

Mathematical Physics

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A difficult problem

Strong and weak interactions are mathematically incorrect.

More precisely: their rigorous construction is one of the most difficult problems of mathematics:

Clay Mathematics Institute Millennium Prize [10^6 \$] Problem

5. Yang-Mills Existence and Mass Gap

Prove that for any compact simple gauge group G , a non-trivial quantum Yang-Mills theory exists on \mathbb{R}^4 and has a mass gap $\Delta > 0$. Existence includes establishing axiomatic properties at least as strong as those of [Wightman, Osterwalder-Schrader].

There is not even a toy model of a 4D QFT!

Quantum geometry

Argument due to Wheeler (1950s), made precise by Doplicher-Fredenhagen-Roberts (1995):

Can we measure, in principle, structures of the size 10^{-60} cm?

NO!

- Need test particles of Compton wavelength $\lambda \approx 10^{-60}$ cm.
- These carry enormous energy, so large that they create a black hole with Schwarzschild horizon $\gg \lambda$!

Lesson from quantum physics:

Never use unobservables!

Our standard QFT violates this principle (admits arbitrarily small distances).

Need to develop **QFT on quantum geometries**.

First success stories

Take toy quantum geometry which admits computation.

Theorem (H. Grosse+RW, 2004)

- 1 4D scalar fields exist perturbatively **in finite volume**.
- 2 At one-loop there is no Landau ghost.

Theorem (Disertori-Gurau-Magnen-Rivasseau, 2006)

Assume **Planck volume = volume of universe**.

Then the **β -function is zero** to all orders in perturbation theory.

This is a precious result!

Immediate question: **Can we construct the model?**

Inside an atom of geometry

- Take toy quantum geometry
- Adjust $V = \text{volume of universe} = \text{volume of Planck cell}$

Quantum fields are infinite matrices (i.e. ∞ -many colours).

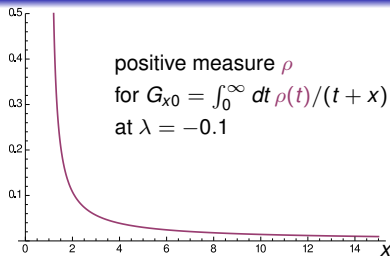
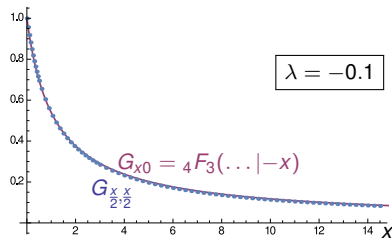
Theorem (H. Grosse+RW, 2012–2014)

- 1 The limit $V \rightarrow \infty$ of the 4D scalar Euclidean QFT is exactly solvable for any coupling constant $> -\frac{1}{\pi}$.
- 2 All correlation functions (which depend on continuous colour) are expressed in terms of the solution of a fixed point problem.
- 3 Diagonal correlation functions have interpretation as Schwinger functions in position space.
 - They have full Euclidean symmetry (of standard, not quantum!) 4D space.
 - They are blind to colour (confinement / darkness).

Work in progress: ① Time ② Quantum gravity

- ① QFT on space-time arises if Schwinger functions are reflection-positive.
 - Overwhelming numerical evidence and partial analytic proof that 2-point function is reflection positive.
 - Higher functions and non-triviality with Jan Schlemmer [SFB 878].
- ② This subject has inspired enormous progress with combinatorial quantum gravity.
Breakthrough by Razvan Gurau (2011), many followers
 - Jins de Jong [SFB 878]: construction of a quartic analogue of the Kontsevich model
 - Carlos Pérez-Sánchez [DAAD]: Schwinger-Dyson equations in coloured tensor models

Supplement: Reflection positivity



- Right: **positive measure ρ** supported on $]1, \infty[$
- Left: **red curve** is auxiliary function $G_{x_0} = \int_0^\infty \frac{dt \rho(t)}{t+x}$ which solves fixed point problem up to 10^{-8}
- Defines diagonal function $G_{\frac{x}{2}, \frac{x}{2}}$ (Schwinger function).
- Reflection positivity is existence on a **blue positive function ϱ** on the right (**the Källén-Lehmann mass spectrum**) such that $G_{\frac{x}{2}, \frac{x}{2}} = \int_0^\infty \frac{dt \varrho(t)}{t+\frac{x}{2}}$