Matricial quantum field theory: renormalisation, integrability & positivity

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based on arXiv:1610.00526 & 1612.07584 with Harald Grosse and Akifumi Sako

and arXiv: 1205.0465, 1306.2816, 1402.1041, 1406.7755 & 1505.05161
with Harald Grosse

Goal: Quantum Field Theory satisfying axioms

- 1932: axioms for quantum mechanics [von Neumann]
- 1950's: unique extension to quantum fields [Wightman]
  = unbounded op.-valued distributions $f \mapsto \Phi(f) : \mathcal{D} \to \mathcal{D} \subset \mathcal{H}$

Theorem: vacuum expectation values $\langle \Omega, \Phi(x_1) \cdots \Phi(x_N)\Omega \rangle$ are boundary values of holomorphic functions

- their restriction to real subspace of Euclidean points (minus diagonals) defines Schwinger functions
- Schwinger functions inherit real analyticity, Euclidean invariance, complete symmetry and reflection positivity

Theorem [Osterwalder-Schrader 1974]

These properties are sufficient to reconstruct Wightman theory!

So far no non-trivial QFT model in 4 dimensions ...

Selected techniques

- exactly solvable 2D-models (e.g. Thirring, Schwinger)
- candidate Schwinger functions as moments of perturbed Gaußian measure (e.g. $P[\phi]_2$, ϕ^4_3, probably not ϕ^4_4)
- fermionic summation techniques (e.g. Gross-Neveu$_2$)
- for all realistic models (e.g. QED$_4$, Standard Model$_4$): renormalised perturbation theory – BPHZ(L)

Z= Wolfhart Zimmermann

BPHZ(L) has two aspects:

1. Renormalisation amounts to normalisation conditions for relevant/marginal correlation functions. These conditions are of non-perturbative nature.
2. When restricted to graphs, these conditions boil down to momentum space Taylor subtraction and forest formula.

Matricial quantum field theory

. . . is the marriage of

1. matrix models for 2D quantum gravity
2. QFT on noncommutative spaces

1. Kontsevich model (1992)
   designed to prove Witten's conjecture that hermitean one-matrix model computes intersection numbers of stable cohomology classes on the moduli space of complex curves
2. Space-time should become a noncommutative manifold at short distances.
   - Euclidean scalar field $\phi \in \mathcal{A}$ (noncommutative algebra)
   - $\mathcal{A}$ often has finite-dimensional approximations

The Kontsevich model

defined by partition function

$$\mathcal{Z}(E) := \frac{\displaystyle\int d\Phi \;\exp\Big(-\mathrm{Tr}\big(E\Phi^2 + \tfrac{\mathrm{i}}{6}\Phi^3\big)\Big)}{\displaystyle\int d\Phi \;\exp\Big(-\mathrm{Tr}\big(E\Phi^2\big)\Big)}$$

- Asymptotic expansion in 'coupling constant' $\frac{\mathrm{i}}{6}$ gives rational function of eigenvalues e_i of E. This rational function generates the intersection numbers.
- Related to Hermitean one-matrix model
$$\mathcal{Z}(E)[[t_n]] = \int DM \;\exp(-\mathcal{N}\sum_n t_n \,\mathrm{tr}(M^n))$$
where $t_n := (2n-1)!!\mathrm{tr}(E^{-(2n-1)})$
- Large-$\mathcal{N}$ limit gives KdV evolution equation. Exact solution related to Virasoro algebra.

QFT on noncommutative geometries

Example: Moyal algebra = Rieffel deformation of $C^\infty(\mathbb{R}^2)$

$$(f\star g)(\xi) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{d\eta\, dk}{(2\pi)^2} f(x+\tfrac{1}{2}\Theta k)\, g(\xi+\eta)\, e^{\mathrm{i}\langle k,\eta\rangle} \qquad \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

- matrix basis $\phi(\xi) = \sum_{m,n=0}^{\infty} \Phi_{mn} f_{mn}(\xi)$

$$f_{mn}(\xi) = 2(-1)^m \sqrt{\frac{m!}{n!}} \Big(\sqrt{\tfrac{2}{\theta}}\xi_1 + \mathrm{i}\xi_2\Big)^{n-m} L_m^{n-m}\Big(\frac{2\|\xi\|^2}{\theta}\Big) e^{-\frac{\|\xi\|^2}{\theta}}$$

- satisies $f_{mn}\star f_{kl} = \delta_{nk} f_{ml}$ and $\int \frac{d\xi}{8\pi} f_{mn}(\xi) = \frac{\theta}{4}\delta_{mn}$

- Consider scalar field theories on Moyal space

$$S(\phi) := \frac{1}{(8\pi)^{D/2}} \int_{\mathbb{R}^D} d\xi \Big(\frac{1}{2}\phi\star(-\Delta + 4\Omega^2\|\Theta^{-1}\xi\|^2)\star\phi + \mathrm{tr}(\mathrm{pol}(\phi))\Big)$$

- f_{mn}-expansion at $\Omega = 1$ yields Kontsevich-type matrix model

$$S(\Phi) = V\,\mathrm{tr}(E\Phi^2 + \mathrm{pol}(\Phi)), \qquad E = \big((\tfrac{\mu^2}{2} + \tfrac{n}{V^{\frac{2}{D}}})\delta_{mn}\big), \quad V = (\tfrac{\theta}{4})^{D/2}$$

Two independent dimensions

1. Topological dimension 2 from expansion of matrix models into ribbon graphs, i.e. simplicial 2-complexes.
   - dual to triangulations (Φ^3) or quadrangulations (Φ^4) of 2D-surfaces
   - partition function counts them = 2D quantum gravity
   - non-planar ribbon graphs suppressed in large-$\mathcal{N}$ limit
2. Dynamical dimension D encoded in spectrum of the unbounded positive operator E,
   $$D = \inf\{p \in \mathbb{R}_+ \;:\; \mathrm{tr}((1+E)^{-\frac{p}{2}}) < \infty\}$$
   - ignored in 2D quantum gravity
   - highly relevant for renormalisation of matricial QFT

polynomial	finite	super-ren	just ren.	not ren.
Φ^3	$D < 2$	$2[\frac{D}{2}] \in \{2,4\}$	$2[\frac{D}{2}] = 6$	$2[\frac{D}{2}] > 6$
Φ^4	$D < 2$	$2[\frac{D}{2}] = 2$	$2[\frac{D}{2}] = 4$	$2[\frac{D}{2}] > 4$

Φ^3_6 matricial QFT

- action $S(\Phi) = V\,\mathrm{tr}(ZE\Phi^2 + (\kappa+\nu E+\zeta E^2)\Phi + \frac{\lambda_{bare}Z^{\frac{3}{2}}}{3}\Phi^3)$
  for $E = \Big(\frac{\mu^2_{bare}}{2} + \mu^2 e\big(\frac{|n|}{\mu^2 V^{2/D}}\big)\delta_{mn}\Big), \quad m,n \in \mathbb{N}^{D/2}$
- $\mu_{bare}, \lambda_{bare}, Z, \kappa, \nu, \zeta$ to be fixed by normalisation conditions
- partition function $\mathcal{Z}(J) = \int d\Phi\ \exp(-S(\Phi) + V\,\mathrm{tr}(\Phi J))$

$$\log\frac{\mathcal{Z}(J)}{\mathcal{Z}(0)} = \sum_{B=1}^{\infty}\sum_{N_B\geq\cdots\geq N_1\geq 1}\frac{V^{2-B}}{S_{N_1\dots N_B}} G_{|p^1_1\dots p^1_{N_1}|\dots|p^B_1\dots p^B_{N_B}|}\prod_{\beta=1}^{B}\Big(\prod_{j_\beta=1}^{N_\beta} J_{p^\beta_{j_\beta}p^\beta_{j_\beta+1}}\Big)_{cycl}$$

Strategy

- $\mathcal{Z}(J)$ is meaningless for $\lambda \in \mathbb{R}$!
- $\mathcal{Z}(J)$ is only used as tool to derive identities (Schwinger-Dyson equations) between $G_{|p^1_1\dots p^1_{N_1}|\dots|p^B_1\dots p^B_{N_B}|}$
- Forget $\mathcal{Z}$, declare SD-equations as exact and search for rigorous solutions $G_{\dots}$ of them!

Schwinger-Dyson equations

Inserting $\mathcal{Z}(J) = \exp\Big(-\frac{Z^{3/2}\lambda_{bare}}{3V^2}\sum \frac{\partial^3}{\partial J_{kl}\partial J_{lm}\partial J_{mk}}\Big)\mathcal{Z}_{\leq 2}(J)$ into $G_{|a|} \equiv \frac{1}{V}\frac{\partial \log \mathcal{Z}[J]}{\partial J_{aa}}\Big|_{J=0}$ gives equation quadratic in $G_{|a|}$, linear in $\sum_m G_{|am|}$ and $G_{|a|a|}$

- typical feature: SD-equation for n-point function depends on $(m > n)$-point function
- Here we are rescued:
  1. $G_{|a|a|}$ comes with $\frac{1}{V^2}$, goes away in limit $V^{2/D} \sim \theta \to \infty$
  2. $G_{|am|}$ expressable in terms of $G_{|a|}, G_{|m|}$ thanks to Ward-Takahashi identity for $U(\infty)$-group action:

Theorem (Disertori-Gurau-Magnen-Rivasseau 2006)

$$\sum_n \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{bn}\partial J_{na}} = \sum_n \frac{V}{Z(E_a - E_b)}\Big(J_{an}\frac{\partial}{\partial J_{bn}} - J_{nb}\frac{\partial}{\partial J_{na}}\Big)\mathcal{Z}[J] - \frac{V}{Z}(\nu + \zeta(E_a + E_b))\frac{\partial \mathcal{Z}[J]}{\partial J_{ba}} \qquad \text{(for } a \neq b\text{)}$$

Scaling limit $\mathcal{N}, V \to \infty$ with $\frac{\mathcal{N}}{V^{2/D}} = \mu^2\Lambda^2$ fixed

Non-linear integral equation for $\tilde{G}(x) = \mu^{1-D/2} G_{|a|}\big|_{|a|=V^{2/D}\mu^2 x}$ similar to equation from Virasoro constraint in Kontsevich model:

Theorem [Makeenko-Semenoff 1991]

$$W^2(X) + \int_a^b dY \rho(Y) \frac{W(X)-W(Y)}{X-Y} = X + \text{const}$$

is solved by $W(X) = \sqrt{X+c} + \frac{1}{2}\int_a^b \frac{dY\ \rho(Y)}{(\sqrt{X+c}+\sqrt{Y+c})\sqrt{Y+c}}$

together with a consistency condition on c.

Identification $X = (2e(x)+1)^2$, $\rho(Y) = \dfrac{2\lambda^2 (e^{-1}(\frac{\sqrt{Y}-1}{2}))^{D/2-1}}{\Gamma(D/2)\sqrt{Y} e'(e^{-1}(\frac{\sqrt{Y}-1}{2}))}$

Ansatz for $\tilde{G}(x) =: \frac{1}{2\lambda}(W(X) - \sqrt{X})$

$$W(X) = \frac{\sqrt{X+c}}{\sqrt{Z}} - \nu + \frac{1}{2}\int_a^b \frac{dY\ \rho(Y)}{(\sqrt{X+c}+\sqrt{Y+c})\sqrt{Y+c}}$$

normalisation conditions on $\tilde{G}_{\dots}$ translate to

$$\underbrace{W(1)=1}_{D\geq 2}, \quad \underbrace{W'(1) = \frac{d}{dX}\sqrt{X}\Big|_{X=1} = \frac{1}{2}}_{D\geq 4}, \quad \underbrace{W''(1) = \frac{d^2}{dX^2}\sqrt{X}\Big|_{X=1} = -\frac{1}{4}}_{D=6}$$

Solution of renormalised equation for $D = 6$

$$\frac{1}{\sqrt{Z[\tilde{\Lambda}]}} = \sqrt{1+c} + \frac{1}{2}\int_1^{\tilde{\Lambda}} dT \frac{\rho(T)}{(\sqrt{1+c}+\sqrt{T+c})^2\sqrt{T+c}}$$

$$-c = \int_1^{\infty} \frac{dT\ \rho(T)}{(\sqrt{1+c}+\sqrt{T+c})^3\sqrt{T+c}}$$

$\Rightarrow$ $Z \in [0,1]$ for $\lambda \in \mathbb{R}$ (see LSZ)

$$W(X) = \sqrt{(X+c)(1+c)} - c + \frac{1}{2}\int_1^{\infty} \frac{dT\ \rho(T)\ (\sqrt{X+c}-\sqrt{1+c})^2}{(\sqrt{X+c}+\sqrt{T+c})(\sqrt{1+c}+\sqrt{T+c})^2\sqrt{T+c}}$$

$$\beta_\lambda := \Lambda^2 \frac{d\lambda_{bare}(\tilde{\Lambda}(\Lambda))}{d\Lambda^2} = \frac{2\lambda^3\Lambda^6}{\left(\sqrt{1+c}+\sqrt{(2e(\Lambda^2)+1)^2+c}\right)^2\sqrt{(2e(\Lambda^2)+1)^2+c}} > 0$$

Perturbative expansion for $e(x) = x$, $\rho(T) = \frac{\lambda^2(\sqrt{T}-1)^2}{4\sqrt{T}}$

$$c = -\frac{2\log 2 - 1}{4}\lambda^2 + \frac{(2\log 2-1)(4\log 2-3)}{32}\lambda^4 + \mathcal{O}(\lambda^6)$$

$$\tilde{G}(x) = \frac{\lambda}{4(2x+1)}\left(2(1+x)^2\log(1+x) - x(2+3x)\right) + \frac{\lambda^3}{16(2x+1)^3}\left(x^3(2+3x)(2\log 2-1)^2\right) + \mathcal{O}(\lambda^5)$$

Higher correlation functions

…satisfy linear integral equations, easily reduced to $(1+\ldots+1)$:

$$G_{|a^1_1\ldots a^1_{N_1}|\ldots|a^B_1\ldots a^B_{N_B}|} = \lambda^{N_1+\cdots+N_B-B} \sum_{k_1=1}^{N_1}\cdots\sum_{k_B=1}^{N_B} G_{|a^1_{k_1}|\ldots|a^B_{k_B}|} \prod_{\beta=1}^{B} \prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta} \frac{1}{F^2_{a^\beta_{k_\beta}} - F^2_{a^\beta_{l_\beta}}}$$

$W_{|a_k|}$ if $B=1$

F_a = renormalisation of E_a

Higher correlation functions

…satisfy linear integral equations, easily reduced to $(1+\ldots+1)$:

$$G_{|a^1_1\ldots a^1_{N_1}|\ldots|a^B_1\ldots a^B_{N_B}|} = \lambda^{N_1+\cdots+N_B-B}\sum_{k_1=1}^{N_1}\cdots\sum_{k_B=1}^{N_B} G_{|a^1_{k_1}|\ldots|a^B_{k_B}|}\prod_{\beta=1}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{1}{F^2_{a^\beta_{k_\beta}}-F^2_{a^\beta_{l_\beta}}}$$

Proposition

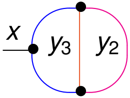
$$G(X|Y)=\frac{4\lambda^2}{\sqrt{X+c}\cdot\sqrt{Y+c}\cdot(\sqrt{X+c}+\sqrt{Y+c})^2}$$

$$G(X^1|\ldots|X^B)=\frac{d^{B-3}}{dt^{B-3}}\Big(\frac{(-2\lambda)^{3B-4}}{(R(t))^{B-2}}\frac{1}{\sqrt{X^1+c-2t}^3}\cdots\frac{1}{\sqrt{X^B+c-2t}^3}\Big)\Big|_{t=0}$$

$$R(T)=\lim_{\tilde\Lambda\to\infty}\Big(\frac{1}{\sqrt{Z(\tilde\lambda)}}-\int_1^{\tilde\Lambda}\frac{dT\rho(T)}{\sqrt{T+c}}\frac{1}{(\sqrt{T+c}+\sqrt{T+c-2t})\sqrt{T+c-2t}}\Big)$$

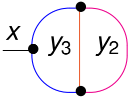
Proof: ansatz for recursion and experience with Bell polynomials

Simplest 6D-ribbon graph with overlapping divergence



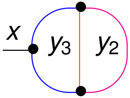
$$= \frac{(-\lambda)^3}{(2x+1)} \int_0^\infty \frac{y_3^2 dy_3}{2} \int_0^\infty \frac{y_2^2 dy_2}{2} \left\{ \frac{1}{(x+y_3+1)^2(y_3+y_2+1)(x+y_2+1)} \right\}$$

Zimmermann's forest formula



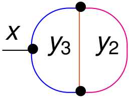
$$
\begin{aligned}
&= \frac{(-\lambda)^3}{(2x+1)} \int_0^\infty \frac{y_3^2\,dy_3}{2} \int_0^\infty \frac{y_2^2\,dy_2}{2} \Bigg\{ \Bigg[\frac{1}{(x+y_3+1)^2(y_3+y_2+1)(x+y_2+1)} \Bigg]_\emptyset \\
&\quad + \Bigg[\Big(-\frac{1}{(y_3+1)^3} \Big) \frac{1}{x+y_2+1} \Bigg]_3 + \Bigg[\frac{1}{(x+y_3+1)^2} \Big(-\frac{1}{(y_2+1)^2} + \frac{y_3+x}{(y_2+1)^3} \Big) \Bigg]_2 \\
&\quad + \Bigg[\frac{1}{(y_3+y_2+1)} \Big(-\frac{1}{(y_3+1)^2(y_2+1)} + \frac{2x}{(y_3+1)^3(y_2+1)} + \frac{x}{(y_3+1)^2(y_2+1)^2} \\
&\qquad - \frac{3x^2}{(y_3+1)^4(y_2+1)} - \frac{x^2}{(y_3+1)^2(y_2+1)^3} - \frac{2x^2}{(y_3+1)^3(y_2+1)^2} \Big) \Bigg]_1 \\
&\quad + \Bigg[\Big(-\frac{1}{(y_3+1)^3} \Big) \Big(-\frac{1}{y_2+1} + \frac{x}{(y_2+1)^2} - \frac{x^2}{(y_2+1)^3} \Big) \Bigg]_{13} \\
&\quad + \Bigg[\Big(\Big(-\frac{1}{(y_3+1)^2} + \frac{2x}{(y_3+1)^3} - \frac{3x^2}{(y_3+1)^4} \Big) \Big(-\frac{1}{(y_2+1)^2} + \frac{y_3}{(y_2+1)^3} \Big) \\
&\qquad + \Big(-\frac{1}{(y_3+1)^2} + \frac{2x}{(y_3+1)^3} \Big) \Big(\frac{x}{(y_2+1)^3} \Big) \Big) \Bigg]_{12} \Bigg\}
\end{aligned}
$$

Zimmermann's forest formula

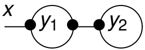


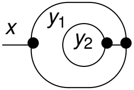
$$
= \frac{-\lambda^3}{4(2x+1)^3}\Big\{(x+1)(2x+1)(3x+2)\log(1+x) + (x+1)^3(3x+1)(\log(1+x))^2
+ x(1+x)(1+3x+3x^2)\Big((\log(1+x))^2 - 2\log(1+x)\log x + 2\mathrm{Li}_2\Big(\frac{1}{1+x}\Big)\Big)
- 3x^3(2+3x)\zeta(2)\Big\} + \frac{\lambda^3 x}{2(2x+1)}\Big(\zeta(2) + 1 - \frac{x}{2}\Big)
$$

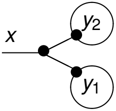
Zimmermann's forest formula



$$= \frac{-\lambda^3}{4(2x+1)^3}\Big\{(x+1)(2x+1)(3x+2)\log(1+x) + (x+1)^3(3x+1)(\log(1+x))^2$$
$$+ x(1+x)(1+3x+3x^2)\Big((\log(1+x))^2 - 2\log(1+x)\log x + 2\mathrm{Li}_2\Big(\frac{1}{1+x}\Big)\Big)$$
$$- 3x^3(2+3x)\zeta(2)\Big\} + \frac{\lambda^3 x}{2(2x+1)}\Big(\zeta(2) + 1 - \frac{x}{2}\Big)$$

adding: 





gives the λ^3-order of the exact formula for $\tilde{G}(x)$!

Schwinger functions

undo the passage to the f_{mn}-matrix basis of Moyal space:

Theorem [HG+RW, 2013]: *connected* **Schwinger functions**

$$S^c_N(\mu\xi_1,\dots,\mu\xi_N) := \lim_{V\mu^2\to\infty} \sum_{m_i,n_i=0}^{\infty} f_{m_1n_1}(\xi_1)\cdots f_{m_Nn_N}(\xi_N)\, \frac{(V\mu^2)^{-2}\mu^{3N}\partial^N \log \mathcal{Z}(J)}{\partial J_{m_1n_1}\cdots\partial J_{m_Nn_N}}\Big|_{J=0}$$

$$= \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta\ \text{even}}} \sum_{\sigma\in\mathcal{S}_N} \Big(\prod_{\beta=1}^{B} \frac{2^{\frac{DN_\beta}{2}}}{N_\beta} \int_{\mathbb{R}^D} \frac{dp_\beta}{(2\pi\mu^2)^{\frac{D}{2}}}\, e^{\mathrm{i}\left\langle p_\beta, \sum_{i=1}^{N_\beta}(-1)^{i-1}\xi_{\sigma(N_1+\dots+N_{\beta-1}+i)}\right\rangle} \Big)$$

$$\times \frac{1}{(8\pi)^{\frac{D}{2}} S_{N_1\dots N_B}}\, \tilde{G}\Big(\underbrace{\frac{\|p_1\|^2}{2\mu^2},\dots,\frac{\|p_1\|^2}{2\mu^2}}_{N_1}\Big|\dots\Big|\underbrace{\frac{\|p_B\|^2}{2\mu^2},\dots,\frac{\|p_B\|^2}{2\mu^2}}_{N_B}\Big)$$

Confinement of noncommutativity: have internal interaction of matrices; commutative subsector propagates to outside world

- Schwinger functions are symmetric and invariant under full Euclidean group (completely unexpected for NCQFT!)
- remains: reflection positivity (…and non-triviality)

Reflection positivity $S(\bar{f}^r \otimes f) \geq 0$

- f stands for sequences of test functions of complicated support
- $f_1^r(\tau,\vec{\xi}) = f_1(-\tau,\vec{\xi})$ is time reflection

Implies for very special f:
The temporal Fourier transform of $\tilde{S}$ (in all independent energies) is, for any spatial momenta, a positive definite function.

Theorem (Hausdorff-Bernstein-Widder, 1921-1912/28-1941)

For a [smooth] function F on $(\mathbb{R}_+)^N \ni t = (t^1,\dots,t^N)$ are equivalent:

1. F is positive definite, i.e. $\sum_{i,j=1}^K \overline{c_i} c_j F(t_i+t_j) \geq 0$
2. F is the joint Laplace transform of a positive measure*
3. F is completely monotonic, $(-1)^{k_1+\dots+k_N}\partial_{t^1}^{k_1}\dots\partial_{t^N}^{k_N} F(t) \geq 0$

*This is 60% of the proof of the Osterwalder-Schrader theorem.

Stieltjes functions

Prototype for $N = 1$

$$\int_{-\infty}^{\infty} \frac{e^{ip^0 t}}{(p^0)+\vec{p}^2+m^2} = \Big(\frac{2\pi t}{\sqrt{\vec{p}^2+m^2}}\Big)^{\frac{1}{2}} K_{\frac{1}{2}}\big(t\sqrt{\vec{p}^2+m^2}\big) = \frac{\pi e^{-t\sqrt{\vec{p}^2+m^2}}}{\sqrt{\vec{p}^2+m^2}}$$

Theorem

Up to integration in m^2 with positive measure, $\frac{1}{(p^0)+\vec{p}^2+m^2}$ is the only function with positive definite Fourier transform for $N = 1$.

- $p^2 \mapsto \int_0^\infty \frac{\varrho(m^2)dm^2}{p^2+m^2}$ forms the class of Stieltjes functions
- in QFT, $\varrho(m^2)$ is the Källén-Lehmann spectral measure

Is $\tilde{G}(\frac{\|p\|^2}{2\mu^2}, \frac{\|p\|^2}{2\mu^2})$ Stieltjes?

- We work on this for Φ_4^4 since 2013. Have some analytic evidence, confirmed by computer, but no complete proof.
- For Φ_D^3 we have the answer:

Reflection positivity of the 2-point function

Theorem (Grosse-Sako-W 2016)

1. The Φ^3_D-matricial QFT is not reflection positive for $\lambda \in \mathrm{i}\mathbb{R}$.
2. The Φ^3_D two-point function is reflection positive for $D \in \{4, 6\}$ and some range of $\lambda \in \mathbb{R}$, but not in $D = 2$.

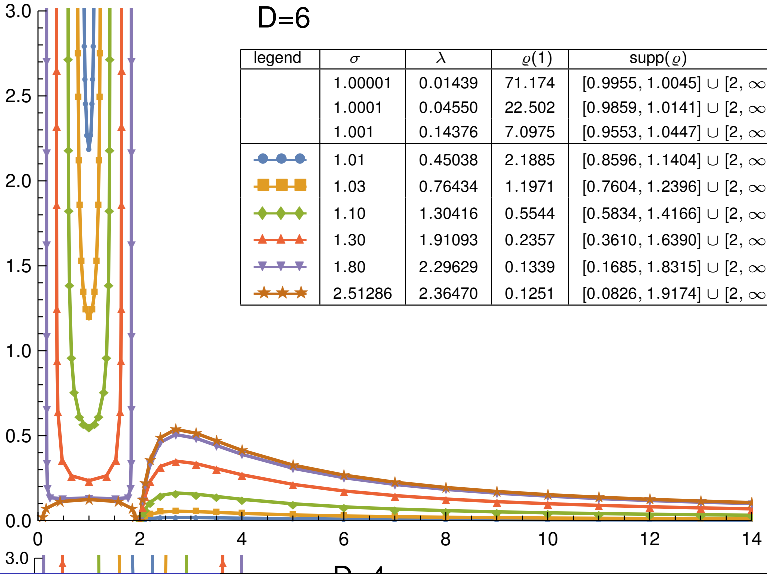
measure supported on fuzzy mass shell plus scattering part:

$$\tilde{G}\Big(\frac{\|p\|^2}{2\mu^2}, \frac{\|p\|^2}{2\mu^2}\Big) \overset{6D}{=} \frac{\lambda^2}{4\pi(\sigma^2-1)} \int_0^\pi d\phi \, \frac{\begin{array}{c}\{2\frac{\log(1+\sigma)}{\sigma} - 1 + \sigma(\sigma-1)\tan^2\phi \\ - \tan\phi(1+\sigma^2\tan^2\phi)\big(\arctan_{[0,\pi]}(\sigma\tan\phi) - \phi\big)\}\end{array}}{1 - \frac{\sqrt{\sigma^2-1}}{\sigma}\cos\phi + \frac{\|p\|^2}{\mu^2}}$$

$$+ \frac{\lambda^2}{4}\int_2^\infty dt \, \frac{t(t-2)/(t-1)^3}{t + \frac{\|p\|^2}{\mu^2}},$$

where $\sigma := \frac{1}{\sqrt{1+c}} \in [1, -2W_{-1}(-\frac{1}{2\sqrt{e}}) - 1]$ is the inverse solution of $\lambda^2 = \frac{4(\sigma^2-1)}{\sigma^2 - 2\sigma + 2\log(1+\sigma)} \in [1, \frac{8W_{-1}(-\frac{1}{2\sqrt{e}})}{1+2W_{-1}(-\frac{1}{2\sqrt{e}})}]$

Källén-Lehmann measure: plots



legend	σ	λ	$\varrho(1)$	$\mathrm{supp}(\varrho)$
	1.00001	0.01439	71.174	$[0.9955, 1.0045] \cup [2, \infty[$
	1.0001	0.04550	22.502	$[0.9859, 1.0141] \cup [2, \infty[$
	1.001	0.14376	7.0975	$[0.9553, 1.0447] \cup [2, \infty[$
●—●—●	1.01	0.45038	2.1885	$[0.8596, 1.1404] \cup [2, \infty[$
■—■—■	1.03	0.76434	1.1971	$[0.7604, 1.2396] \cup [2, \infty[$
◆—◆—◆	1.10	1.30416	0.5544	$[0.5834, 1.4166] \cup [2, \infty[$
▲—▲—▲	1.30	1.91093	0.2357	$[0.3610, 1.6390] \cup [2, \infty[$
▼—▼—▼	1.80	2.29629	0.1339	$[0.1685, 1.8315] \cup [2, \infty[$
★—★—★	2.51286	2.36470	0.1251	$[0.0826, 1.9174] \cup [2, \infty[$

Reflection positivity of higher Schwinger functions?

- Connected Schwinger functions $S^c_{N\geq 4}$ are not positive!
- Anyway too much, one needs positivity of FT of full functions

e.g. $\tilde{G}(\frac{\|p\|^2}{2\mu^2},\frac{\|p\|^2}{2\mu^2})\tilde{G}(\frac{\|q\|^2}{2\mu^2},\frac{\|q\|^2}{2\mu^2})+\tilde{G}(\frac{\|p\|^2}{2\mu^2},\frac{\|p\|^2}{2\mu^2}|\frac{\|q\|^2}{2\mu^2},\frac{\|q\|^2}{2\mu^2})$

- Difficult for $N=4$,
  but $G(2|2|2)+G(2)G(2)G(2)$ is not positive.

Very probable conclusion

The Φ^3_D matricial QFT does not satisfy Osterwalder-Schrader.

- Reason: Higher functions too much localised in p-space!
  already $\tilde{G}(\frac{\|p\|^2}{2\mu^2},\frac{\|p\|^2}{2\mu^2})\propto\frac{C_1\log(\|p\|^2+\mu^2)+C_2}{\|p\|^2+\mu^2}$ almost fails
- For Φ^4_4 we expect $\tilde{G}(\frac{\|p\|^2}{2\mu^2},\frac{\|p\|^2}{2\mu^2})\propto\frac{C}{(\|p\|^2+\mu^2)^{1-\frac{1}{\pi}\arcsin(|\lambda|\pi)}}$ (hope!)
- Keeps us busy for the next time!

Backup: 2-point function $G(x,y)$ of Φ^4_4

after renormalisation in large-$(V,\mathcal{N})$ limit:

1. $$\lambda x \int_0^\infty \frac{G(x,0)G(p,y) - G(p,0)G(x,y)}{p-x} = (1 + yG(x,0))G(x,y) - (1+y)G(x,0)G(0,y)$$
2. $1 + \lambda \int_0^\infty dp(G(p,y) - G(p,0)) = (1+y)G(0,y)$
3. $G(x,y) = G(y,x)$

- using Riemann-Hilbert techniques we solved (1)+(2) up to one unknown function
- one-sided Hilbert transform $\mathcal{H}_a(f) = \frac{1}{\pi}\mathcal{P}\int_0^\infty \frac{f(p)\,dp}{p-a}$ arises
- remains (3): a single integral equation $G(x,0) = G(0,x)$

Solution of $\lambda\phi^4_4$ on extreme Moyal space

Theorem (2012/13)

Given boundary function $G(x,0)$,

define $\tau_y(x) := \underset{[0,\pi]}{\arctan}\left(\dfrac{|\lambda|\pi x}{y + \frac{1+\lambda\pi x\mathcal{H}_x[G(\bullet,0)]}{G(x,0)}}\right)$. Then

$$G(x,y) = \frac{\sin(\tau_y(x))}{|\lambda|\pi x} e^{\mathrm{sign}(\lambda)(\mathcal{H}_0[\tau_0(\bullet)] - \mathcal{H}_x[\tau_y(\bullet)])} \begin{cases} 1 & \lambda < 0 \\ \left(1+\frac{Cx+yF(y)}{\Lambda^2-x}\right) & \lambda > 0 \end{cases}$$

From symmetry $G(x,0) = G(0,x)$:

Fixed point equation for boundary function (assuming $\lambda < 0$)

$$G(x,0) = \frac{1}{1+x}\exp\left(-\lambda\int_0^x dt \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p\mathcal{H}_p[G(\bullet,0)]}{G(p,0)}\right)^2}\right)$$

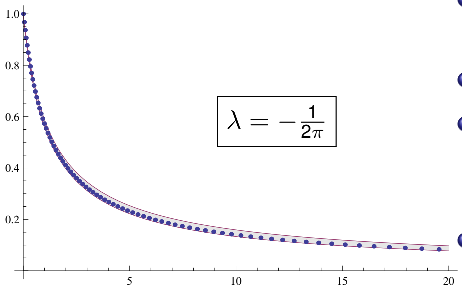
Fixed point theorem

Reflection positivity = Stieltjes property is excluded for $\lambda > 0$

Theorem [H.Grosse+RW, 2015]

Let $-\frac{1}{6} \leq \lambda \leq 0$. Then the equation has a C_0^1-solution

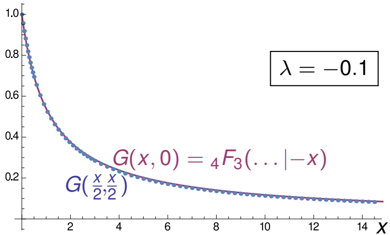
$$\frac{1}{(1+x)^{1-|\lambda|}} \leq G(x,0) \leq \frac{1}{(1+x)^{1-\frac{|\lambda|}{1-2|\lambda|}}}$$

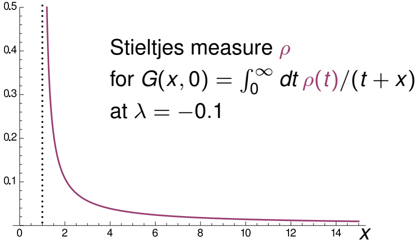


- proof via Schauder fixed point theorem
- compactness via Arzelà-Ascoli
- Banach is slightly missed: $\|Tf - Tg\| \leq (1 + \frac{1}{e} + \mathcal{O}(\lambda))\|f - g\|$
- need exact asymptotics!

Approximation by 4F3 hypergeometric function

ansatz $G(x,0) = {}_4F_3\left(\begin{smallmatrix} a,b_1,b_2,b_3 \\ c_1,c_2,c_3 \end{smallmatrix}\middle| -x\right)$; matching a, b_i, c_i at one point x result in global error $\sup_x |\dots| \approx 10^{-8}$ in fixed point eq.





reflection positivity equivalent to existence of a blue curve on the right whose Stieltjes transform is $G(\frac{x}{2},\frac{x}{2})$ on the left

- measure for $G(x,0)$ (and almost surely for $G(\frac{x}{2},\frac{x}{2})$) has mass gap $[0,1[$, but no further gap (remnant of UV/IR-mixing)
- absence of the second gap (usually $]1,4[$) circumvents triviality theorems