Matricial quantum field theory

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based on arXiv:1610.00526 & 1612.07584 with Harald Grosse and Akifumi Sako

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with Harald Grosse

Matricial quantum field theory

. . . is the marriage of

1. matrix models for 2D quantum gravity
2. QFT on noncommutative spaces

to progress on a longstanding problem of constructive QFT:

construct a (even simple, but not trivial) quantum field theory in 4 dimensions satisfying axioms of Wightman, Haag-Kastler or Osterwalder-Schrader

Background I: 2D quantum gravity

1. topological quantum gravity (1988/89)
   [Labastida-Pernici-Witten, Montano-Sonnenschein, Myers-Periwal]
2. Hermitean one-matrix model (1989/90)
   [Brezin-Kazakov, Douglas-Shenker, Gross-Migdal]

Conjecture [Witten 1991]: Both are the same!

The matrix model computes intersection numbers of stable cohomology classes on the moduli space of complex curves

Proved by Kontsevich 1991/92 via matrix Airy function

$$\mathcal{Z}(E) := \frac{\int d\Phi \;\exp\big(-\mathrm{Tr}(E\Phi^2 + \frac{\mathrm{i}}{6}\Phi^3)\big)}{\int d\Phi \;\exp\big(-\mathrm{Tr}(E\Phi^2)\big)}, \qquad E = \mathrm{diag}(e_i)$$

1. rational function of $\{e_i\}$ generates intersection numbers
2. $\mathcal{Z}(E)[[t_n]] = \int DM \;\exp(-\mathcal{N}\sum_n t_n \,\mathrm{tr}(M^n))$ where $t_n := (2n-1)!!\mathrm{tr}(E^{-(2n-1)})$

Exact solution related to Virasoro constraints and KdV evolution eq.

Background II: QFT on noncommutative geometries

1. Compactification of M-theory on noncommutative tori [Connes-Douglas-Schwarz 1997]
2. Limits of string theory in presence of magnetic fields [Schomerus, Seiberg-Witten 1999]
3. QFT-models on Moyal space (B =const) show UV/IR-mixing [Minwalla-van Raamsdonk-Seiberg 1999]

UV/IR cured in $\lambda\phi^{\star 4}_4$ with harmonic propagation

$$S(\phi) := \frac{1}{64\pi^2}\int_{\mathbb{R}^4} d\xi\Big(\frac{1}{2}\phi\star(-\Delta+4\Omega^2\|\Theta^{-1}\xi\|^2)\star\phi + \frac{\lambda}{4}\phi^{\star 4}\Big)(\xi)$$

- renormalisable to all orders in λ [Grosse-W 2004]
- β-function is zero to all orders [Disertori-Gurau-Magnen-Rivasseau 2006]

Can it be constructed?

- Yes, up to solution of a fixed point problem [Grosse-W 2012]
- Complete solution for $\phi^{\star 3}_{\{2,4,6\}}$ [Grosse-Sako-W 2016]

Background III: QFT axioms

1. Wightman (1956): quantum fields are unbounded operator-valued distributions $f \mapsto \Phi(f) : \mathcal{D} \to \mathcal{D} \subset \mathcal{H}$
2. Haag-Kastler (1964): reformulation in terms of C^*-algebras
3. Osterwalder-Schrader (1974): use

Theorem: vacuum expectation values $\langle \Omega, \Phi(x_1) \cdots \Phi(x_N)\Omega\rangle$ are boundary values of holomorphic functions

- determined by real space of Schwinger functions
- Schwinger functions inherit real analyticity, Euclidean invariance, complete symmetry and reflection positivity
- conversely, this suffices to reconstruct Wightman theory

So far no non-trivial QFT model in 4 dimensions . . .
Matricial QFT looks promising . . .

From NCQFT to matrix models

Noncommutative manifold is a noncommutative algebra $\mathcal{A}$ which often has finite-dimensional approximations: matrices

Example: Moyal algebra = Rieffel deformation of $C^\infty(\mathbb{R}^2)$

$$(f \star g)(\xi) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{d\eta\, dk}{(2\pi)^2} f(x+\tfrac{1}{2}\Theta k)\, g(\xi+\eta)\, e^{\mathrm{i}\langle k,\eta\rangle}, \qquad \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

- $f_{mn}(\xi) = 2(-1)^m \sqrt{\frac{m!}{n!}} \Big(\sqrt{\tfrac{2}{\theta}}\xi_1 + \mathrm{i}\xi_2\Big)^{n-m} L_m^{n-m}\Big(\frac{2\|\xi\|^2}{\theta}\Big) e^{-\frac{\|\xi\|^2}{\theta}}$

  satisies $f_{mn} \star f_{kl} = \delta_{nk} f_{ml}$ and $\int \frac{d\xi}{8\pi} f_{mn}(\xi) = \frac{\theta}{4}\delta_{mn}$

Correspondence NCQFT - Kontsevich-type matrix models

1. Topological dimension 2 from expansion of matrix models into ribbon graphs, i.e. simplicial 2-complexes.
2. Dynamical dimension D encoded in spectrum of the unbounded positive operator E,

$$D = \inf\{p \in \mathbb{R}_+ \;:\; \mathrm{tr}((1+E)^{-\frac{p}{2}}) < \infty\}$$

Φ^3_6 matricial QFT

- action $S(\Phi) = V\,\mathrm{tr}(ZE\Phi^2 + (\kappa+\nu E+\zeta E^2)\Phi + \frac{\lambda_{bare}Z^{\frac{3}{2}}}{3}\Phi^3)$
  for $E = \Big(\big(\frac{\mu^2_{bare}}{2} + \mu^2 e(\frac{|n|}{\mu^2 V^{2/D}})\big)\delta_{mn}\Big), \quad m,n \in \mathbb{N}^{D/2}$
- $\mu_{bare}, \lambda_{bare}, Z, \kappa, \nu, \zeta$ to be fixed by normalisation conditions
- NCQFT yields $e(x) = x$ and $V = \theta^{D/2}$
- partition function $\mathcal{Z}(J) = \int d\Phi\ \exp(-S(\Phi) + V\,\mathrm{tr}(\Phi J))$

$$\log\frac{\mathcal{Z}(J)}{\mathcal{Z}(0)} = \sum_{B=1}^{\infty}\sum_{N_B\geq\cdots\geq N_1\geq 1}\frac{V^{2-B}}{S_{N_1\dots N_B}} G_{|p_1^1\dots p_{N_1}^1|\dots|p_1^B\dots p_{N_B}^B|}\prod_{\beta=1}^{B}\Big(\prod_{j_\beta=1}^{N_\beta} J_{p^\beta_{j_\beta}p^\beta_{j_\beta+1}}\Big)_{cycl}$$

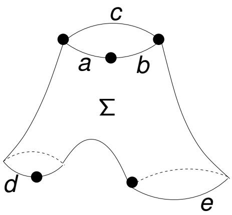
Strategy

- $\mathcal{Z}(J)$ is meaningless for $\lambda\in\mathbb{R}$!
- $\mathcal{Z}(J)$ is only used as tool to derive identities (Schwinger-Dyson equations) between $G_{|p_1^1\dots p_{N_1}^1|\dots|p_1^B\dots p_{N_B}^B|}$
- Forget $\mathcal{Z}$, declare SD-equations as exact and search for rigorous solutions $G_{\dots}$ of them!

Interlude: link to topological quantum field theory?

$G_{|p^1_1\dots p^1_{N_1}|\dots|p^B_1\dots p^B_{N_B}|}$ represents functor $G : \mathfrak{Bord}_* \to \mathfrak{Vect}$

- $\mathfrak{Bord}_*$ – category of 2D bordisms which preserve marked points on boundary
- $G : (\Sigma : S^1_{*n_1} \sqcup \dots \sqcup S^1_{*n_B} \to \varnothing) \to (A^{\otimes_c n_1} \otimes_s \dots \otimes_s A^{\otimes_c n_B} \stackrel{\text{lin}}{\to} \mathbb{C})$ where S^1_{*n} circle with n marked points, $S^1_{*0} := \varnothing$, $A = M_{\mathcal{N}}(\mathbb{C})$



$$G(\Sigma)((J^1 \otimes_c J^2 \otimes_c J^3) \otimes_s J^4 \otimes_s J^5) = \sum_{a,b,c,d,e} G_{|abc|d|e|} J^1_{ab} J^2_{bc} J^3_{ca} J^4_{dd} J^5_{ee}$$

- bordisms$_*$ glued along marked points (which disappear), not along entire boundary components

Schwinger-Dyson equations

1-point function in dimension $D \leq 6$, $\quad \underline{a} = (a_1, \dots, a_{D/2})$:

$$G_{|\underline{a}|} = \frac{1}{2ZE_{\underline{a}}} \Big\{ -\kappa - \nu E_{\underline{a}} - \zeta E^2_{\underline{a}} - \lambda_{bare} Z^{\frac{3}{2}} \Big(G^2_{|\underline{a}|} + \frac{1}{V} \sum_{\underline{m} \in \mathbb{N}^{D/2}_{\mathcal{N}}} G_{|\underline{am}|} + \frac{G_{|\underline{a}|\underline{a}|}}{V^2} \Big) \Big\}$$

- typical feature: SD-equation for n-point function depends on $(m > n)$-point function
- Here we are rescued:
  1. $G_{|a|a|}$ comes with $\frac{1}{V^2}$, goes away in limit $V^{2/D} \sim \theta \to \infty$
  2. $G_{|am|}$ expressable in terms of $G_{|a|}, G_{|m|}$ thanks to Ward-Takahashi identity for $U(\infty)$-group action:

Theorem (Disertori-Gurau-Magnen-Rivasseau 2006)

$$\sum_n \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{bn} \partial J_{na}} = \sum_n \frac{V}{Z(E_a - E_b)} \Big(J_{an} \frac{\partial}{\partial J_{bn}} - J_{nb} \frac{\partial}{\partial J_{na}} \Big) \mathcal{Z}[J] - \frac{V}{Z} (\nu + \zeta(E_a + E_b)) \frac{\partial \mathcal{Z}[J]}{\partial J_{ba}} \qquad \text{(for } a \neq b\text{)}$$

Scaling limit $\mathcal{N}, V \to \infty$ with $\frac{\mathcal{N}}{V^{2/D}} = \mu^2\Lambda^2$ fixed

Non-linear integral equation for $\tilde{G}(x) = \mu^{1-D/2} G_{|a|}\big|_{|a|=V^{2/D}\mu^2 x}$
similar to equation from Virasoro constraint in Kontsevich model:

Theorem [Makeenko-Semenoff 1991]

$$W^2(X) + \int_a^b dY \rho(Y) \frac{W(X)-W(Y)}{X-Y} = X + \text{const}$$

is solved by $W(X) = \sqrt{X+c} + \frac{1}{2}\int_a^b \frac{dY\ \rho(Y)}{(\sqrt{X+c}+\sqrt{Y+c})\sqrt{Y+c}}$
together with a consistency condition on c.

Identification $X = (2e(x)+1)^2$, $\rho(Y) = \frac{2\lambda^2 (e^{-1}(\frac{\sqrt{Y}-1}{2}))^{D/2-1}}{\Gamma(D/2)\sqrt{Y} e'(e^{-1}(\frac{\sqrt{Y}-1}{2}))}$

Ansatz for $\tilde{G}(x) =: \frac{1}{2\lambda}(W(X) - \sqrt{X})$

$$W(X) = \frac{\sqrt{X+c}}{\sqrt{Z}} - \nu + \frac{1}{2}\int_1^{\tilde{\Lambda}} \frac{dY\ \rho(Y)}{(\sqrt{X+c}+\sqrt{Y+c})\sqrt{Y+c}}$$

normalisation conditions on $\tilde{G}_{\dots}$ translate to

$$\underbrace{W(1)=1,}_{D\geq 2}\quad \underbrace{W'(1) = \frac{d}{dX}\sqrt{X}\Big|_{X=1} = \frac{1}{2},}_{D>4}\quad \underbrace{W''(1) = \frac{d^2}{dX^2}\sqrt{X}\Big|_{X=1} = -\frac{1}{4}}_{D=6}$$

Solution of renormalised equation for $D = 6$

Theorem

Given coupling constant λ, eigenvalue function $e(x)$ of 6D degeneracy. Then the renormalised 1-point function of $\lambda\Phi^3_6$ is

$$\tilde{G}(x) = \frac{\sqrt{(X+c)(1+c)} - c - \sqrt{X}}{2\lambda} + \frac{\lambda}{4}\int_1^\infty \frac{dT\, (e^{-1}(\frac{\sqrt{T}-1}{2}))^2\, (\sqrt{X+c} - \sqrt{1+c})^2}{\sqrt{T} e'(e^{-1}(\frac{\sqrt{T}-1}{2}))(\sqrt{X+c} + \sqrt{T+c})(\sqrt{1+c} + \sqrt{T+c})^2 \sqrt{T+c}}$$

where $\sqrt{X} := 2e(x) + 1$ and $c(\lambda)$ is implicit solution of

$$-c = \lambda^2 \int_1^\infty \frac{dT\, (e^{-1}(\frac{\sqrt{T}-1}{2}))^2}{\sqrt{T} e'(e^{-1}(\frac{\sqrt{T}-1}{2}))(\sqrt{1+c} + \sqrt{T+c})^3 \sqrt{T+c}}$$

- explicit integrals for NCQFT with $e(x) = x$
- matches perfectly the expansion into ribbon graphs, although expansion neither convergent nor Borel summable

Higher correlation functions

…satisfy linear integral equations, easily reduced to $(1+\ldots+1)$:

$$G_{|a^1_1\ldots a^1_{N_1}|\ldots|a^B_1\ldots a^B_{N_B}|} = \lambda^{N_1+\cdots+N_B-B} \sum_{k_1=1}^{N_1} \cdots \sum_{k_B=1}^{N_B} G_{|a^1_{k_1}|\ldots|a^B_{k_B}|} \prod_{\beta=1}^{B} \prod_{\substack{l_\beta=1 \\ l_\beta\neq k_\beta}}^{N_\beta} \frac{1}{F^2_{a^\beta_{k_\beta}} - F^2_{a^\beta_{l_\beta}}}$$

$W_{|a_k|}$ if $B=1$

F_a = renormalisation of E_a

Higher correlation functions

...satisfy linear integral equations, easily reduced to $(1+\ldots+1)$:

$$G_{|a^1_1\ldots a^1_{N_1}|\ldots|a^B_1\ldots a^B_{N_B}|} = \lambda^{N_1+\cdots+N_B-B}\sum_{k_1=1}^{N_1}\cdots\sum_{k_B=1}^{N_B} G_{|a^1_{k_1}|\ldots|a^B_{k_B}|}\prod_{\beta=1}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{1}{F^2_{a^\beta_{k_\beta}}-F^2_{a^\beta_{l_\beta}}}$$

Theorem [Grosse-Sako-W 2016]

$$G(X|Y) = \frac{4\lambda^2}{\sqrt{X+c}\cdot\sqrt{Y+c}\cdot(\sqrt{X+c}+\sqrt{Y+c})^2}$$

$$G(X^1|\ldots|X^B) = \frac{d^{B-3}}{dt^{B-3}}\Big(\frac{(-2\lambda)^{3B-4}}{(R(t))^{B-2}}\frac{1}{\sqrt{X^1+c-2t}^3}\cdots\frac{1}{\sqrt{X^B+c-2t}^3}\Big)\Big|_{t=0}$$

$$R(T) = \lim_{\tilde\Lambda\to\infty}\Big(\frac{1}{\sqrt{Z(\tilde\Lambda)}}-\int_1^{\tilde\Lambda}\frac{dT\rho(T)}{\sqrt{T+c}}\frac{1}{(\sqrt{T+c}+\sqrt{T+c-2t})\sqrt{T+c-2t}}\Big)$$

Proof: ansatz for recursion and experience with Bell polynomials

Schwinger functions

undo the passage to the f_{mn}-matrix basis of Moyal space:

Theorem [Grosse-W 2013]: *connected* Schwinger functions

$$S^c_N(\mu\xi_1,\dots,\mu\xi_N)$$
$$:= \lim_{V\mu^2\to\infty} \sum_{m_i,n_i=0}^{\infty} f_{m_1n_1}(\xi_1)\cdots f_{m_Nn_N}(\xi_N) \frac{(V\mu^2)^{-2}\mu^{3N}\partial^N \log \mathcal{Z}(J)}{\partial J_{m_1n_1}\dots\partial J_{m_Nn_N}}\Big|_{J=0}$$
$$= \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \Big(\prod_{\beta=1}^{B} \frac{2^{\frac{DN_\beta}{2}}}{N_\beta}\int_{\mathbb{R}^D} \frac{dp_\beta}{(2\pi\mu^2)^{\frac{D}{2}}} e^{\mathrm{i}\langle p_\beta,\sum_{i=1}^{N_\beta}(-1)^{i-1}\xi_{\sigma(N_1+\dots+N_{\beta-1}+i)}\rangle}\Big)$$
$$\times \frac{1}{(8\pi)^{\frac{D}{2}} S_{N_1\dots N_B}} \tilde{G}\Big(\underbrace{\frac{\|p_1\|^2}{2\mu^2},\dots,\frac{\|p_1\|^2}{2\mu^2}}_{N_1}\Big|\dots\Big|\underbrace{\frac{\|p_B\|^2}{2\mu^2},\dots,\frac{\|p_B\|^2}{2\mu^2}}_{N_B}\Big)$$

Confinement of noncommutativity: have internal interaction of matrices; commutative subsector propagates to outside world

- Schwinger functions are symmetric and invariant under full Euclidean group (completely unexpected for NCQFT!)
- remains: reflection positivity (...and non-triviality)

Reflection positivity $S(\bar{f}^r \otimes f) \geq 0$

- f stands for sequences of test functions of complicated support
- $f_1^r(\tau, \vec{\xi}) = f_1(-\tau, \vec{\xi})$ is time reflection

Implies for very special f:
The temporal Fourier transform of $\tilde{S}$ (in all independent energies) is, for any spatial momenta, a positive definite function.

Theorem [Hausdorff-Bernstein-Widder, 1921-1912/28-1941]

For a [smooth] function F on $(\mathbb{R}_+)^N \ni t = (t^1, \dots, t^N)$ are equivalent:

1. F is positive definite, i.e. $\sum_{i,j=1}^K \overline{c_i} c_j F(t_i + t_j) \geq 0$
2. F is the joint Laplace transform of a positive measure*
3. F is completely monotonic, $(-1)^{k_1+\dots+k_N} \partial_{t^1}^{k_1} \dots \partial_{t^N}^{k_N} F(t) \geq 0$

*This is 60% of the proof of the Osterwalder-Schrader theorem.

Stieltjes functions

Prototype for $N = 1$

$$\int_{-\infty}^{\infty} \frac{e^{ip^0 t}}{(p^0)+\vec{p}^2+m^2} = \Big(\frac{2\pi t}{\sqrt{\vec{p}^2+m^2}}\Big)^{\frac{1}{2}} K_{\frac{1}{2}}\big(t\sqrt{\vec{p}^2+m^2}\big) = \frac{\pi e^{-t\sqrt{\vec{p}^2+m^2}}}{\sqrt{\vec{p}^2+m^2}}$$

Theorem

Up to integration in m^2 with positive measure, $\frac{1}{(p^0)+\vec{p}^2+m^2}$ is the only function with positive definite Fourier transform for $N = 1$.

- $p^2 \mapsto \int_0^\infty \frac{\varrho(m^2)dm^2}{p^2+m^2}$ forms the class of Stieltjes functions
- in QFT, $\varrho(m^2)$ is the Källén-Lehmann spectral measure

Is $\tilde{G}(\frac{\|p\|^2}{2\mu^2}, \frac{\|p\|^2}{2\mu^2})$ Stieltjes?

- We work on this for Φ^4_4 since 2013. Have some analytic evidence, confirmed by computer, but no complete proof.
- For Φ^3_D we have the answer:

Reflection positivity of the 2-point function

Theorem (Grosse-Sako-W 2016)

1. The Φ^3_D-matricial QFT is not reflection positive for $\lambda \in \mathrm{i}\mathbb{R}$.
2. The Φ^3_D two-point function is reflection positive for $D \in \{4,6\}$ and some range of $\lambda \in \mathbb{R}$, but not in $D = 2$.

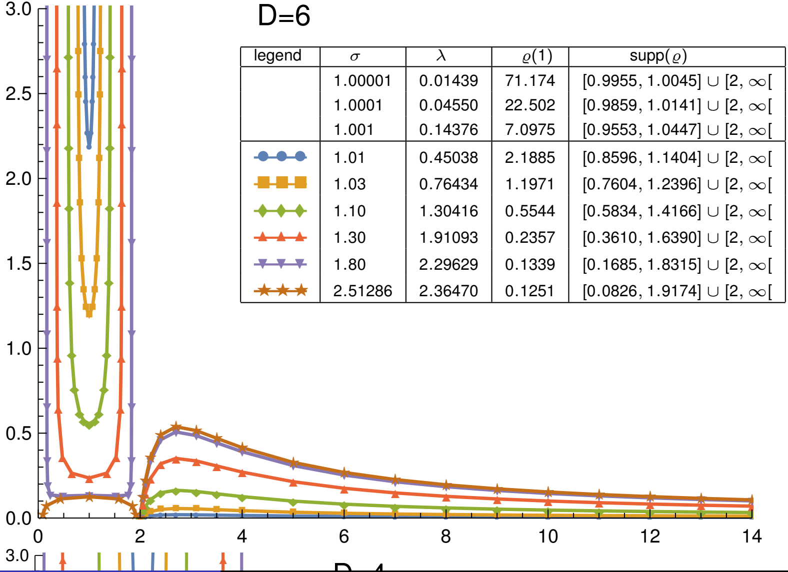
measure supported on fuzzy mass shell plus scattering part:

$$\tilde{G}\Big(\frac{\|p\|^2}{2\mu^2},\frac{\|p\|^2}{2\mu^2}\Big) \overset{6D}{=} \frac{\lambda^2}{4\pi(\sigma^2-1)}\int_0^\pi d\phi \, \frac{\begin{array}{c}\{2\frac{\log(1+\sigma)}{\sigma}-1+\sigma(\sigma-1)\tan^2\phi \\ -\tan\phi(1+\sigma^2\tan^2\phi)\,(\arctan_{[0,\pi]}(\sigma\tan\phi)-\phi)\}\end{array}}{1-\frac{\sqrt{\sigma^2-1}}{\sigma}\cos\phi+\frac{\|p\|^2}{\mu^2}}$$

$$+\frac{\lambda^2}{4}\int_2^\infty dt\, \frac{t(t-2)/(t-1)^3}{t+\frac{\|p\|^2}{\mu^2}},$$

where $\sigma := \frac{1}{\sqrt{1+c}} \in [1, -2W_{-1}(-\frac{1}{2\sqrt{e}})-1]$ is the inverse solution of $\lambda^2 = \frac{4(\sigma^2-1)}{\sigma^2-2\sigma+2\log(1+\sigma)} \in [1, \frac{8W_{-1}(-\frac{1}{2\sqrt{e}})}{1+2W_{-1}(-\frac{1}{2\sqrt{e}})}]$

Källén-Lehmann measure: plots



legend	σ	λ	$\varrho(1)$	$\mathrm{supp}(\varrho)$
	1.00001	0.01439	71.174	$[0.9955, 1.0045] \cup [2, \infty[$
	1.0001	0.04550	22.502	$[0.9859, 1.0141] \cup [2, \infty[$
	1.001	0.14376	7.0975	$[0.9553, 1.0447] \cup [2, \infty[$
●—●—●	1.01	0.45038	2.1885	$[0.8596, 1.1404] \cup [2, \infty[$
■—■—■	1.03	0.76434	1.1971	$[0.7604, 1.2396] \cup [2, \infty[$
◆—◆—◆	1.10	1.30416	0.5544	$[0.5834, 1.4166] \cup [2, \infty[$
▲—▲—▲	1.30	1.91093	0.2357	$[0.3610, 1.6390] \cup [2, \infty[$
▼—▼—▼	1.80	2.29629	0.1339	$[0.1685, 1.8315] \cup [2, \infty[$
★—★—★	2.51286	2.36470	0.1251	$[0.0826, 1.9174] \cup [2, \infty[$

Reflection positivity of higher Schwinger functions?

- Connected Schwinger functions $S^c_{N\geq 4}$ are not positive!
- Anyway too much, one needs positivity of FT of full functions

e.g. $$\tilde{G}(\tfrac{\|p\|^2}{2\mu^2},\tfrac{\|p\|^2}{2\mu^2})\tilde{G}(\tfrac{\|q\|^2}{2\mu^2},\tfrac{\|q\|^2}{2\mu^2}) + \tilde{G}(\tfrac{\|p\|^2}{2\mu^2},\tfrac{\|p\|^2}{2\mu^2}|\tfrac{\|q\|^2}{2\mu^2},\tfrac{\|q\|^2}{2\mu^2})$$

- Difficult for $N=4$,
  but $G(2|2|2)+G(2)G(2)G(2)$ is not positive.

Very probable conclusion

The Φ^3_D matricial QFT does not satisfy Osterwalder-Schrader.

- Reason: Higher functions too much localised in p-space!
  already $\tilde{G}(\frac{\|p\|^2}{2\mu^2},\frac{\|p\|^2}{2\mu^2}) \propto \frac{C_1\log(\|p\|^2+\mu^2)+C_2}{\|p\|^2+\mu^2}$ almost fails
- For Φ^4_4 we expect $\tilde{G}(\frac{\|p\|^2}{2\mu^2},\frac{\|p\|^2}{2\mu^2}) \propto \frac{C}{(\|p\|^2+\mu^2)^{1-\frac{1}{\pi}\arcsin(|\lambda|\pi)}}$ (hope!)
- Keeps us busy for the next time!