Integrability in a 4D QFT model

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Quantum Field Theory

- 1932: mathematical foundation of quantum mechanics [von Neumann]
- 1950's: unique extension to quantum fields [Wightman]

  Theorem: vacuum expectation values of field operators are boundary values of holomorphic functions
- their restriction to real subspace of Euclidean points (minus diagonals) defines Schwinger functions
- Schwinger functions inherit real analyticity, Euclidean invariance, complete symmetry and reflection positivity

Theorem [Osterwalder-Schrader, 1974]

These properties are sufficient to reconstruct Wightman theory!

So far no non-trivial QFT model in 4 dimensions . . .

The anomalous dimension η

effective 2-point function in momentum space

$$\hat{S}_2(p) \propto \frac{1}{(p^2+m^2)^{1-\eta/2}}$$

- reflection positivity requires $\eta \geq 0$
- ...but convergence in 4D needs $\eta \leq 0$

proposed way out: require decay of effective vertex functions $\hat{\mathcal{V}}(p_i) \to 0$ for $p_i \to \infty$ (asymptotic freedom, $\beta < 0$)

- tendency to produce infrared problems ($\to$ confinement), no solution so far
- put models with $\beta = 0$ back to agenda: hope to solve convergence by integrability

4D QFT models with vanishing β-function

1. $\mathcal{N} = 4$ super Yang-Mills theory
   - very active subject with many strong results: overwhelming evidence for integrability [see Lett. Math. Phys. **99** (2012), $>$ 500pp]
   - Wightman axioms are not made for gauge theory, but $\mathcal{N} = 4$ SYM might suggest reasonable axioms
2. self-dual noncommutative ϕ_4^4-theory
   - perturbatively renormalisable [Grosse-W., 2004]
   - β-function vanishes to all orders in perturbation theory [Disertori-Gurau-Magnen-Rivasseau, 2006]

$\rightarrow$ Can we construct it? . . . YES

- Could have failed at 1000 steps, but didn't. Something makes everything work: Integrability?
- Osterwalder-Schrader axioms might be satisfied. . .

Dismiss the partition function

- successful approach to $P[\Phi]_2$ [Glimm-Jaffe, Simon, ...] consists in Feynman-Kac-type perturbation of free measure

  already difficult: renormalisation=normal ordering destroys boundedness from below (can be repaired)

- renormalisation in 4D is more expensive:
  – (bare) anomalous dimension is in fact $\eta = +\infty$
  – subtraction may produce $\eta_{ren} > 0$ for unstable $\lambda < 0$

Message

Consider the possibility that 4D Schwinger functions do not arise from a measure (for $\beta = 0$)!

(i.e. one cannot give a rigorous meaning to partition function)

What else?

Take partition function as guiding principle to suggest quantum equations of motion (Schwinger-Dyson equations), and make them rigorous (renormalisation)

- old idea which usually leads nowhere
- our model on noncommutative space is essentially a matrix model which carries $U(\infty)$ group action
- produces infinite number of constraints, leading to collapse of Schwinger-Dyson tower
- get self-consistency equation for 2-point function alone (similar to Thirring model),

  plus hierarchy of affine equations for higher N-point functions

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4}\frac{dx}{64\pi^2}\Big(\frac{Z}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2_{bare})\phi+\frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with Moyal product $(f\star g)(x)=\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dy\,dk}{(2\pi)^4}f(x+\tfrac{1}{2}\Theta k)\,g(x+y)\,e^{\mathrm{i}\langle k,y\rangle}$

has matrix basis $\phi(x)=\sum_{\underline{m},\underline{n}\in\mathbb{N}^2}\Phi_{\underline{mn}}f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\frac{2}{\theta}}y\Big)^{n-m}L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

satisies $f_{\underline{mn}}\star f_{\underline{kl}}=\delta_{\underline{nk}}f_{\underline{ml}}$ and $\int\frac{dx}{64\pi^2}f_{\underline{mn}}(x)=V\delta_{\underline{mn}}$, take cut-off $\mathcal{N}$

Simplification at $\Omega=1$, describes $V_{\text{Planck}}=V_{\text{universe}}=\left(\frac{\theta}{4}\right)^2$

$$S[\Phi]=V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}ZE_{\underline{m}}\Phi_{\underline{mn}}\Phi_{\underline{nm}}+\frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Big)$$

$$E_{\underline{m}}=\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu^2_{bare}}{2},\qquad |\underline{m}|:=m_1+m_2\leq\mathcal{N}$$

More generally: field-theoretical matrix models

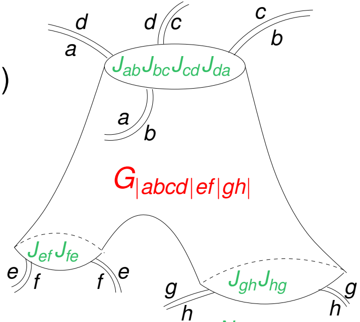
Euclidean quantum field theory (formal!)

- action $S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + P[\Phi])$ for unbounded positive selfadjoint operator E with compact resolvent, and $P[\Phi]$ a polynomial
- partition function $\mathcal{Z}[J] = \int \mathcal{D}[\Phi]\ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))$

- For $P[\Phi] = \frac{\mathrm{i}}{6}\Phi^3$ this is the Kontsevich model which computes the intersection theory on the moduli space of complex curves. We choose $P[\Phi] = \frac{\lambda}{4}\Phi^4$.
- Perturbative expansion $e^{V\,\mathrm{tr}(P[\Phi])} = \sum_{n=0}^{\infty} \frac{1}{n!}(V\,\mathrm{tr}(P[\Phi])^n$ leads to ribbon graphs. They encode genus-g Riemann surface with B boundary components.
- We avoid the expansion, but keep the topological structure:

Topological expansion

- Riemann surfaces with B boundary components (avoid genus expansion)
- k^{th} boundary component carries a cycle $\boldsymbol{J}^{N_k}_{p_1 \dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$



- expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p_1^1 \dots p_{N_1}^1| \dots |p_1^B \dots p_{N_B}^B|} \prod_{\beta=1}^{B} \boldsymbol{J}^{N_\beta}_{p_1^\beta \dots p_{N_\beta}^\beta}$ according to the cycle structure
- QFT of matrix models determines the weights of Riemann surfaces with decorated boundary components compatible with
  1. gluing (of fringes, not boundaries!)
  2. covariance (under $\Phi \mapsto U^* \Phi U$, which is not a symmetry!)

Ward identity

Proposition [Disertori-Gurau-Magnen-Rivasseau, 2006]

$$0 = \sum_{n\in I} \Big(\frac{(E_a - E_p)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an}\partial J_{np}} + J_{pn}\frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}}{\partial J_{np}}\Big)$$

Theorem [Grosse-W. 2012]

$$\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} = \delta_{ap}\Big\{ V^2 \sum_{(K)} \frac{J_{P_1}\cdots J_{P_K}}{S_K}\Big(\sum_{n\in I}\frac{G_{|an|P_1|\dots|P_K|}}{V^{|K|+1}} + \frac{G_{|a|a|P_1|\dots|P_K|}}{V^{|K|+2}}$$
$$+\sum_{r\geq 1}\sum_{q_1,\dots,q_r\in I} \frac{G_{|q_1 a q_1 \dots q_r|P_1|\dots|P_K|}J^r_{q_1\dots q_r}}{V^{|K|+1}}\Big)$$
$$+V^4 \sum_{(K),(K')} \frac{J_{P_1}\cdots J_{P_K} J_{Q_1}\cdots J_{Q_{K'}}}{S_K S_{K'}} \frac{G_{|a|P_1|\dots|P_K|}}{V^{|K|+1}} \frac{G_{|a|Q_1|\dots|Q_{K'}|}}{V^{|K'|+1}}\Big\}\mathcal{Z}[J]$$
$$+\frac{V}{E_p - E_a}\sum_{n\in I}\Big(J_{pn}\frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}[J]}{\partial J_{np}}\Big)$$

- in Schwinger-Dyson equations resulting from $\frac{\partial^N \mathcal{Z}}{\partial J \dots \partial J}$, two J-derivatives are killed

Schwinger-Dyson equations

in a scaling limit $V \to \infty$ and $\frac{1}{V}\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}}$ finite, we have:

1. a closed non-linear equation for $G_{|\underline{ab}|}$

$$G_{|\underline{ab}|} = \frac{1}{Z(E_{\underline{a}}+E_{\underline{b}})} - \frac{Z\lambda}{(E_{\underline{a}}+E_{\underline{b}})} \frac{1}{V}\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}} \Big(G_{|\underline{ab}|} G_{|\underline{ap}|} - \frac{G_{|\underline{pb}|} - G_{|\underline{ab}|}}{Z(E_{\underline{p}}-E_{\underline{a}})} \Big)$$

- limit $\mathcal{N} \to \infty$ needs renormalisation!

2. affine equations for higher $(N_1+\ldots+N_B)$-point functions

- algebraic for $\max(N_\beta) > 2$
- e.g. $G_{|\underline{abcd}|} = (-\lambda)\frac{G_{|\underline{ab}|}G_{|\underline{cd}|}-G_{|\underline{ad}|}G_{|\underline{cb}|}}{(E_{\underline{a}}-E_{\underline{c}})(E_{\underline{b}}-E_{\underline{d}})}$ $\Rightarrow$ β-function zero!
- complicated equation for $G_{|\underline{ab}|\underline{cd}|}$, makes model not completely trivial

Limits and renormalisation

Thermodynamic limit $\sqrt{V}=\frac{\theta}{4}\to\infty$ with $\frac{\mathcal{N}}{\sqrt{V\mu^4}}=\Lambda^2$ fixed

- 'continuous' matrix indices $a,b,\dots\in[0,\Lambda^2]$ and sums↦integrals
- finite Hilbert transform arises: $\mathcal{H}^\Lambda_a(f)=\frac{1}{\pi}\mathcal{P}\int_0^{\Lambda^2}\frac{f(p)\,dp}{p-a}$
- renormalisation $\mu_{bare}\mapsto\mu$ and $Z^1\mapsto(1+\mathcal{Y})$ by normalisation conditions $G_{00}=1$ and $\frac{\partial G_{0b}}{\partial b}\Big|_{b=0}=-(1+\mathcal{Y})$, $\mu^2 G_{|\underline{ab}|}\mapsto G_{ab}$
- note: consistent only because of $\beta=0$!

key: split problem into eqns for $D_{ab}:=\frac{b}{a}(G_{ab}-G_{a0})$ and for G_{a0}:

Carleman-type singular linear integral equation

$$\left(\frac{b}{a}+\frac{1+\lambda\pi a\mathcal{H}^\Lambda_a\big[G_{\bullet 0}\big]}{aG_{a0}}\right)D_{ab}-\lambda\pi\mathcal{H}^\Lambda_a\big[D_{\bullet b}\big]=-G_{a0}$$

solution theory by Carleman, Tricomi, Muskhelishvili,...

Solution of $\lambda\phi^4_4$ on extreme Moyal space

Theorem (2012/13)

Given boundary function G_{a0},

define $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\dfrac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a\mathcal{H}^{\Lambda}_a[G_{\bullet 0}]}{G_{a0}}}\right)$. Then

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\operatorname{sign}(\lambda)(\mathcal{H}_0[\tau_0(\bullet)] - \mathcal{H}_a[\tau_b(\bullet)])} \begin{cases} 1 & \text{for } \lambda < 0 \\ \left(1 + \frac{Ca + bF(b)}{\Lambda^2 - a}\right) & \text{for } \lambda > 0 \end{cases}$$

- surprisingly, instantons live at $\lambda > 0$
- remaining eqn G_{a0} reduces to symmetry $G_{b0} = G_{0b}$:

Fixed point equation for boundary function (assuming $\lambda < 0$)

$$G_{b0} = G_{0b} = \frac{1}{1+b}\exp\left(-\lambda\int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p\mathcal{H}^{\Lambda}_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

Discussion

together with explicit (but complicated for $G_{ab|cd}$, $G_{ab|cd|ef}$, ...) formulae for higher correlation functions, we have exact solution of $\lambda\phi^4_4$ on extreme Moyal space in terms of ($\Lambda \to \infty$ now safe!)

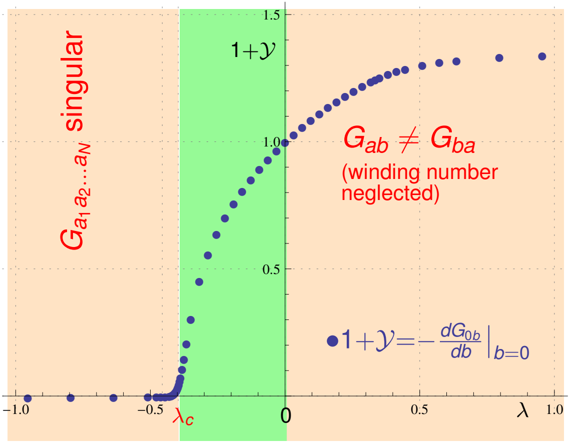
$$G_{b0} = G_{0b} = \frac{1}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}_p^\infty[G_{\bullet 0}]}{G_{p0}}\right)^2} \right)$$

possible treatments

1. perturbative solution: reproduces all Feynman graphs, generates polylogarithms and ζ-functions
2. iterative solution on computer: nicely convergent, find interesting phase structure
3. rigorous existence proof of a solution
4. work in progress: try to guess the solution; should give uniqueness as by-product

Computer simulation: evidence for phase transitions

piecewise linear approximation of G_{0b}, G_{ab} for $\Lambda^2{=}10^7$ and 2000 sample points. Consider $1+\mathcal{Y}:=-\frac{dG_{0b}}{db}\Big|_{b=0}$



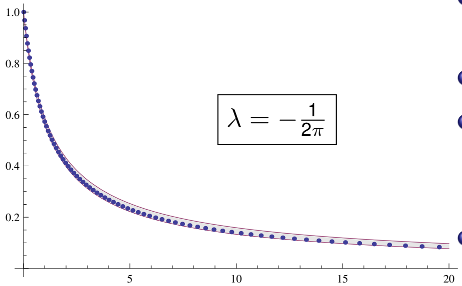
- inconsistency for $\lambda > 0$
- $(1+\mathcal{Y})'(\lambda)$ discontinuous at $\lambda_c = -0.39$
- next steps: reflection positivity of Schwinger functions excluded outside $]\lambda_c, 0]$!

Fixed point theorem

Theorem [H.Grosse+RW, 2015]

Let $-\frac{1}{6} \leq \lambda \leq 0$. Then the equation has a C^1_0-solution

$$\frac{1}{(1+b)^{1-|\lambda|}} \leq G_{0b} \leq \frac{1}{(1+b)^{1-\frac{|\lambda|}{1-2|\lambda|}}}$$



- proof via Schauder fixed point theorem
- compactness via Arzelà-Ascoli
- Banach is slightly missed: $\|Tf - Tg\| \leq (1 + \frac{1}{e} + \mathcal{O}(\lambda))\|f - g\|$
- need exact asymptotics!

From matrix model to Schwinger functions on $\mathbb{R}^4$

Definition (formal!) of connected Schwinger functions

$$\mu^N S_c(\mu x_1,\dots,\mu x_N)$$
$$:= \lim_{V\mu^4\to\infty} \sum_{\underline{m}_i,\underline{n}_i\in\mathbb{N}^2} f_{\underline{m}_1\underline{n}_1}(x_1)\cdots f_{\underline{m}_N\underline{n}_N}(x_N) \frac{(V\mu^4)^{-2}\mu^{4N}\partial^N \log\mathcal{Z}[J]}{\partial J_{\underline{m}_1\underline{n}_1}\dots\partial J_{\underline{m}_N\underline{n}_N}}\Big|_{J=0}$$

- produces $f_{\underline{mn}}$-cycles for every face
- write $G_{|\dots|\underline{m}_1\dots\underline{m}_j|\dots|}$ as Laplace-Fourier transform

Lemma

(with $J+i\equiv i$, $\quad |z_i|<1$)

$$\sum_{m_1,\dots,m_J=0}^{\infty}\prod_{i=1}^{J} z_i^{m_i} L_{m_i}^{m_{i+1}-m_i}(r_i) = \frac{\exp\Big(-\dfrac{\sum_{i,k=1}^{J} r_i(z_{k+i}\cdots z_{J+i})}{1-(z_1\cdots z_J)}\Big)}{1-(z_1\cdots z_J)}$$

- $r_i \propto \frac{x_i^2}{\sqrt{V}}$, $z_i \propto -e^{-\frac{t+\mathrm{i}\omega_{kl}}{\sqrt{V}}}$ gives factor $V^{\#(\text{even faces})}$

Schwinger functions

topological expansion into boundary components survives:

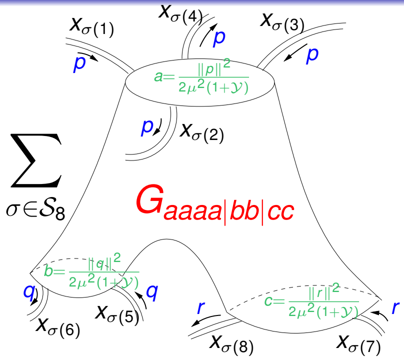
Theorem [HG+RW, 2013]: *connected* **Schwinger functions**

$$S_c(\mu x_1,\dots,\mu x_N) = \frac{1}{64\pi^2} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \Big(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4 p_\beta}{4\pi^2\mu^4}\, e^{\mathrm{i}\big\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta}(-1)^{i-1}\mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)}\big\rangle} \Big) \times G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1} \Big| \dots \Big| \underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}}$$

confinement of noncommutativity: have internal interaction of matrices; commutative subsector propagates to outside world

- Schwinger functions are symmetric and invariant under full Euclidean group (completely unexpected for NCQFT!)
- remains: reflection positivity
- finally: is it non-trivial?

Connected (4+2+2)-point function



1. no clustering (remnant of non-locality)
2. particle scattering without momentum exchange
   - in 4D a sign of triviality (mind assumptions!)
   - familiar in 2D models with factorising S-matrix
   - a consequence of integrability [Moser, 1975] & [Kulish, 1976]

road to non-triviality (proposed by J. Schlemmer)

- *if* our Schwinger functions do *not* arise from a measure, there is no reason they satisfy Nelson-Symanzik positivity
- . . . but the free field is Nelson-Symanzik positive, so our Schwinger functions should not be trivial

Osterwalder-Schrader reflection positivity

Proposition [H. Grosse+RW, 2013]

$S(x_1, x_2)$ is reflection positive iff $a \mapsto G_{aa}$ is a Stieltjes function,

$$G_{aa} = \int_0^\infty \frac{\rho(t)\,dt}{a+t}, \qquad \rho \text{ – positive measure}$$

excluded for any $\lambda > 0$ (unless rescued by winding number)

- purely real conditions [Widder, 1938]:

$$L_{k,t}[f(\bullet)] := \frac{(-t)^{k-1}}{c_k} \frac{d^{2k-1}}{dt^{2k-1}}\big(t^k f(t)\big) \geq 0\,, \qquad \begin{array}{c} c_1 = 1 \\ c_{k>1} = k!(k-2)! \end{array}$$

  $\rightarrow$ lowest orders verified numerically

- complex conditions [Krein]:
  1. $f(t) \geq 0$ for $t > 0$
  2. $f : \mathbb{C} \setminus]-\infty, 0] \to \mathbb{C}$ holomorphic
  3. $\mathrm{Im}(f(x + \mathrm{i}y)) < 0$ for $y > 0$ (anti-Herglotz)

  $\rightarrow$ verified on subset $x > -\frac{4}{5}$ of $\mathbb{C}$

Reflection positivity simplifies the problem

if G_{x0} is Stieltjes, then Hilbert transform can be avoided:

$$\frac{G_{xy}}{G_{x0}} = \exp\left(-\frac{1}{\pi}\int_1^\infty \frac{dt}{t+x} \operatorname*{arctan}_{[0,\pi]}\left(\frac{y\,\mathrm{Im}(G_{-(t+\mathrm{i}\epsilon),0})}{1-\lambda t\int_0^\infty ds\,\frac{G_{s0}}{t+s}+y\,\mathrm{Re}(G_{-(t+\mathrm{i}\epsilon),0})}\right)\right)$$

Which class of functions has desired positivity+holomorphicity and manageable integral transforms?

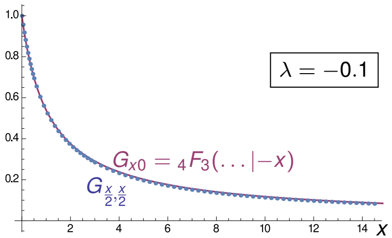
hypergeometric functions $G_{x0} = {}_nF_{n-1}\left(\begin{matrix}a,b_1,\dots b_{n-1}\\ c_1,\dots,c_{n-1}\end{matrix}\middle| -x\right)$ if $a \in [0,1]$ and $c_i > b_i > a$

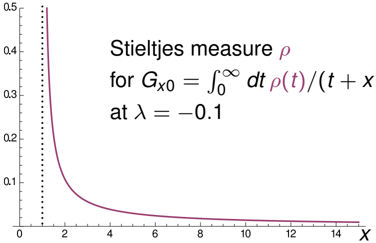
- holomorphicity at $y > 0$: determine a, b_i, c_i by $G^{(k)}_{0y} = G^{(k)}_{y0}$
- find: $a = 1 + \frac{1}{\pi}\arcsin(\lambda\pi)$, subsequently proven!

1. critical coupling constant is $\lambda_c = -\frac{1}{\pi} = -0.3183\dots$
2. anomalous dimension is $\eta = -\frac{2}{\pi}\arcsin(\lambda\pi)$

Källén-Lehmann spectrum

- matching conditions for $G_{x,0} = {}_4F_3(\dots|-x)$ at one point x result in global error $\sup_x|\dots| \approx 10^{-8}$ in fixed point equation





reflection positivity equivalent to existence of a blue curve on the right whose Stieltjes transform is $G_{\frac{x}{2},\frac{x}{2}}$ on the left

- measure for G_{x0} (and almost surely for $G_{\frac{x}{2},\frac{x}{2}}$) has mass gap $[0,1[$, but no further gap (remnant of UV/IR-mixing)
- absence of the second gap (usually $]1,4[$) circumvents triviality theorems

Summary

1. regularisation on extreme noncommutative space defines 4D QFT that is exactly solvable in terms of fixed point problem
   - theory defined by quantum equations of motion (= Schwinger-Dyson equations), not by a measure
   - existence proved within small region
   - phase transitions and critical phenomena
2. projection to Schwinger functions for scalar field on $\mathbb{R}^4$ = confinement of noncommutativity
   - full Euclidean symmetry (completely unexpected)
   - no momentum exchange (close to triviality), possibly a consequence of integrability
   - numerical approach with tiny error: leaves no doubt that Schwinger 2-point function is reflection positive for $-\frac{1}{\pi}<\lambda\leq 0$
3. ready to embark on higher Schwinger functions

Generalisation of the method

1. Φ^3-model
   - = Kontsevich model, related to KdV hierarchy
   - work in progress [H. Grosse, A. Sako, R. Kullock, RW], first for superrenoralisable 2D case
   - linear equations for 1- and 2-point functions, direct solution of Carleman-type equation
2. Generalisation to coloured tensor models (see next talk by Vincent Rivasseau)
   - initial graphical treatment of Ward identities and Schwinger-Dyson equations [D. Ousmane-Samary, 2014]
   - rigourous treatment for rank-3 models using surgery on boundary graphs [C. Pérez-Sánchez, 2016], all topologies now included
   - might give complementary insight to the constructive renormalisation programme for tensor models