Exact solution of quantum field theory toy models

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based on arXiv:1610.00526 with Harald Grosse and Akifumi Sako

and arXiv: 1205.0465, 1306.2816, 1402.1041, 1406.7755 & 1505.05161
with Harald Grosse

Goal: Quantum Field Theory satisfying axioms

- 1932: mathematical foundation of quantum mechanics [von Neumann]
- 1950's: unique extension to quantum fields [Wightman]

  Theorem: vacuum expectation values of field operators are boundary values of holomorphic functions
- their restriction to real subspace of Euclidean points (minus diagonals) defines Schwinger functions
- Schwinger functions inherit real analyticity, Euclidean invariance, complete symmetry and reflection positivity

Theorem [Osterwalder-Schrader, 1974]

These properties are sufficient to reconstruct Wightman theory!

So far no non-trivial QFT model in 4 dimensions ...

Current reality: QFT defined by Feynman graphs

Consider graphs (= simplicial 1-complexes) Γ consisting of

1. n vertices, of valence determined by QFT model
2. ℓ edges; their number per vertex is bounded by valence of the vertex

Can restrict to (path-)connected graphs.

Spanning trees and momenta

1. choose any spanning tree $T \subseteq \Gamma$ of $n-1$ edges
2. unconnected valences of Γ permit incoming or outgoing momenta $q_1, \ldots, q_e \in \mathbb{R}^D$
3. assign oriented loop momenta $p_1, \ldots, p_{\ell-n+1} \in \mathbb{R}^D$ to the $\ell - n + 1$ edges of $\Gamma \setminus T$

Observe: q_i, p_j are entering/leaving momenta for T, sum to 0.

Every edge in T (thus in Γ) has uniquely determined momentum

Weights and renormalisation

QFT model in dimension D assigns momentum-dependent weights to vertices and edges

1. multiply all weight factors
2. integrate over the loop momenta $p_1, \dots, p_{\ell-n+1} \in B_\Lambda \subset \mathbb{R}^D$

amplitude $G_\Gamma^\Lambda(q_1, \dots, q_e)$ depends on choice of T

- continuum limit $B_\Lambda \to \mathbb{R}^D$ eliminates all ambiguities, but is often divergent

Renormalisation

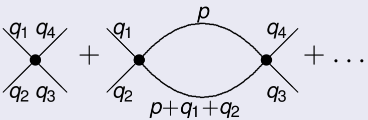
- metric structure: edges with large momenta are short
- compare short loops with well-chosen multiple Z_Γ of vertex: $\bigcirc = \{\bigcirc - Z_\Gamma \bullet\} + Z_\Gamma \bullet$
- $\{\ \}$ has limit, $Z_\Gamma \bullet$ effectively changes (=renormalises) vertex weight when summing over all graphs
- in favourable cases, $\lim_{\Lambda\to\infty} G_\Gamma^\Lambda(q_1, \dots, q_e)$ exists

Example: Euclidean $\lambda\phi^4_4$ model

1. 4-valent vertices  of constant weight $(-\lambda)Z$
2. edges $\underline{\quad p \quad}$ of weight $\frac{1}{Z_2(p^2+m_b^2)}$

4-point function (with $q_1 + q_2 + q_3 + q_4 = 0$)

[below we ignore $Z_2 \neq 1$ and $m_b \neq m$]



$$= (-\lambda)Z + (-\lambda)^2 Z^2 \int_{B_\Lambda \subset \mathbb{R}^4} \frac{dp}{(2\pi)^4} \frac{1}{(p^2+m^2)((p+q_1+q_2)^2+m^2)} + \ldots$$

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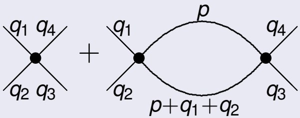

$+\dots$

$$= (-\lambda)Z + (-\lambda)^2 Z^2 \int_{B_\Lambda \subset \mathbb{R}^4} \frac{dp}{(2\pi)^4} \frac{1}{(p^2+m^2)((p+q_1+q_2)^2+m^2)} + \dots$$

$$= (-\lambda)\Big(Z - \lambda Z^2 \int_{B_\Lambda \subset \mathbb{R}^4} \frac{dp}{(2\pi)^4} \frac{1}{(p^2+m^2)^2}\Big)$$

$$+ (-\lambda)^2 Z^2 \underbrace{\int_{B_\Lambda \subset \mathbb{R}^4} \frac{dp}{(2\pi)^4} \Big(\frac{1}{(p^2+m^2)((p+q_1+q_2)^2+m^2)} - \frac{1}{(p^2+m^2)^2}\Big)}_{\text{convergent}} + \dots$$

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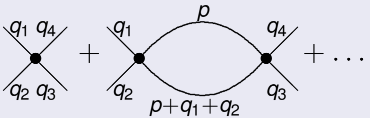


$$= (-\lambda)Z + (-\lambda)^2 Z^2 \int_{B_\Lambda\subset\mathbb{R}^4} \frac{dp}{(2\pi)^4}\frac{1}{(p^2+m^2)((p+q_1+q_2)^2+m^2)} + \dots$$

$$= (-\lambda)\Big(Z - \lambda Z^2 \int_{B_\Lambda\subset\mathbb{R}^4} \frac{dp}{(2\pi)^4}\frac{1}{(p^2+m^2)^2}\Big)$$

$\leftarrow$ {when including all $\sum_\Gamma$ in ( ), normalize to 1

$$+ (-\lambda)^2 Z^2 \int_{B_\Lambda\subset\mathbb{R}^4} \frac{dp}{(2\pi)^4}\Big(\frac{1}{(p^2+m^2)((p+q_1+q_2)^2+m^2)} - \frac{1}{(p^2+m^2)^2}\Big) + \dots$$

convergent

rearranged into $(\ )^2 = 1$ when including $\sum_\Gamma$

The question of convergence

For nice QFT models (incl. standard model of particle physics):

Theorem (Bogoliubov, Parasiuk, Hepp, Zimmermann)

At the level of formal power series there is a recursive reorganisation of graph amplitudes into

1. renormalised weights of edges & vertices
2. Taylor-subtracted (in external momenta for the loop) integrands of the loop integral

- In this way we assign to any graph a well-defined amplitude G_Γ independent of the choice of T.
- $\exists$ generating function for such graphs (partition function).

Physical quantities can be computed from $\sum_\Gamma G_\Gamma$.
But: giving sense to this sum is an open problem!

A QFT toy model

Consider not graphs, but simplicial 2-compexes $\mathcal{K}$ (also called ribbon graphs) consisting of

1. two sorts of vertices:
   – n black [internal] 3-valent vertices
   – B white [external] vertices of (any) valences $N_1, \dots, N_B$
2. edges which saturate all valences
3. faces; we require every face to have at most one white vertex

Under these conditions we have:

- $N := N_1 + \cdots + N_B$ external edges connecting black–white
- ℓ internal edges connecting black–black
- $N := N_1 + \cdots + N_B$ external faces with one white vertex
- F internal faces with all vertices black

The simplicial 2-complex (= ribbon graph) defines a Riemann surface on which it can be prescribed, of Euler characteristics

$$\chi = 2 - 2g = (B + n) - (N + \ell) + (N + F)$$

Planar ribbon graphs

- recall $\chi = 2 - 2g = (B + n) - \ell + F$ from previous slide
- write as $F = \ell - n + (2 - B - 2g)$
- take ribbon graph encoding 2-sphere ($g = 0$) with single $B = 1$ white vertex, then:

$$\left.\begin{array}{l}\text{number F of}\\ \text{internal faces}\end{array}\right\} = \left\{\begin{array}{l}\text{number of loops in 1-subcomplex made of}\\ \text{black vertices and internal edges}\end{array}\right.$$

Message

With appropriate weights for edges and (black) vertices, planar ribbon graphs define a QFT toy model which shares all interesting properties with a true QFT.

- labels of the N external faces analogous to N external momenta
- labels of F internal analogous to loop momenta

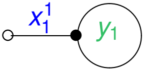
Weights and renormalisation

1. label the faces:
   – external faces by $x^1_1, \dots, x^1_{N_1}, \dots, x^B_1, \dots, x^B_{N_B} \in \mathbb{R}_+$
   – internal faces by $y_1, \dots, y_F \in \mathbb{R}_+$
2. black vertices receive weight $(-\lambda)$, white vertices weight 1
3. edges separating faces labelled z_1, z_2 have weight $\frac{1}{z_1+z_2+1}$ (z_i can be internal or external, also $z_1 = z_2$ is possible)
4. integrate internal face variables over $[0, \Lambda^2]$

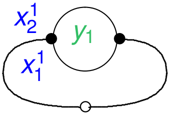
renormalisation = Taylor subtraction of 1-valent faces
corresponding 1-valent vertex is given weight 0 and disappears

- gives well-defined amplitude $G_{\mathcal{K}}$ to any simplicial complex $\mathcal{K}$
- $\exists$ generating function for all $\mathcal{K}$: partition function for random matrix theory

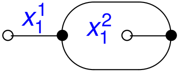
Examples



$$G_{\mathcal{K}_1}(x_1^1) = \frac{(-\lambda)}{2x_1^1+1}\int_0^\infty dy_1 \Big(\frac{1}{x_1^1+y_1+1} - \frac{1}{y_1+1}\Big)$$



$$G_{\mathcal{K}_2}(x_1^1, x_2^1) = \frac{(-\lambda)^2}{(x_1^1+x_2^1+1)^2}\int_0^\infty \frac{dy_1}{(x_1^1+y_1+1)(x_2^1+y_1+1)}$$



$$G_{\mathcal{K}_3}(x_1^1|x_1^2) = \frac{(-\lambda)^2}{(2x_1^1+1)(2x_1^2+1)(x_1^1+x_1^2+1)^2}$$

Can one sum up the power series in λ?

Consider the set $\mathcal{M}(N_1,\dots,N_B;n)$ of all planar simplicial complexes with

- same numbers (B, n) and types (=valences) of vertices
- different topology (due to different connecting edges & faces)

1. Fix external face labels x_i^β, assign weights, compute amplitudes

$$\sum_{\mathcal{K}\in\mathcal{M}(N_1,\dots,N_B;n)} G_{\mathcal{K}}(x_1^1\dots x_{N_1}^1|\dots|x_1^B\dots x_{N_B}^B) =: (-\lambda)^n G^{(n)}(x_1^1\dots x_{N_1}^1|\dots|x_1^B\dots x_{N_B}^B)$$

2. Now keep white vertices fixed, but sum over n.

   What can be said about convergence of the series

$$\sum_{n\geq n_0}(-\lambda)^n G^{(n)}(x_1^1\dots x_{N_1}^1|\dots|x_1^B\dots x_{N_B}^B)?$$

[And *if* it converges, what is its limit?]

Can one sum up the power series in λ?

1. Absolute convergence cannot be expected!
   - for amplitudes which do not need renormalisation one proves uniform bounds $|G_{\mathcal{K}}(\dots)| \leq |\lambda|^n C^n$
   - uniform lower bounds not much different, maybe there is $\frac{1}{n}$ improvement, but not much more
   - for absent renormalisation: all $\mathcal{K}$ of same n have uniform sign, hence no cancellation
   - The problem is: There are too many (ribbon) graphs! Roughly, there are as many planar (ribbon) graphs with n vertices as there are trees with n vertices, n^{n-2} (Cayley).
2. Reasonable convergence in disk tangent to imaginary axis: Borel resummation and Nevanlinna-Sokal theorem

Exact formulae for any correlation function

In arXiv:1610.00526 we (H.Grosse, A.Sako, RW) prove exact formulae for any $G(x^1_1 \ldots x^1_{N_1}|\ldots|x^B_1 \ldots x^B_{N_B})$. They are analytic in λ^2!

Examples: ($B = 1$, $N = 2$) and ($B = 3$, $N = 3$)

Let $c(\lambda)$ be the inverse solution of $2\lambda^2 = \frac{1-\sqrt{c+1}}{\log(1+\frac{1}{\sqrt{c+1}})}$ through $c(0) = 0$. Then

$$G(x,y) = \frac{2}{\sqrt{(2x+1)^2+c} + \sqrt{(2y+1)^2+c}} \Bigg\{ 1 + \frac{\frac{2\lambda^2}{2x+1}\log\frac{(2x+2)(\sqrt{(2x+1)^2+c}+2x+1)}{(2x+1)\sqrt{c+1}+\sqrt{(2x+1)^2+c}} - \frac{2\lambda^2}{2y+1}\log\frac{(2y+2)(\sqrt{(2y+1)^2+c}+2y+1)}{(2y+1)\sqrt{c+1}+\sqrt{(2y+1)^2+c}}}{\sqrt{(2x+1)^2+c} - \sqrt{(2y+1)^2+c}} \Bigg\}$$

$$G(x|y|z) = \frac{(-2\lambda)^5}{1 - \frac{2\lambda^2}{1+c+\sqrt{1+c}}} \left[\frac{1}{((2x+1)^2+c)((2y+1)^2+c)((2z+1)^2+c)} \right]^{\frac{3}{2}}$$

Preparation for the solution: Partition function

Euclidean quantum field theory of matrix model (formal!)

- action $S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + \alpha\Phi + \frac{\lambda}{3}\Phi^3)$ for $E = E^*, \Phi = \Phi^* \in M_{\mathcal{N}}(\mathbb{C})$ and $\alpha, \lambda \in \mathbb{R}$
- partition function $\mathcal{Z}[J] = \int d\Phi\ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))$

This is essentially ($\alpha = 0, \lambda = \frac{i}{2}$) the Kontsevich model which computes intersection numbers of the stable cohomology classes on (Deligne-Mumford compactification of) the moduli space $\mathcal{M}_{g,B}$ of complex curves.

exact formulae can only be expected for $V, \mathcal{N} \to \infty$ (such limits also required by symmetries as part of QFT axioms)

- Kontsevich model is known to be solvable, with solution related to Virasoro algebra and KdV hierarchy.

The Harish-Chandra–Itzykson–Zuber formula

Standard solution technique of Kontsevich model diagonalises $\Phi = \Phi^* = UXU^*$, with $U \in U(\mathcal{N})$ and $X = (x_i \delta_{ij})$ diagonal. Need:

Theorem (Harish-Chandra 1957, Itzykson–Zuber 1980)

Let $\Phi = U(x_i\delta_{ij})U^*$, $\Delta(x) := \prod_{j<i}(x_i - x_j) = \det(x_i^{j-1})$ and dU the Haar measure on $U(\mathcal{N})$. Then

1. $d\Phi = \frac{(2\pi)^{\mathcal{N}(\mathcal{N}-1)/2}}{\prod_{p=1}^{\mathcal{N}} p!}(\Delta(x))^2\big(\prod_{i=1}^{\mathcal{N}} dx_i\big)dU$
2. For diagonal $\mathcal{N} \times \mathcal{N}$ matrices X, Y one has

$$\int_{U(\mathcal{N})} dU\, e^{\frac{\mathcal{N}}{s}\mathrm{tr}(XUYU^*)} = \frac{\prod_{p=1}^{\mathcal{N}-1} p!}{(\frac{N}{s})^{N(N-1)/2}} \frac{\det\big((e^{\frac{\mathcal{N}}{s}x_i y_j})_{1\le i,j\le \mathcal{N}}\big)}{\Delta(x)\Delta(y)}$$

Resulting $\{x_i\}$-integral is treatable for Φ^3, but becomes (even for special cases) very difficult for Φ^4.

New approach: Schwinger-Dyson equations

Logarithm of $\mathcal{Z}$ expands into cycles $J^{N_k}_{p_1\dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$:

$$\log \frac{\mathcal{Z}(J)}{\mathcal{Z}(0)} = \sum_{B=1}^{\infty} \sum_{N_B \geq \dots \geq N_1 \geq 1} \frac{V^{2-B}}{S_{N_1\dots N_B}} G_{|p^1_1\dots p^1_{N_1}|\dots|p^B_1\dots p^B_{N_B}|} \prod_{\beta=1}^{B} J^{N_\beta}_{p^\beta_1\dots p^\beta_{N_\beta}}$$

The $G_{|p^1_1\dots p^1_{N_1}|\dots|p^B_1\dots p^B_{N_B}|}$ collect connected simplical complexes with

- $\{N_1,\dots,N_B\}$-valent white vertices, external face var's p^β_i
- black vertex weight $(-\lambda)$, edge weight $\frac{1}{E_m+E_n}$
- internal face sum $\sum_{m=0}^{\mathcal{N}-1}$

Strategy

1. use $\mathcal{Z}(J) = K \exp\Big(-\frac{\lambda}{3V^2}\sum_{k,l,m=0}^{\mathcal{N}-1} \frac{\partial^3}{\partial J_{kl}\partial J_{lm}\partial J_{mk}}\Big)\mathcal{Z}_{free}(J)$ (*)

   $$\mathcal{Z}_{free}(J) = \exp\Big(\frac{V}{2}\sum_{m,n=0}^{\mathcal{N}-1} \frac{(\alpha\delta_{mn}+J_{mn})(\alpha\delta_{mn}+J_{nm})}{E_m+E_n}\Big)$$

   to derive equations between the formal power series $G_{\dots}$
2. forget $\mathcal{Z}$ and declare these equations as exact
3. search for rigorous solutions $G_{\dots}$ of these equations

1-point function

Inserting (*) into $G_{|a|} \equiv \frac{1}{V}\frac{\partial \log Z(J)}{\partial J_{aa}}\Big|_{J=0}$ gives

$$G_{|a|} = \frac{1}{2E_a}\Big(\alpha - \lambda G_{|a|}^2 - \frac{\lambda}{V}\sum_{m=0}^{\mathcal{N}-1} G_{|am|} - \frac{\lambda}{V^2} G_{|a|a|}\Big)$$

- general lesson: N-point functions depend on $(> N)$-point functions, making this approach in general useless
- the equation is non-linear, making its solution difficult
- $\sum_{m=0}^{\infty} G_{|am|}$ diverges. This is where α enters the game: Choose α such that $G_{|0|} = 0$:

$$G_{|a|} = \frac{1}{2E_a}\Big(-\lambda G_{|a|}^2 - \frac{\lambda}{V}\sum_{m=0}^{\mathcal{N}-1} (G_{|am|} - G_{|0m|}) - \frac{\lambda}{V^2}(G_{|a|a|} - G_{|0|0|}\Big)$$

Ward-Takahashi identity

Same computation gives for $N = 2$ and $a \neq b$:

$$G_{|ab|} = \frac{1}{E_a + E_b}\Big(1 - \frac{\lambda}{V^2 \mathcal{Z}(0)} \sum_{m=0}^{\mathcal{N}-1} \frac{\partial}{\partial J_{ab}} \frac{\partial}{\partial J_{bm}} \frac{\partial}{\partial J_{ma}} \mathcal{Z}(J)\Big|_{J=0}\Big) \quad (*)$$

Proposition (Disertori, Gurau, Magnen, Rivasseau, 2006)

$$\sum_m \frac{\partial^2 \mathcal{Z}(J)}{\partial J_{bm} \partial J_{ma}} = \delta_{ab} \mathrm{W}_a(J) + \frac{V}{E_a - E_b} \sum_m \Big(J_{am} \frac{\partial \mathcal{Z}[J]}{\partial J_{bm}} - J_{mb} \frac{\partial \mathcal{Z}[J]}{\partial J_{ma}}\Big)$$

Proof: $\int d\Phi\, e^{-S(\Phi) + V\,\mathrm{tr}(\Phi J)} = \int d\Psi\, e^{-S(\Psi) + V\,\mathrm{tr}(\Psi J)}, \quad \Psi = U^* \Phi U$ □

inserted into (*): $$G_{|ab|} = \frac{1}{E_a + E_b}\Big(1 + \lambda \frac{G_{|a|} - G_{|b|}}{E_a - E_b}\Big)$$

Inserting $G_{|ab|}$ into formula on previous slide gives non-linear equation for $G_{|a|}$ alone, up to $\frac{1}{V^2}(G_{|a|a|} - G_{|0|0|})$ corrections which vanish for $V \to \infty$

Summary: All N-point functions with $B = 1$

Introduce $W_{|a|} := 2(\lambda G_{|a|} + E_a)$. Then:

1. $W_{|a|}^2 = 4E_a^2 - \frac{4\lambda^2}{V^2}(G_{|a|a|} - G_{|0|0|}) - \frac{2\lambda^2}{V}\sum_{m=0}^{\mathcal{N}-1}\Big(\frac{W_{|a|} - W_{|m|}}{E_a^2 - E_m^2} - \frac{W_{|0|} - W_{|m|}}{E_0^2 - E_m^2}\Big)$
2. $G_{|ab|} = \frac{1}{2}\frac{W_{|a|} - W_{|b|}}{E_a^2 - E_b^2}$
3. $G_{|a_1 \dots a_N|} = \lambda \frac{G_{|a_1 a_3 \dots a_N|} - G_{|a_2 a_3 \dots a_N|}}{(E_{a_1}^2 - E_{a_2}^2)}$

Proposition: Solution of (2)+(3)

$$G_{|a_1 a_2, \dots a_N|} = \frac{\lambda^{N-2}}{2}\sum_{k=1}^{N} W_{|a_k|}\prod_{l=1, l\neq k}^{N}\frac{1}{E_{a_k}^2 - E_{a_l}^2}$$

Remains to solve (1) for $W_{|a|}$, which is possible in a combined limit $(V, \mathcal{N}) \to \infty$ with $\frac{\mathcal{N}}{V}$ fixed. This eliminates $(G_{|a|a|} - G_{|0|0|})$ and produces a non-linear integral equation for $W_{|a|}$ alone.

The non-linear integral equation and its solution

Assume $E_n = \mu^2(\frac{1}{2} + e(\frac{n}{\mu^2 V}))$ for increasing C^1-function with $e(0) = 0$. In limit $(V, \mathcal{N}) \to \infty$ with $\frac{\mathcal{N}}{V} = \mu^2 \Lambda^2$, after rescaling in μ and change of variables $X := (2e(x) + 1)^2$, we have

$$W^2(X) + \int_1^{\Xi} dY \rho(Y) \frac{W(X)-W(Y)}{X-Y} = X + \int_1^{\Xi} dY \rho(Y) \frac{W(1)-W(Y)}{1-Y} \qquad (*)$$

where $\rho(Y) := \frac{2\lambda^2}{\sqrt{Y}\cdot e'(e^{-1}(\frac{\sqrt{Y}-1}{2}))}$ and $\Xi := (1 + 2e(\Lambda^2))^2$

Theorem (inspired by [Makeenko-Semenoff, 1991])

The non-linear integral equation (*) is solved by

$$W(X) := \sqrt{X+c} + \frac{1}{2}\int_1^{\Xi} dY \frac{\rho(Y)}{(\sqrt{X+c} + \sqrt{Y+c})\sqrt{Y+c}}$$

where $c(\lambda, e)$ is the inverse solution of

$$W(1) = 1 = \sqrt{1+c} + \frac{1}{2}\int_1^{\Xi} dY \frac{\rho(Y)}{(\sqrt{1+c} + \sqrt{Y+c})\sqrt{Y+c}}$$

$(N_1 + \ldots + N_B)$-point function

Using Ward-Takahashi identity we explicitly reduce $(N_1 + \ldots + N_B)$-point function to $(1 + \ldots + 1)$-point function:

Proposition

$$G_{|a^1_1 \ldots a^1_{N_1}|\ldots|a^B_1 \ldots a^B_{N_B}|}$$

$$= \lambda^{N_1+\cdots+N_B-B} \sum_{k_1=1}^{N_1} \cdots \sum_{k_B=1}^{N_B} G_{|a^1_{k_1}|\ldots|a^B_{k_B}|} \prod_{\beta=1}^{B} \prod_{\substack{l_\beta=1 \\ l_\beta \neq k_\beta}}^{N_\beta} \frac{1}{E^2_{a^1_{k_1}} - E^2_{a^1_{l_1}}}$$

large-$(V, \mathcal{N})$ limit $a \mapsto x$ with $X = (2e(x) + 1)^2$:

$$G(X^1_1, \ldots, X^1_{N_1} | \ldots | X^B_1, \ldots, X^B_{N_B})$$

$$= \lambda^{N_1+\cdots+N_B-B} \sum_{k_1=1}^{N_1} \cdots \sum_{k_B=1}^{N_B} G(X^1_{k_1} | \ldots | X^B_{k_B}) \prod_{\beta=1}^{B} \prod_{\substack{l_\beta=1 \\ l_\beta \neq k_\beta}}^{N_\beta} \frac{4}{X^\beta_{k_\beta} - X^\beta_{l_\beta}}$$

Equation for (1+ ... +1)-point function

Proposition

In large-$(V,\mathcal{N})$ limit, with $G(X|Y_{\triangleleft}^{\{i_1,\dots,i_p\}}) := G(X|Y^{i_1}|\dots|Y^{i_p})$:

$$W(X^1)G(X^1|X_{\triangleleft}^{\{2,\dots B\}})+\frac{1}{2}\int_1^\infty dT\rho(T)\frac{G(X^1|X_{\triangleleft}^{\{2,\dots B\}})-G(T|X_{\triangleleft}^{\{2,\dots B\}})}{(X-T)}$$

$$= -\lambda\sum_{\beta=2}^{B} G(X^1,X^\beta,X^\beta|X_{\triangleleft}^{\{2,\dots\check{\beta}\dots B\}})$$

$$-\lambda\sum_{\substack{J\subset\{2,\dots,B\}\\ 1\leq|J|\leq B-2}} G(X^1|X_{\triangleleft}^{J})G(X^1|X_{\triangleleft}^{\{2,\dots,B\}\setminus J}) \qquad \text{[lower order]}$$

Proposition ($B=2$)

$$W(X)G(X|Y) = -\lambda G(X,Y,Y) - \frac{1}{2}\int_1^\infty dZ\rho(Z)\,\frac{G(X|Y)-G(Z|Y)}{X-Z}$$

solved by $G(X|Y) = \dfrac{4\lambda^2}{\sqrt{X+c}(\sqrt{X+c}+\sqrt{Y+c})^2\sqrt{Y+c}}$

Ansatz for $B \geq 3$

Explicit solution for $B = 3$ and *a lot of work* suggest:

Ansatz

$$G(X^1|\dots|X^B) = \frac{(-2\lambda)^{3B-4}}{\rho_0} \sum_{M=0}^{B-3} \gamma_B^M \frac{d^M}{dt^M} \Big(\prod_{\beta=1}^{B} \frac{1}{\sqrt{X^\beta + c - 2t}^{\,3}} \Big)\Big|_{t=0}$$

where $\rho_0 := 1 - \int_1^\infty \frac{dT\rho(T)}{2\sqrt{(T+c)}^{\,3}}$, $\quad \gamma_3^0 = 1$

Equations for γ_B^M

Lemma

The ansatz amounts to a system of equations for $l \geq -2$ and tuples $\mathcal{M} = (m_2, \dots, m_B)$ with $M := m_2 + \dots + m_B$:

$$\sum_{j=0}^{B-5-M-l} (M+2+l+j)!\gamma_B^{M+2+l+j} \frac{(2j+2l+5)!!\rho_j}{(l+2+j)!}$$

$$= (M+l+1)!\gamma_{B-1}^{M+l+1} \sum_{\beta=2}^{B} \frac{(2l+2m_\beta+3)!!(2m_\beta+1)m_\beta!}{(l+m_\beta+1)!(2m_\beta+1)!!}$$

$$+ \sum_{l'+l''=l} \frac{(2l'+1)!!(2l''+1)!!}{2\rho_0 l'!l''!} \sum_{\mathcal{M}'\cup\mathcal{M}''=\mathcal{M}} (M'+l')!\gamma_{\#(\mathcal{M}')+1}^{M'+l'}(M''+l'')!\gamma_{\#(\mathcal{M}'')+1}^{M''+l''}$$

where $\rho_j = \delta_{j,0} - \frac{1}{2}\int_1^\infty \frac{dT\,\rho(T)}{\sqrt{T+c}^{3+2j}}$

For $l{=}-2$: $$\sum_{j=0}^{B-3-M} \binom{M+j}{j}(2j+1)!!\rho_j\gamma_B^{M+j} = \gamma_{B-1}^{M-1}, \qquad \gamma_3^0 = 1$$

Bell polynomials

Definition

The Bell polynomials $B_{n,k}$ are defined by $B_{0,k}(\{\ \}) = \delta_{k,0}$ and

$$B_{n,k}(\{x_j\}_{j=1}^{n-k+1}) = \sum \frac{n!}{j_1!j_2!\cdots j_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{j_1}\left(\frac{x_2}{2!}\right)^{j_2}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}$$

for $n\geq 1$, where the sum is over non-negative integers j_i with $j_1+j_2+...+j_{n-k+1} = k$ and $1j_1+2j_2+...+(n-k-1)j_{n-k+1} = n$.

They arise in all sorts of partition problems, such as

Theorem (Faà di Bruno)

$$\frac{d^n}{dx^n}(f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x))\, B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x))$$

Solution for γ_B^M

Proposition

The equations for γ_B^M, $M \in \{0, \dots, B-3\}$, at $l = -2$ and $l = -1$,

$$\sum_{j=0}^{B-3-M} \binom{M+j}{j} (2j+1)!! \rho_j \gamma_B^{M+j} = \gamma_{B-1}^{M-1}$$

$$\sum_{j=0}^{B-4-M} \binom{M+1+j}{j+1} (2j+3)!! \rho_j \gamma_B^{M+1+j} = (2M+B-1)\gamma_{B-1}^{M}$$

together with initial condition $\gamma_3^M = \delta_{M,0}$, have the solution

$$\gamma_B^M = \frac{1}{\rho_0^{B-3}} \sum_{K=0}^{B-3-M} \frac{(B-3+K)!}{(B-3-M)!M!} B_{B-3-M,K}\Big(\Big\{-\frac{(2r+1)!!\rho_r}{(r+1)\rho_0}\Big\}_{r=1}^{B-2-M-K}\Big)$$

A (new?) identity for Bell polynomials

For any $l, n_0, \dots, n_p \in \mathbb{N}$, the Bell polynomials satisfy

$$
\begin{aligned}
&\frac{(2l+5)!!}{(l+2)!}\sum_{K\geq 0}(N-2+K)!\frac{B_{N-M-l-4,K}(\{x_r\})}{(N-M-l-4)!} \\
&-\sum_{K\geq 0}(N-3+K)!\frac{B_{N-M-l-4,K}(\{x_r\})}{(N-M-l-4)!}\sum_{i=0}^{p} n_i\frac{(2l+2i+3)!!(2i+1)i!}{(2i+1)!!(l+i+1)!} \\
&=\sum_{j\geq 1}\sum_{K\geq 0}(N-2+K)!\frac{(2j+2l+5)!!(j+1)!}{(2j+1)!!(j+l+2)!}\cdot\frac{x_j}{j!}\cdot\frac{B_{N-M-l-j-4,K}(\{x_r\})}{(N-M-l-j-4)!} \\
&+\frac{1}{2}\sum_{l'=0}^{l}\sum_{n'_0=0}^{n_0}\cdots\sum_{n'_p=0}^{n_p}\sum_{K',K''\geq 0}\frac{(2l'+1)!!(2l''+1)!!}{l'!\,l''!}\binom{n_0}{n'_0}\cdots\binom{n_p}{n'_p} \\
&\times(N'-2+K')!\frac{B_{N'-M'-l'-2,K'}(\{x_r\})}{(N'-M'-l'-2)!}(N''-2+K'')!\frac{B_{N''-M''-l''-2,K''}(\{x_r\})}{(N''-M''-l''-2)!}
\end{aligned}
$$

where $l'' := l - l'$, $N'' := N - N'$ and $M'' := M - M'$ and for $? = \{\varnothing, ', ''\}$: $n_0^? + \dots + n_p^? = N^?$ and $0n_0^? + 1n_1^? + \dots + pn_p^? = M^?$

Exact solution of (1+ . . . +1)-point function

Using generating function

$$\exp\left(u\sum_{j=1}^{\infty}\frac{x_j t^j}{j!}\right)=\sum_{n,k\geq 0}u^k\frac{t^n}{n!}B_{n,k}\left(\{x_r\}_{r=1}^{n-k+1}\right)$$

it is straightforward to prove:

Theorem

$$G(X^1|\dots|X^B)$$

$$=(-2\lambda)^{3B-4}\frac{d^{B-3}}{dt^{B-3}}\left(\frac{\frac{1}{\sqrt{X^1+c-2t}^3}\cdots\frac{1}{\sqrt{X^B+c-2t}^3}}{\left(1-\int_1^\infty\frac{dT\,\rho(T)}{\sqrt{T+c}}\frac{1}{(\sqrt{T+c}+\sqrt{T+c-2t})\sqrt{T+c-2t}}\right)^{B-2}}\right)\Bigg|_{t=0}$$

This completes the exact solution of the QFT toy model.

QFT on noncommutative geometries

Quantum Physics and General Relativity are incompatible if space-time is a semi-Riemannian manifold

- modest approach: noncommutative geometry
  Euclidean quantum fields are elements of operator algebra
- finite-dimensional approximation: matrices arise generically
- key question: describe the large-$(V, \mathcal{N})$ limit in NCG

Example: Moyal algebra = Rieffel deformation of $C^\infty(\mathbb{R}^2)$

$$(f \star g)(\xi) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{d\eta\, dk}{(2\pi)^2} f(x+\tfrac{1}{2}\Theta k)\, g(\xi+\eta)\, e^{\mathrm{i}\langle k,\eta\rangle} \qquad \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

- matrix basis $\phi(\xi) = \sum_{m,n=0}^{\infty} \Phi_{mn} f_{mn}(\xi)$

$$f_{mn}(\xi) = 2(-1)^m \sqrt{\frac{m!}{n!}} \left(\sqrt{\frac{2}{\theta}}\xi_1 + \mathrm{i}\xi_2\right)^{n-m} L_m^{n-m}\left(\frac{2\|\xi\|^2}{\theta}\right) e^{-\frac{\|\xi\|^2}{\theta}}$$

- satisies $f_{mn} \star f_{kl} = \delta_{nk} f_{ml}$ and $\int \frac{d\xi}{8\pi} f_{mn}(\xi) = V\delta_{mn}$
  with $V = \frac{\theta}{4}$

From QFT on NCG to Schwinger functions on $\mathbb{R}^2$

$$S(\phi) := \frac{1}{8\pi}\int_{\mathbb{R}^2} d\xi \Big(\frac{1}{2}\phi\star(-\Delta+4\|\Theta^{-1}\xi\|^2)\star\phi + \alpha\phi + \frac{\lambda}{3}\phi\star\phi\star\phi\Big)(\xi)$$

$$= V\,\mathrm{tr}(E\Phi^2 + \alpha\Phi + \tfrac{\lambda}{3}\Phi^3), \qquad E = ((\tfrac{\mu^2}{2} + \tfrac{n}{V})\delta_{mn})$$

Definition (formal!) of connected Schwinger functions

$$\mu^N S_c(\mu\xi_1, \dots, \mu\xi_N)$$
$$:= \lim_{V\mu^2\to\infty} \sum_{m_i,n_i=0}^{\infty} f_{m_1n_1}(\xi_1)\cdots f_{m_Nn_N}(\xi_N) \frac{(V\mu^2)^{-2}\mu^{4N}\partial^N \log \mathcal{Z}(J)}{\partial J_{m_1n_1}\dots\partial J_{m_Nn_N}}\Big|_{J=0}$$

- produces f_{mn}-cycles for every face
- sum over Laguerre polynomials yields factor V for every even face
- previously $\frac{1}{V^B}$-suppressed contributions of multiple boundaries B are amplified!

Schwinger functions

large-$(V,\mathcal{N})$ limit yields topological expansion into boundary components (=white vertices)

Theorem [HG+RW, 2013]: *connected* Schwinger functions

$$S_c(\mu\xi_1,\dots,\mu\xi_N) = \frac{1}{8\pi} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \Bigg(\prod_{\beta=1}^{B} \frac{2^{N_\beta}}{N_\beta} \int_{\mathbb{R}^2} \frac{dp_\beta}{2\pi\mu^2}\, e^{\mathrm{i}\left\langle p_\beta, \sum_{i=1}^{N_\beta}(-1)^{i-1}\xi_{\sigma(N_1+\dots+N_{\beta-1}+i)}\right\rangle} \Bigg) \times G\Big(\underbrace{\frac{\|p_1\|^2}{2\mu^2},\dots,\frac{\|p_1\|^2}{2\mu^2}}_{N_1}\Big|\dots\Big|\underbrace{\frac{\|p_B\|^2}{2\mu^2},\dots,\frac{\|p_B\|^2}{2\mu^2}}_{N_B}\Big)$$

confinement of noncommutativity: have internal interaction of matrices; commutative subsector propagates to outside world

- Schwinger functions are symmetric and invariant under full Euclidean group (completely unexpected for NCQFT!)
- remains: reflection positivity
- finally: is it non-trivial?

Osterwalder-Schrader reflection positivity

Proposition [H. Grosse+RW, 2013]

$S(\xi_1, \xi_2)$ is reflection positive iff $x \mapsto G(x,x)$ is Stieltjes function,

$$G(x,x) = \int_0^\infty \frac{\rho(t)\,dt}{x+t}, \qquad \rho - \text{positive measure}$$

Stieltjes functions are positive in a very strong sense (e.g. stronger than completely monotonic functions)

Theorem (Krein)

f is Stieltjes iff

1. $f(t) \geq 0$ for $t > 0$
2. $f : \mathbb{C} \setminus]-\infty, 0] \to \mathbb{C}$ holomorphic
3. $\mathrm{Im}(f(x+\mathrm{i}y)) < 0$ for $y > 0$ (anti-Herglotz)

For the solved model, (2) fails for $\lambda \in \mathrm{i}\mathbb{R}$ and (3) for $\lambda \in \mathbb{R}$. Hence, the model does not define a true quantum field theory.

Better model: quartic interaction

We [H. Grosse, RW] actually started here and developed the presented techniques for

$$S(\phi)=\frac{1}{64\pi^2}\int_{\mathbb{R}^4} d\xi\Big(\frac{Z}{2}\phi\star(-\Delta+4\|\Theta^{-1}\xi\|^2+\mu_{bare}^2)\star\phi+\frac{Z^2\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(\xi)$$

$$= V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2} ZE_{\underline{m}}\Phi_{\underline{mn}}\Phi_{\underline{nm}}+\frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Big)$$

where $E_{\underline{m}}=\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu_{bare}^2}{2}$, $|\underline{m}|:=m_1+m_2\leq\mathcal{N}$, $\quad V=(\frac{\theta}{4})^2$

- simplest function is 2-point function $G_{|\underline{ab}|}$; fulfils closed non-linear equation (before renormalisation)

$$G_{|\underline{ab}|}=\frac{1}{Z(E_{\underline{a}}+E_{\underline{b}})}-\frac{Z\lambda}{(E_{\underline{a}}+E_{\underline{b}})}\frac{1}{V}\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}}\Big(G_{|\underline{ab}|}G_{|\underline{ap}|}-\frac{G_{|\underline{pb}|}-G_{|\underline{ab}|}}{Z(E_{\underline{p}}-E_{\underline{a}})}\Big)$$

- higher $(N_1+\ldots+N_B)$-point functions satisfy algebraic equations if one $N_i\geq 3$, otherwise affine integral equations

2-point function $G(x, y)$

after renormalisation in large-$(V, \mathcal{N})$ limit:

1. $$\lambda x \int_0^\infty \frac{G(x,0)G(p,y) - G(p,0)G(x,y)}{p-x} = (1 + yG(x,0))G(x,y) - (1+y)G(x,0)G(0,y)$$
2. $1 + \lambda \int_0^\infty dp(G(p,y) - G(p,0)) = (1+y)G(0,y)$
3. $G(x,y) = G(y,x)$

- using Riemann-Hilbert techniques we solved (1)+(2) up to one unknown function
- one-sided Hilbert transform $\mathcal{H}_a(f) = \frac{1}{\pi}\mathcal{P}\int_0^\infty \frac{f(p)\,dp}{p-a}$ arises
- remains (3): a single integral equation $G(x,0) = G(0,x)$

Solution of $\lambda\phi^4_4$ on extreme Moyal space

Theorem (2012/13)

Given boundary function $G(x,0)$,

define $\tau_y(x) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi x}{y + \frac{1+\lambda\pi x\mathcal{H}_x[G(\bullet,0)]}{G(x,0)}}\right)$. Then

$$G(x,y)=\frac{\sin(\tau_y(x))}{|\lambda|\pi x}e^{\mathrm{sign}(\lambda)(\mathcal{H}_0[\tau_0(\bullet)]-\mathcal{H}_x[\tau_y(\bullet)])}\begin{cases} 1 & \lambda<0 \\ \left(1+\frac{Cx+yF(y)}{\Lambda^2-x}\right) & \lambda>0 \end{cases}$$

From symmetry $G(x,0)=G(0,x)$:

Fixed point equation for boundary function (assuming $\lambda<0$)

$$G(x,0)=\frac{1}{1+x}\exp\left(-\lambda\int_0^x dt\int_0^\infty\frac{dp}{(\lambda\pi p)^2+\left(t+\frac{1+\lambda\pi p\mathcal{H}_p[G(\bullet,0)]}{G(p,0)}\right)^2}\right)$$

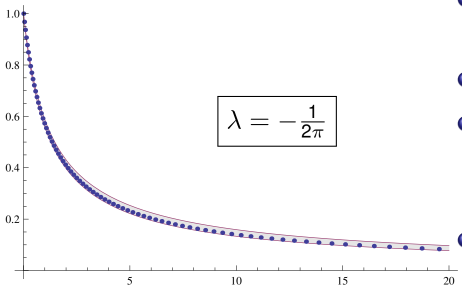
Fixed point theorem

Reflection positivity = Stieltjes property is excluded for $\lambda > 0$

Theorem [H.Grosse+RW, 2015]

Let $-\frac{1}{6} \leq \lambda \leq 0$. Then the equation has a C_0^1-solution

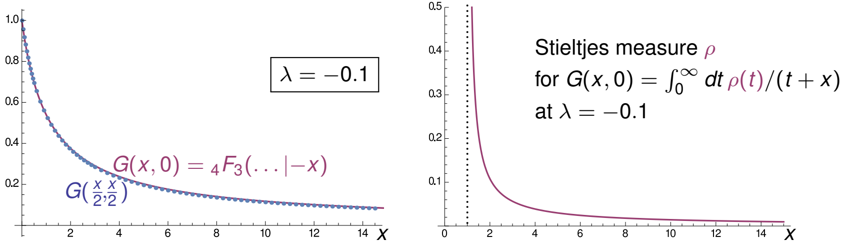
$$\frac{1}{(1+x)^{1-|\lambda|}} \leq G(x,0) \leq \frac{1}{(1+x)^{1-\frac{|\lambda|}{1-2|\lambda|}}}$$



- proof via Schauder fixed point theorem
- compactness via Arzelà-Ascoli
- Banach is slightly missed: $\|Tf - Tg\| \leq (1 + \frac{1}{e} + \mathcal{O}(\lambda))\|f - g\|$
- need exact asymptotics!

Approximation by 4F3 hypergeometric function

ansatz $G(x,0) = {}_4F_3\left({a,b_1,b_2,b_3 \atop c_1,c_2,c_3}\middle| -x\right)$; matching a, b_i, c_i at one point x result in global error $\sup_x |\dots| \approx 10^{-8}$ in fixed point eq.



reflection positivity equivalent to existence of a blue curve on the right whose Stieltjes transform is $G(\frac{x}{2},\frac{x}{2})$ on the left

- measure for $G(x,0)$ (and almost surely for $G(\frac{x}{2},\frac{x}{2})$) has mass gap $[0,1[$, but no further gap (remnant of UV/IR-mixing)
- absence of the second gap (usually $]1,4[$) circumvents triviality theorems