

A solvable quantum field theory in 4 dimensions

Raimar Wolkenhaar

Mathematisches Institut, Westfälische Wilhelms-Universität Münster



(based on joint work with Harald Grosse,
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Introduction

- **Quantum field theory (QFT)** is the theory that describes Nature at very high energy density.
- One famous such experiment measures the magnetic moment g of the electron: $\frac{g_{\text{experiment}}}{2} = 1.001\,159\,652\,180\,7$
- QFT predicts that number in terms of the **electron charge e** measured to $e^{-2} = 137.035\,999\,084$:

$$\begin{aligned}
 \frac{g_{\text{QFT}}}{2} &= 1 + \frac{1}{2} \frac{e^2}{\pi} + \left\{ \frac{197}{144} + \left(\frac{1}{2} - 3 \log 2 \right) \zeta(2) + \frac{3}{4} \zeta(3) \right\} \left(\frac{e^2}{\pi} \right)^2 \\
 &+ \left\{ \frac{28259}{5184} + \left(\frac{17101}{235} - \frac{596}{2} \cdot \log 2 \right) \zeta(2) + \frac{139}{18} \zeta(3) + \frac{100}{3} \text{Li}_4 \left(\frac{1}{2} \right) \right. \\
 &+ \left. \frac{25}{3} \left(\frac{\log^4 2}{6} - \zeta(2) \log^2 2 \right) - \frac{239 \zeta(4) - 166 \zeta(2) \cdot \zeta(3) + 215 \zeta(5)}{24} \right\} \left(\frac{e^2}{\pi} \right)^3 \\
 &+ \left\{ \dots \right\} \left(\frac{e^2}{\pi} \right)^4 \\
 &= 1.001\,159\,652\,153\,5
 \end{aligned}$$

Problems

Clay Mathematics Institute Millennium Prize Problem (2000)

5. Yang-Mills Existence and Mass Gap

Prove that for any compact simple gauge group G , a **non-trivial quantum Yang-Mills theory exists on \mathbb{R}^4** and has a **mass gap $\Delta > 0$** . Existence includes establishing **axiomatic properties** at least as strong as those of **[Wightman, Osterwalder-Schrader]**.

- This problem is out of reach for our century.
- There is no solution for a far more modest problem:

Prove that a **non-trivial toy model** for a quantum field theory **on \mathbb{R}^4 exists and satisfies [Wightman, Osterwalder-Schrader]**.

- We have got a candidate for such a toy model.
Its construction involves several areas of mathematics.

Regularisation of $\lambda\phi_4^4$ on noncommutative space

$$S[\phi] = \int_{\mathbb{R}^4} \frac{dx}{64\pi^2} \left(\frac{1}{2} \phi (-\Delta + \mu^2) \phi + \frac{\lambda}{4} \phi \phi \phi \phi \right)(x)$$

Regularisation of $\lambda\phi_4^4$ on noncommutative space

$$S[\phi] = \int_{\mathbb{R}^4} \frac{dx}{64\pi^2} \left(\frac{Z}{2} \phi \star (-\Delta + \Omega^2 (2\Theta^{-1}x)^2 + \mu_{bare}^2) \phi + \frac{\lambda Z^2}{4} \phi \star \phi \star \phi \star \phi \right) (x)$$

with **Moyal product** $(f \star g)(x) = \int_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{dy dk}{(2\pi)^4} f(x + \frac{1}{2}\Theta k) g(x+y) e^{i(k,y)}$

takes at $\Omega = 1$ in matrix basis $f_{\underline{mn}}(x) = f_{m_1 n_1}(x^0, x^1) f_{m_2 n_2}(x^3, x^4)$

$$f_{mn}(y^0, y^1) = 2(-1)^m \sqrt{\frac{m!}{n!}} \left(\sqrt{\frac{2}{\theta}} y \right)^{n-m} L_m^{n-m} \left(\frac{2|y|^2}{\theta} \right) e^{-\frac{|y|^2}{\theta}}$$

due to $f_{\underline{mn}} \star f_{\underline{kl}} = \delta_{\underline{nk}} f_{\underline{ml}}$ and $\int dx f_{\underline{mn}}(x) = 64\pi^2 V \delta_{\underline{mn}}$ the form

$$S[\Phi] = V \left(\sum_{\underline{m}, \underline{n} \in \mathbb{N}_{\mathcal{N}}^2} E_{\underline{m}} \Phi_{\underline{mn}} \Phi_{\underline{nm}} + \frac{Z^2 \lambda}{4} \sum_{\underline{m}, \underline{n}, \underline{k}, \underline{l} \in \mathbb{N}_{\mathcal{N}}^2} \Phi_{\underline{mn}} \Phi_{\underline{nk}} \Phi_{\underline{kl}} \Phi_{\underline{lm}} \right)$$

$$E_{\underline{m}} = Z \left(\frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu_{bare}^2}{2} \right), \quad |\underline{m}| := m_1 + m_2 \leq \mathcal{N}$$

- $V = \left(\frac{\theta}{4}\right)^2$ is for $\Omega = 1$ the **volume** of the nc manifold.

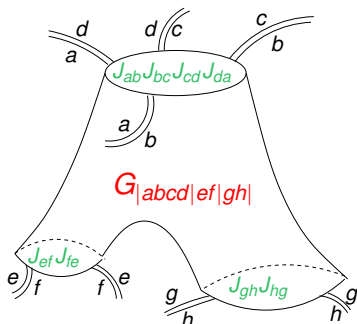
More generally: field-theoretical matrix models

Euclidean quantum field theory

- action $S[\Phi] = V \operatorname{tr}(E\Phi^2 + P[\Phi])$
for unbounded positive selfadjoint operator E with compact resolvent, and $P[\Phi]$ a polynomial
- partition function $\mathcal{Z}[J] = \int \mathcal{D}[\Phi] \exp(-S[\Phi] + V \operatorname{tr}(\Phi J))$
- For $P[\Phi] = \frac{i}{6}\Phi^3$ this is the **Kontsevich model** which computes the intersection theory on the moduli space of complex curves. We choose $P[\Phi] = \frac{\lambda}{4}\Phi^4$.
- Perturbative expansion $e^{V \operatorname{tr}(P[\Phi])} = \sum_{n=0}^{\infty} \frac{1}{n!} (V \operatorname{tr}(P[\Phi]))^n$ leads to **ribbon graphs**. They encode **genus- g** Riemann surface with **B boundary components**.
- We avoid the expansion, but keep the topological structure:

Topological expansion

- Choosing $E = \text{diag}(E_a)$, matrix index conserved along every strand.
- The k^{th} boundary component carries a cycle $J_{p_1 \dots p_{N_k}}^{N_k} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.



- Expand $\log \mathcal{Z}[\mathcal{J}] = \sum \frac{1}{S} V^{2-B} G_{|p_1^1 \dots p_{N_1}^1| \dots |p_1^B \dots p_{N_B}^B|} \prod_{\beta=1}^B J_{p_1^\beta \dots p_{N_\beta}^\beta}^{N_\beta}$ according to the cycle structure.
- QFT of matrix models determines the **weights of Riemann surfaces** with **decorated boundary components** compatible with
 - gluing (of fringes, not boundaries!)
 - covariance (under $\Phi \mapsto U^* \Phi U$, which is not a symmetry!)

Consistency conditions (for $S_{int}[\Phi] = \frac{\lambda}{4}\text{tr}(\Phi^4)$)

In a scaling limit $V \rightarrow \infty$ and $\frac{1}{V} \sum_{p \in I}$ finite, we have

1. A closed non-linear equation for $G_{|ab|}$

$$G_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a + E_b)} \frac{1}{V} \sum_{p \in I} \left(G_{|ab|} G_{|ap|} - \frac{G_{|pb|} - G_{|ab|}}{E_p - E_a} \right)$$

2. For $N \geq 4$ a universal algebraic recursion formula

$$G_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|} G_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G_{|b_{2l} b_1 \dots b_{2l-1}|} G_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

- scaling limit corresponds to restriction to genus $g = 0$
- similar formulae for $B \geq 2$
- no index summation in $G_{|abcd|} \Rightarrow \beta\text{-function zero!}$

Back to $\lambda\Phi_4^4$ on Moyal space

- Infinite volume limit (i.e. $\theta \rightarrow \infty$) turns discrete matrix indices into continuous variables $a, b, \dots \in \mathbb{R}_+$ and sums into integrals
- Need energy cutoff $a, b, \dots \in [0, \Lambda^2]$ and normalisation of lowest Taylor terms of two-point function $G_{|nm|} \mapsto G_{ab}$
- **Carleman-type singular integral equation** for $G_{ab} - G_{a0}$

Theorem (2012/13) (for $\lambda < 0$, using $G_{b0} = G_{0b}$)

Let $\mathcal{H}_a^\Lambda(f) = \frac{1}{\pi} \mathcal{P} \int_0^{\Lambda^2} \frac{f(p) dp}{p-a}$ be the *finite Hilbert transform*. Then

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\text{sign}(\lambda)(\mathcal{H}_0^\Lambda[\tau_0(\bullet)] - \mathcal{H}_a^\Lambda[\tau_b(\bullet)])}$$

where $\tau_b(a) := \arctan_{[0, \pi]} \left(\frac{|\lambda|\pi a}{b + \frac{1 + \lambda\pi a \mathcal{H}_p^\Lambda[G_{\bullet 0}]}{G_{a0}}} \right)$ and G_{0b} solution of

$$G_{b0} = G_{0b} = \frac{1}{1+b} \exp \left(-\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1 + \lambda\pi p \mathcal{H}_p^\Lambda[G_{\bullet 0}]}{G_{p0}} \right)^2} \right)$$

Discussion

Together with explicit (but **complicated** for $G_{ab|cd}$, $G_{ab|cd|ef}$, ...) formulae for higher correlation functions, we have **exact solution** of $\lambda\phi_4^4$ on extreme Moyal space in terms of

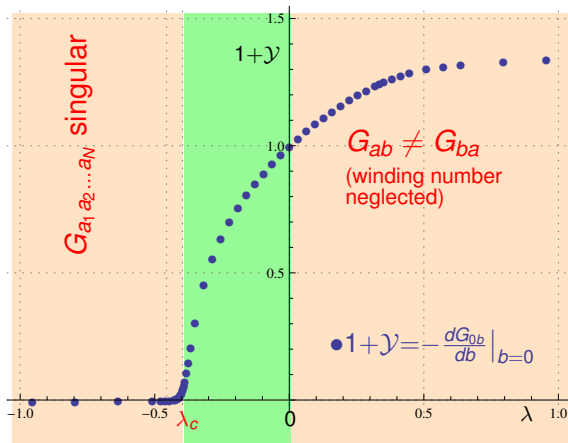
$$G_{b0} = G_{0b} = \frac{1}{1+b} \exp \left(-\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1 + \lambda\pi p \tau_p^\Lambda [G_{\bullet 0}]}{G_{p0}} \right)^2} \right)$$

Possible treatments

- 1 Perturbative solution: **reproduces all Feynman graphs**, generates **polylogarithms and ζ -functions**.
Cannot expect convergence: τ_b not small for $\lambda \searrow 0$
- 2 Iterative solution on computer: **nicely convergent**, find interesting phase structure
- 3 **Rigorous existence proof** of a solution, uniqueness still open (but obvious for computer)

Computer simulation: evidence for phase transitions

piecewise linear approximation of G_{0b} , G_{ab} for $\Lambda^2=10^7$ and 2000 sample points. Consider $1+\mathcal{Y} := -\left.\frac{dG_{0b}}{db}\right|_{b=0}$



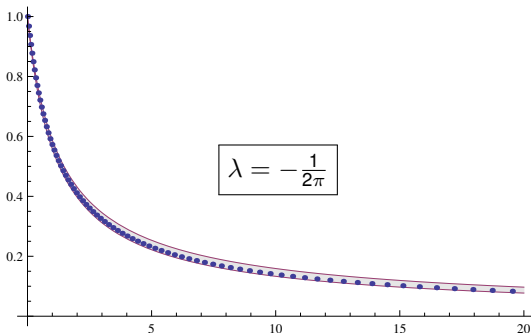
- $(1 + \mathcal{Y})'(\lambda)$ discontinuous at $\lambda_c = -0.39$
- order parameter $b_\lambda = \sup\{b : G_{0b}=1\}$ non-zero for $\lambda < \lambda_c$
- A key property for Schwinger functions is realised in $]\lambda_c, 0]$, not outside!
The critical couplings coincide!

Fixed point theorem

Theorem (2015)

Let $-\frac{1}{6} \leq \lambda \leq 0$. Then the equation has a C_0^1 -solution

$$\frac{1}{(1+b)^{1-|\lambda|}} \leq G_0 b \leq \frac{1}{(1+b)^{1-\frac{|\lambda|}{1-2|\lambda|}}}$$



Proof via **Schauder fixed point theorem**.

This involves **continuity and compactness** of a certain operator (in norm topology)

Relativistic and Euclidean quantum field theory

- We view QFT as defined by **Wightman's axioms** for distributions $\mathcal{W}_N(x_1, \dots, x_N) = W_N(x_1 - x_2, \dots, x_{N-1} - x_N)$
- Theorem: The W_N are **boundary values of holomorphic functions** (on permuted extended forward tube $\subset \mathbb{C}^{4(N-1)}$)
- Restriction of W_N to real subspace of **Euclidean points** (minus diagonals) defines **Schwinger functions**
- Schwinger functions inherit properties such as real analyticity, Euclidean invariance and **complete symmetry**
- Hence, moments of probability distributions provide candidate Schwinger functions ([link to statistical physics](#))

Theorem (Osterwalder-Schrader, 1974)

One additional requirement, **reflection positivity**, leads back to **Wightman theory**

From matrix model to Schwinger functions on \mathbb{R}^4

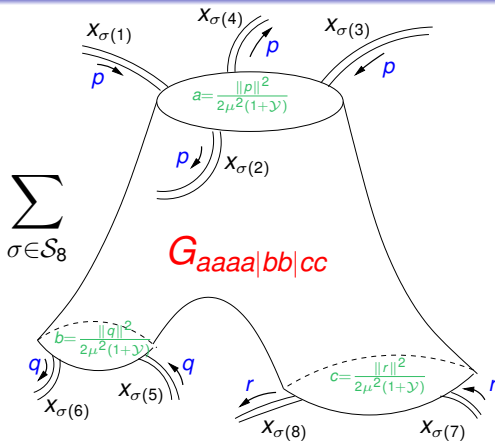
reverting harmonic oscillator basis \blacktriangleright , $1 + \mathcal{Y} := -\frac{dG_{0b}}{db} \Big|_{b=0} \dots$

Theorem (2013): *connected* Schwinger functions

$$\begin{aligned}
 S_c(\mu X_1, \dots, \mu X_N) &= \frac{1}{64\pi^2} \sum_{\substack{N_1 + \dots + N_B = N \\ N_\beta \text{ even}}} \sum_{\sigma \in S_N} \left(\prod_{\beta=1}^B \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4 p_\beta}{4\pi^2 \mu^4} e^{i \langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta} (-1)^{i-1} \mu X_{\sigma(N_1 + \dots + N_{\beta-1} + i)} \rangle} \right) \\
 &\quad \times \mathbf{G} \underbrace{\left(\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})} \right)}_{N_1} \dots \underbrace{\left(\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})} \right)}_{N_B}
 \end{aligned}$$

- Schwinger functions are symmetric and **invariant under full Euclidean group** (completely unexpected for NCQFT!)
- remains: **reflection positivity**
- finally: Is it **non-trivial**?

Connected (4+2+2)-point function



- 1 individual Euclidean symmetry in every boundary component (no clustering)
- 2 particle scattering without momentum exchange
 - in 4D a sign of **triviality** (mind assumptions!)
 - familiar in 2D models with **factorising S-matrix**
 - a consequence of **integrability** [Moser, 1975] & [Kulish, 1976]

Is there a precise link between **exact solution of our 4D model** and **traditional integrability** in 2D? What about Yang-Baxter?

Osterwalder-Schrader reflection positivity

Proposition (2013)

$S(x_1, x_2)$ is reflection positive iff $a \mapsto G_{aa}$ is a **Stieltjes function**,

$$G_{aa} = \int_0^\infty \frac{d(\rho(t))}{a+t}, \quad \rho - \text{positive measure.}$$

- **Excluded for any $\lambda > 0$** (due to renormalisation)!

- Purely real conditions [Widder, 1938]:

$$L_{k,t}[f(\bullet)] := \frac{(-t)^{k-1}}{c_k} \frac{d^{2k-1}}{dt^{2k-1}} (t^k f(t)) \geq 0, \quad c_1=1, c_{k>1}=k!(k-2)!$$

→ lowest orders verified numerically

- Complex conditions [Krein]:

- 1 $f(t) \geq 0$ for $t > 0$
- 2 $f : \mathbb{C} \setminus]-\infty, 0] \rightarrow \mathbb{C}$ holomorphic
- 3 $\text{Im}(f(x+iy)) < 0$ for $y > 0$ (anti-Herglotz)

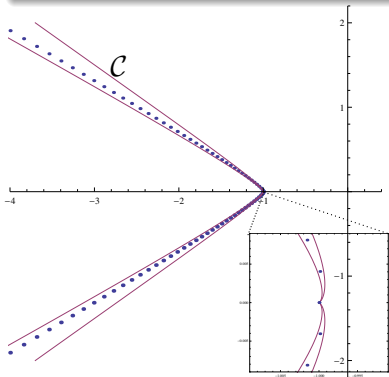
→ work in progress

Analyticity of the 2-point function

Proposition

Fixed point equation $\Leftrightarrow \frac{d}{dz} \log G_{0b} = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(t) dt}{t-z},$

where \mathcal{C}, f depend on G_{0b} with $b \in \mathbb{R}^+$



Theorem

$z \mapsto G_{0z}$ holomorphic and
anti-Herglotz right of \mathcal{C}

remaining steps:

- analytic continuation beyond \mathcal{C}
- prove same results for $z \mapsto G_{zz}$

Summary

- ① $\lambda\phi_4^4$ on nc Moyal space is, at infinite noncommutativity, **exactly solvable** in terms of a fixed point problem
 - $\lambda < 0$ is the **right choice!** (gives positive anomalous dimension – a renormalisation effect)
 - theory **defined by quantum equations of motion** (= Schwinger-Dyson equations), **not by a measure**
 - **existence proved** within small region, uniqueness open
 - **phase transitions and critical phenomena**, hence interesting statistical physics model
- ② Projection to **Schwinger functions for scalar field on \mathbb{R}^4** :
 - **full Euclidean symmetry** (completely unexpected)
 - no clustering
 - **no momentum exchange** (close to triviality), **possibly a consequence of integrability**

Summary (II) and Outlook

- ③ **Reflection positivity** of 2-point function equivalent to Stieltjes functions
 - real condition holds numerically to low order
 - **complex conditions proved in domain right of curve \mathcal{C}**

Next steps

- Ⓐ **extension to $\mathbb{C} \setminus]-\infty, -1]$**
(true for boundaries of fixed point solution)
- Ⓑ **positivity of $2+ \dots +2$ -point functions:** not impossible
(2-point function contains them as subgraphs)
... easy if fixed point solution known explicitly
- Ⓒ **non-triviality:** probably true!
 $\lambda < 0$ expected to violate Nelson-Symanzik positivity,
but the free field has it [due to J. Schlemmer]