A solvable quantum field theory in 4 dimensions

Raimar Wulkenhaar

Mathematisches Institut, Westfälische Wilhelms-Universität Münster

(based on joint work with Harald Grosse,
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Prehistory

X$^{\text{th}}$ International Congress on Mathematical Physics Leipzig (1991)

one of the exciting topics: understand the Standard Model of particle physics as a noncommutative geometry

- Gerd's intuition: a kind of Kaluza-Klein theory[1]
- some student should work this out . . .
- intensive study of noncommutative geometry within the Leipzig Mathematical Physics group in 1993–1996
- also produced some results . . .

[1] Yu.A.Kubyshin, J.M.Mourão, G.Rudolph, I.P.Volobuev: “Dimensional Reduction of Gauge Theories, Spontaneous Compactification and Model Building,” LNP 349 (1989)

History I

…these results convinced the DAAD to finance a transfer to CPT Marseille (1998+1999):

- Nov. 97: Connes-Douglas-Schwarz paper about compactification of M-theory on noncommutative tori
- Thomas Krajewski: Can we do quantum field theory on noncommutative spaces?
- early 1999: achieve one-loop renormalisation of U(1)-Yang-Mills on noncommutative 4-torus
- subject receives enormous boost after Seiberg-Witten (1999) paper: models arise as limits of string theory

a problem arises: Minwalla, van Raamsdonk & Seiberg discover UV/IR-mixing (Dec. 1999)

History II

after failed attempts in Vienna (2000+2001),
solve UV/IR-mixing problem at MPI-MIS Leipzig (2002–. . .):

Theorem [H. Grosse+RW, 2004]

1. 4D scalar model on noncommutative Moyal space with harmonic propagation is perturbatively renormalisable
2. at one-loop there is no Landau ghost

Theorem [Disertori-Gurau-Magnen-Rivasseau, 2006]

Adjust (in above model) Planck volume = volume of universe.
Then the β-function is zero to all orders in perturbation theory.

immediate question: Can we construct the model?
. . . YES (in limit $V_{\text{Planck}} = V_{\text{universe}} \to \infty$)
To our big surprise, Wightman axioms seem to be satisfied!

What is Quantum Field Theory?

- 1932: mathematical foundation of quantum mechanics [von Neumann]
- 1950's: unique extension to quantum fields [Wightman]

  Theorem: vacuum expectation values of field operators are boundary values of holomorphic functions
- their restriction to real subspace of Euclidean points (minus diagonals) defines Schwinger functions
- Schwinger functions inherit properties such as real analyticity, Euclidean invariance, complete symmetry and reflection positivity

Theorem [Osterwalder-Schrader, 1974]

These properties are sufficient to reconstruct Wightman theory!

The anomalous dimension η

effective 2-point function in momentum space

$$\hat{\mathcal{S}}_2(p) \propto \frac{1}{(p^2+m^2)^{1-\eta/2}}$$

- reflection positivity requires $\eta \geq 0$
- ...but convergence in 4D needs $\eta \leq 0$

proposed way out: require decay of effective vertex functions $\hat{\mathcal{V}}(p_i) \to 0$ for $p_i \to \infty$ (asymptotic freedom, $\beta < 0$)

- tendency to produce infrared problems ($\to$ confinement), no solution so far
- put models with $\beta = 0$ back to agenda; maybe the convergence problem is simpler...

Dismiss the measure approach

- successful 2D-approach [Glimm-Jaffe, Simon, . . .] consists in Feynman-Kac-type perturbation of free measure

  already difficult: renormalisation=normal ordering destroys boundedness from below (can be repaired)

- renormalisation in 4D is more expensive:
  – (bare) anomalous dimension is in fact $\eta = +\infty$
  – subtraction may produce $\eta_{ren} > 0$ for unstable $\lambda < 0$

Message

Consider the possibility that 4D Schwinger functions do not arise from a measure (for $\beta = 0$)!

(i.e. one cannot give a rigorous meaning to partition function)

What else?

Take partition function as guiding principle to suggest quantum equations of motion (Schwinger-Dyson equations).

- old idea which usually leads nowhere
- our model on noncommutative space is essentially a matrix model which carries $U(\infty)$ group action
- produces infinite number of constraints, leading to collapse of Schwinger-Dyson tower
- get self-consistency equation for 2-point function alone (similar to Thirring model),

  plus hierarchy of affine equations for higher N-point functions

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4}\frac{dx}{64\pi^2}\Big(\frac{Z}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2_{bare})\phi+\frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with Moyal product $(f\star g)(x)=\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dy\,dk}{(2\pi)^4}f(x+\frac{1}{2}\Theta k)\,g(x+y)\,e^{i\langle k,y\rangle}$

has matrix basis $\phi(x)=\sum_{\underline{m},\underline{n}\in\mathbb{N}^2}\Phi_{\underline{mn}}f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\frac{2}{\theta}}y\Big)^{n-m}L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

satisies $f_{\underline{mn}}\star f_{\underline{kl}}=\delta_{\underline{nk}}f_{\underline{ml}}$ and $\int\frac{dx}{64\pi^2}f_{\underline{mn}}(x)=V\delta_{\underline{mn}}$, take cut-off $\mathcal{N}$

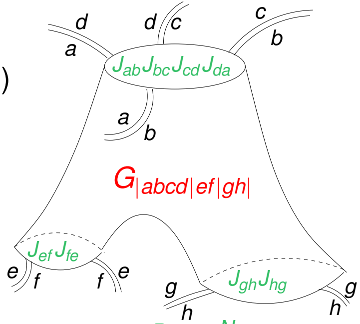
Simplification at $\Omega=1$, describes $V_{\text{Planck}}=V_{\text{universe}}=\left(\frac{\theta}{4}\right)^2$

$$S[\Phi]=V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}ZE_{\underline{m}}\Phi_{\underline{mn}}\Phi_{\underline{nm}}+\frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Big)$$

$$E_{\underline{m}}=\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu^2_{bare}}{2}\,,\qquad |\underline{m}|:=m_1+m_2\leq\mathcal{N}$$

Topological expansion

- Riemann surfaces with B boundary components (avoid genus expansion)
- k^{th} boundary component carries a cycle $\boldsymbol{J}^{N_k}_{p_1\dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$



- expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p_1^1\dots p_{N_1}^1|\dots|p_1^B\dots p_{N_B}^B|} \prod_{\beta=1}^{B} \boldsymbol{J}^{N_\beta}_{p_1^\beta\dots p_{N_\beta}^\beta}$ according to the cycle structure
- QFT of matrix models determines the weights of Riemann surfaces with decorated boundary components compatible with
  1. gluing (of fringes, not boundaries!)
  2. covariance (under $\Phi \mapsto U^* \Phi U$, which is not a symmetry!)

Schwinger-Dyson equations

in a scaling limit $V \to \infty$ and $\frac{1}{V}\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}}$ finite, we have:

1. a closed non-linear equation for $G_{|\underline{ab}|}$

$$G_{|\underline{ab}|} = \frac{1}{Z(E_{\underline{a}}+E_{\underline{b}})} - \frac{Z\lambda}{(E_{\underline{a}}+E_{\underline{b}})} \frac{1}{V}\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}} \Big(G_{|\underline{ab}|}G_{|\underline{ap}|} - \frac{G_{|\underline{pb}|} - G_{|\underline{ab}|}}{Z(E_{\underline{p}}-E_{\underline{a}})}\Big)$$

- limit $\mathcal{N} \to \infty$ needs renormalisation!

2. affine equations for higher $(N_1+\ldots+N_B)$-point functions

- algebraic for $\max(N_\beta) > 2$
- e.g. $G_{|\underline{abcd}|} = (-\lambda)\frac{G_{|\underline{ab}|}G_{|\underline{cd}|} - G_{|\underline{ad}|}G_{|\underline{cb}|}}{(E_{\underline{a}}-E_{\underline{c}})(E_{\underline{b}}-E_{\underline{d}})}$ $\Rightarrow$ β-function zero!
- complicated equation for $G_{|\underline{ab}|\underline{cd}|}$, makes model not completely trivial

Integral equations

- thermodynamic limit $\frac{\theta^2}{16} = V \to \infty$ turns discrete matrix indices into continuous variables $a, b, \dots \in \mathbb{R}_+$ and sums into integrals
- Need energy cutoff $a, b, \dots \in [0, \Lambda^2]$ and normalisation of lowest Taylor terms of 2-point function $G_{|\underline{ab}|} \mapsto G_{ab}$
- Carleman-type singular integral equation for $G_{ab} - G_{a0}$

Theorem [H.Grosse+RW, 2012/13] (for $\lambda < 0$, using $G_{b0} = G_{0b}$)

Let $\mathcal{H}^\Lambda_a(f) = \frac{1}{\pi}\mathcal{P}\int_0^{\Lambda^2} \frac{f(p)\,dp}{p-a}$ be the *finite Hilbert transform*. Then

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\mathrm{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)] - \mathcal{H}^\Lambda_a[\tau_b(\bullet)])}$$

where $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a \mathcal{H}^\Lambda_a[G_{\bullet 0}]}{G_{a0}}}\right)$ and G_{a0} solution of

$$G_{b0} = G_{0b} = \frac{1}{1+b}\exp\left(-\lambda\int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

Discussion

together with explicit (but complicated for $G_{ab|cd}, G_{ab|cd|ef}, \dots$) formulae for higher correlation functions, we have exact solution of $\lambda\phi^4_4$ on extreme Moyal space in terms of ($\Lambda \to \infty$ now safe!)

$$G_{b0} = G_{0b} = \frac{1}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

possible treatments

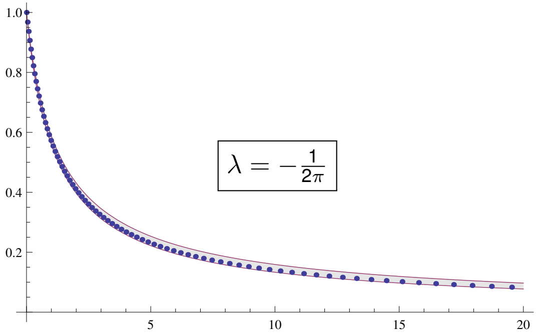
1. perturbative solution: reproduces all Feynman graphs, generates polylogarithms and ζ-functions
2. iterative solution on computer: nicely convergent, find interesting phase structure
3. rigorous existence proof of a solution
4. work in progress: try to guess the solution; should give uniqueness as by-product

Fixed point theorem

Theorem [H.Grosse+RW, 2015]

Let $-\frac{1}{6} \leq \lambda \leq 0$. Then the equation has a C^1_0-solution

$$\frac{1}{(1+b)^{1-|\lambda|}} \leq G_{0b} \leq \frac{1}{(1+b)^{1-\frac{|\lambda|}{1-2|\lambda|}}}$$



Proof via Schauder fixed point theorem.

This involves continuity and compactness of a certain operator (in norm topology)

From matrix model to Schwinger functions on $\mathbb{R}^4$

reverting harmonic oscillator basis, $1+\mathcal{Y} := -\frac{dG_{0b}}{db}\Big|_{b=0}$

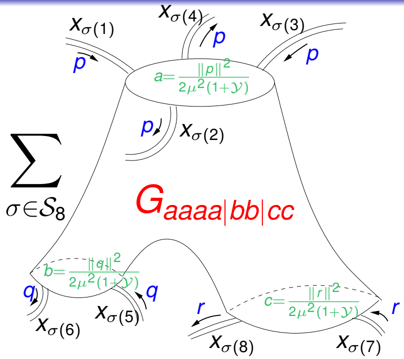
Theorem [HG+RW, 2013]: *connected* Schwinger functions

$$S_c(\mu x_1, \dots, \mu x_N) = \frac{1}{64\pi^2} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma \in \mathcal{S}_N} \Bigg(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4 p_\beta}{4\pi^2 \mu^4} e^{\mathrm{i}\left\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta} (-1)^{i-1} \mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)} \right\rangle} \Bigg)$$
$$\times G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1} \Big| \dots \Big| \underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}}$$

confinement of noncommutativity: have internal interaction of matrices; commutative subsector propagates to outside world

- Schwinger functions are symmetric and invariant under full Euclidean group (completely unexpected for NCQFT!)
- remains: reflection positivity
- finally: is it non-trivial?

Connected (4+2+2)-point function



1. no clustering
2. particle scattering without momentum exchange
   - in 4D a sign of triviality (mind assumptions!)
   - familiar in 2D models with factorising S-matrix
   - a consequence of integrability [Moser, 1975] & [Kulish, 1976]

road to non-triviality (proposed by J. Schlemmer)

- *if* our Schwinger functions do *not* arise from a measure, there is no reason they satisfy Nelson-Symanzik positivity
- . . . but the free field is Nelson-Symanzik positive, so our Schwinger functions should not be trivial

Osterwalder-Schrader reflection positivity

Proposition [H. Grosse+RW, 2013]

$S(x_1, x_2)$ is reflection positive iff $a \mapsto G_{aa}$ is a Stieltjes function,

$$G_{aa} = \int_0^\infty \frac{\rho(t)\,dt}{a+t}, \qquad \rho \text{ – positive measure}$$

excluded for any $\lambda > 0$ (unless rescued by winding number)

- purely real conditions [Widder, 1938]:

$$L_{k,t}[f(\bullet)] := \frac{(-t)^{k-1}}{c_k} \frac{d^{2k-1}}{dt^{2k-1}}\big(t^k f(t)\big) \geq 0\,, \qquad \begin{matrix} c_1 = 1 \\ c_{k>1} = k!(k-2)! \end{matrix}$$

$\rightarrow$ lowest orders verified numerically

- complex conditions [Krein]:
  1. $f(t) \geq 0$ for $t > 0$
  2. $f : \mathbb{C} \setminus]-\infty, 0] \to \mathbb{C}$ holomorphic
  3. $\mathrm{Im}(f(x + \mathrm{i}y)) < 0$ for $y > 0$ (anti-Herglotz)

$\rightarrow$ verified on subset $x > -\frac{4}{5}$ of $\mathbb{C}$

Reflection positivity simplifies the problem

if G_{x0} is Stieltjes, then Hilbert transform can be avoided:

$$\frac{G_{xy}}{G_{x0}} = \exp\Big(-\frac{1}{\pi}\int_1^\infty \frac{dt}{t+x} \operatorname*{arctan}_{[0,\pi]}\Big(\frac{y\,\mathrm{Im}(G_{-(t+\mathrm{i}\epsilon),0})}{1-\lambda t\int_0^\infty ds\,\frac{G_{s0}}{t+s} + y\,\mathrm{Re}(G_{-(t+\mathrm{i}\epsilon),0})}\Big)\Big)$$

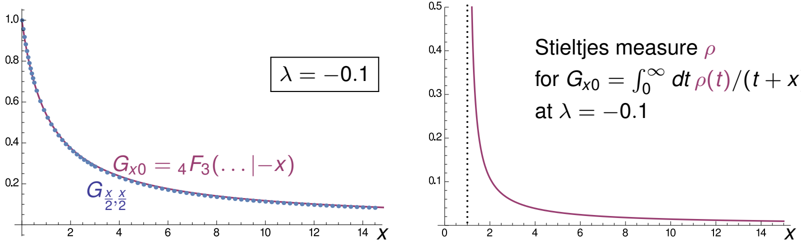
Which class of functions has desired positivity+holomorphicity and manageable integral transforms?

hypergeometric functions $G_{x0} = {}_nF_{n-1}\left(\begin{smallmatrix} a, b_1, \dots b_{n-1} \\ c_1, \dots, c_{n-1}\end{smallmatrix}\middle| -x\right)$ if $a \in [0,1]$ and $c_i > b_i > a$

- holomorphicity at $y > 0$: determine a, b_i, c_i by $G^{(k)}_{0y} = G^{(k)}_{y0}$
- find: $a = 1 + \frac{1}{\pi}\arcsin(\lambda\pi)$, $\prod_{i=1}^n \frac{c_i - 1}{b_i - 1} = \frac{\arcsin(\lambda\pi)}{\lambda\pi}$
- critical coupling constant is $\lambda_c = -\frac{1}{\pi} = -0.3183\ldots$
- also take $c_1 = 2$ (nicer Stieltjes and Hilbert transforms)

Källén-Lehmann spectrum

- matching conditions for $G_{x,0} = {}_4F_3(\dots|-x)$ at one point x result in global error $\sup_x |\dots| \approx 10^{-8}$ in fixed point equation



reflection positivity equivalent to existence of a blue curve on the right whose Stieltjes transform is $G_{\frac{x}{2},\frac{x}{2}}$ on the left

- measure for G_{x0} (and almost surely for $G_{\frac{x}{2},\frac{x}{2}}$) has mass gap $[0,1[$, but no further gap (remnant of UV/IR-mixing)
- absence of the second gap (usually $]1,4[$) circumvents triviality theorems

Summary

1. regularisation on extreme noncommutative space defines 4D-QFT that is exactly solvable in terms of fixed point problem
   - theory defined by quantum equations of motion (= Schwinger-Dyson equations), not by a measure
   - existence proved within small region
   - phase transitions and critical phenomena
2. projection to Schwinger functions for scalar field on $\mathbb{R}^4$ = confinement of noncommutativity
   - full Euclidean symmetry (completely unexpected)
   - no momentum exchange (close to triviality), possibly a consequence of integrability
   - numerical approach with tiny error: leaves no doubt that Schwinger 2-point function is reflection positive for $-\frac{1}{\pi}<\lambda\leq 0$
3. ready to embark on higher Schwinger functions

@ audience:

Thanks for listening!
If you like, please join us.

@ Gerd:

Take advandage of your free time to progress with your favourite Mathematical Physics problem.

Happy retirement!