{V,\mathcal{N}\to\infty} \sum_{m_i,n_i=0}^{\infty} \omega_{\xi_1,\dots,\xi_N}(e_{m_1n_1}\otimes\dots\otimes e_{m_Nn_N}) \frac{\partial^N \log \hat{\mathcal{Z}}(J)}{\mathrm{i}^N \partial J_{m_1n_1}\dots\partial J_{m_Nn_N}}\Big|_{J=0}$$

- so far $\omega_{\xi_1,\dots,\xi_N}(e_{m_1n_1}\otimes\dots\otimes e_{m_Nn_N}) = \frac{\mu^{3N}}{(V\mu^D)^2} f_{m_1n_1}(\xi_1)\cdots f_{m_Nn_N}(\xi_N)$
- maybe quantum diagonal map (removes center of motion) [Bahns-Doplicher-Fredenhagen-Piacitelli, 2003]
- simple integration over $\xi_1+\dots+\xi_N$ considably weakens p-space localisation (but probably not enough)
- future project: systematic investigation of this freedom, and whether this rescues reflection positivity
- Can this be generalised to much larger classes of noncommutative geometries, e.g. all AF-algebras?