A solvable quantum field theory in 4 dimensions

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(based on joint work with Harald Grosse,
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Introduction

Clay Mathematics Institute Millennium Prize Problem (2000)

5. Yang-Mills Existence and Mass Gap

Prove that for any compact simple gauge group G, a non-trivial quantum Yang-Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$. Existence includes establishing axiomatic properties at least as strong as those of [Wightman, Osterwalder-Schrader].

- This problem is out of reach for our century.
- There is no solution for a far more modest problem:

Prove that a non-trivial toy model for a quantum field theory on $\mathbb{R}^4$ exists and satisfies [Wightman, Osterwalder-Schrader].

Vanishing β-function

It is probably a good idea to try first QFT models with vanishing β-function. They are nice both in UV and IR.

Candidates

1. $\mathcal{N} = 4$ super Yang-Mills theory
   - very active subject with many strong results
   - Wightman axioms are not made for gauge theory, but $\mathcal{N} = 4$ SYM might suggest reasonable axioms
2. $\lambda\phi^4_4$-theory on noncommutative Moyal space
   - perturbatively renormalisable (Grosse-W. 2004)
   - β-function vanishes to all orders in perturbation theory (Disertori-Gurau-Magnen-Rivasseau, 2006)
   - model is relatively simple

$\rightarrow$ Can we construct it? ... YES (as statistical physics model)
To our big surprise, Wightman axioms seem to be satisfied!

Regularisation of $\lambda\phi_4^4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4}\frac{dx}{64\pi^2}\Big(\frac{1}{2}\phi\ \big(-\Delta \qquad\qquad\qquad + \mu^2 \quad\big)\phi+\frac{\lambda}{4}\phi\ \ \phi\ \ \phi\ \ \phi\Big)(x)$$

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4}\frac{dx}{64\pi^2}\Big(\frac{Z}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2_{bare})\phi+\frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with Moyal product $(f\star g)(x)=\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dy\,dk}{(2\pi)^4}f(x+\tfrac{1}{2}\Theta k)\,g(x+y)\,e^{\mathrm{i}\langle k,y\rangle}$

takes at $\Omega=1$ in matrix basis $f_{\underline{mn}}(x)=f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\tfrac{2}{\theta}}y\Big)^{n-m}L^{n-m}_m\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

due to $f_{\underline{mn}}\star f_{\underline{kl}}=\delta_{\underline{nk}}f_{\underline{ml}}$ and $\int dx\,f_{\underline{mn}}(x)=64\pi^2V\delta_{\underline{mn}}$ the form

$$S[\Phi]=V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}E_{\underline{m}}\Phi_{\underline{mn}}\Phi_{\underline{nm}}+\frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Big)$$

$$E_{\underline{m}}=Z\Big(\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu^2_{bare}}{2}\Big),\qquad |\underline{m}|:=m_1+m_2\le\mathcal{N}$$

- $V=\left(\frac{\theta}{4}\right)^2$ is for $\Omega=1$ the volume of the nc manifold.

More generally: field-theoretical matrix models

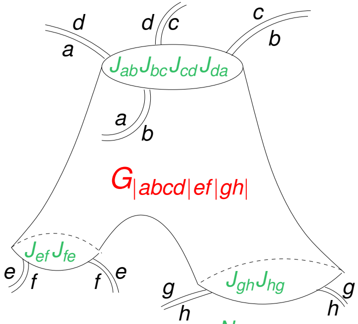
Euclidean quantum field theory

- action $S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + P[\Phi])$
  for unbounded positive selfadjoint operator E with compact resolvent, and $P[\Phi]$ a polynomial
- partition function $\mathcal{Z}[J] = \int \mathcal{D}[\Phi]\ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))$

- For $P[\Phi] = \frac{\mathrm{i}}{6}\Phi^3$ this is the Kontsevich model which computes the intersection theory on the moduli space of complex curves. We choose $P[\Phi] = \frac{\lambda}{4}\Phi^4$.
- Perturbative expansion $e^{-V\,\mathrm{tr}(P[\Phi])} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(V\,\mathrm{tr}(P[\Phi])^n$ leads to ribbon graphs. They encode genus-g Riemann surface with B boundary components.
- We avoid the expansion, but keep the topological structure:

Topological expansion

- Choosing $E = \mathrm{diag}(E_a)$, matrix index conserved along every strand.
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1\dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.



- Expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p^1_1\dots p^1_{N_1}|\dots|p^B_1\dots p^B_{N_B}|} \prod_{\beta=1}^{B} J^{N_\beta}_{p^\beta_1\dots p^\beta_{N_\beta}}$ according to the cycle structure.
- QFT of matrix models determines the weights of Riemann surfaces with decorated boundary components compatible with
  1. gluing (of fringes, not boundaries!)
  2. covariance (under $\Phi \mapsto U^* \Phi U$, which is not a symmetry!)

Ward identity

Proposition [Disertori-Gurau-Magnen-Rivasseau, 2006]

$$0 = \sum_{n\in I} \Big(\frac{(E_a - E_p)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \Big)$$

Theorem [Grosse-W. 2012]

$$\begin{aligned}
\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an} \partial J_{np}} = \delta_{ap} \Big\{ & V^2 \sum_{(K)} \frac{J_{P_1} \cdots J_{P_K}}{S_K} \Big(\sum_{n\in I} \frac{G_{|an|P_1|\dots|P_K|}}{V^{|K|+1}} + \frac{G_{|a|a|P_1|\dots|P_K|}}{V^{|K|+2}} \\
& + \sum_{r\geq 1} \sum_{q_1,\dots,q_r \in I} \frac{G_{|q_1 a q_1 \dots q_r|P_1|\dots|P_K|} J^r_{q_1\dots q_r}}{V^{|K|+1}} \Big) \\
& + V^4 \sum_{(K),(K')} \frac{J_{P_1} \cdots J_{P_K} J_{Q_1} \cdots J_{Q_{K'}}}{S_K S_{K'}} \frac{G_{|a|P_1|\dots|P_K|}}{V^{|K|+1}} \frac{G_{|a|Q_1|\dots|Q_{K'}|}}{V^{|K'|+1}} \Big\} \mathcal{Z}[J] \\
& + \frac{V}{E_p - E_a} \sum_{n\in I} \Big(J_{pn} \frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}[J]}{\partial J_{np}} \Big)
\end{aligned}$$

- in Schwinger-Dyson equations resulting from $\frac{\partial^N \mathcal{Z}}{\partial J \dots \partial J}$, two J-derivatives are killed

Schwinger-Dyson equations (for $S_{int}[\Phi] = \frac{\lambda}{4}\mathrm{tr}(\Phi^4)$)

In a scaling limit $V \to \infty$ and $\frac{1}{V}\sum_{p\in I}$ finite, we have:

1. A closed non-linear equation for $G_{|ab|}$

$$G_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a + E_b)} \frac{1}{V} \sum_{p\in I} \Big(G_{|ab|} G_{|ap|} - \frac{G_{|pb|} - G_{|ab|}}{E_p - E_a} \Big)$$

2. For $N \geq 4$ a universal algebraic recursion formula

$$G_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|} G_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G_{|b_{2l} b_1 \dots b_{2l-1}|} G_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

- scaling limit corresponds to restriction to genus $g = 0$
- similar formulae for $B \geq 2$
- no index summation in $G_{|abcd|}$ $\Rightarrow$ β-function zero!

Back to $\lambda\Phi^4_4$ on Moyal space

- Infinite volume limit (i.e. $\theta \to \infty$) turns discrete matrix indices into continuous variables $a, b, \cdots \in \mathbb{R}_+$ and sums into integrals
- Need energy cutoff $a, b, \cdots \in [0, \Lambda^2]$ and normalisation of lowest Taylor terms of two-point function $G_{|nm|} \mapsto G_{ab}$
- Carleman-type singular integral equation for $G_{ab} - G_{a0}$

Theorem (2012/13) (for $\lambda < 0$, using $G_{b0} = G_{0b}$)

Let $\mathcal{H}^\Lambda_a(f) = \frac{1}{\pi}\mathcal{P}\int_0^{\Lambda^2} \frac{f(p)\,dp}{p-a}$ be the *finite Hilbert transform*. Then

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\mathrm{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)] - \mathcal{H}^\Lambda_a[\tau_b(\bullet)])}$$

where $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a \mathcal{H}^\Lambda_a[G_{\bullet 0}]}{G_{a0}}}\right)$ and G_{a0} solution of

$$G_{b0} = G_{0b} = \frac{1}{1+b}\exp\left(-\lambda\int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

Discussion

Together with explicit (but complicated for $G_{ab|cd}$, $G_{ab|cd|ef}$, ...) formulae for higher correlation functions, we have exact solution of $\lambda\phi^4_4$ on extreme Moyal space in terms of

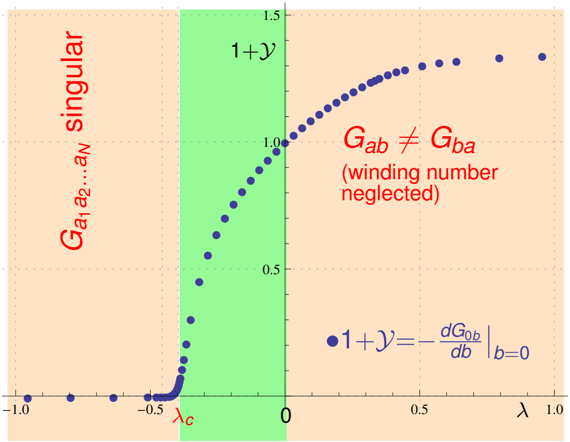
$$G_{b0} = G_{0b} = \frac{1}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}^{\Lambda}_p[G_{\bullet 0}]}{G_{p0}}\right)^2} \right)$$

Possible treatments

1. perturbative solution: reproduces all Feynman graphs, generates polylogarithms and ζ-functions
2. iterative solution on computer: nicely convergent, find interesting phase structure
3. rigorous existence proof of a solution
4. work in progress: try to guess the solution; should give uniqueness as by-product

Computer simulation: evidence for phase transitions

piecewise linear approximation of G_{0b}, G_{ab} for $\Lambda^2 = 10^7$ and 2000 sample points. Consider $1+\mathcal{Y} := -\frac{dG_{0b}}{db}\Big|_{b=0}$



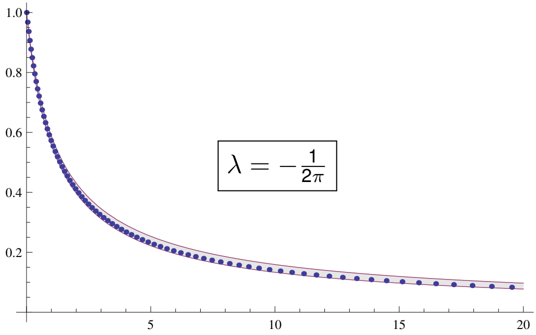
- $(1+\mathcal{Y})'(\lambda)$ discontinuous at $\lambda_c = -0.39$
- order parameter $b_\lambda = \sup\{b : G_{0b} = 1\}$ non-zero for $\lambda < \lambda_c$
- A key property for Schwinger functions is realised in $]\lambda_c, 0]$, not outside!
  The critical couplings coincide!

Fixed point theorem

Theorem (2015)

Let $-\frac{1}{6} \leq \lambda \leq 0$. Then the equation has a C^1_0-solution

$$\frac{1}{(1+b)^{1-|\lambda|}} \leq G_{0b} \leq \frac{1}{(1+b)^{1-\frac{|\lambda|}{1-2|\lambda|}}}$$



Proof via Schauder fixed point theorem.

This involves continuity and compactness of a certain operator (in norm topology)

Relativistic and Euclidean quantum field theory

- We view QFT as defined by Wightman's axioms for distributions $\mathcal{W}_N(x_1, \dots, x_N) = W_N(x_1 - x_2, \dots, x_{N-1} - x_N)$
- Theorem: The W_N are boundary values of holomorphic functions (on permuted extended forward tube $\subset \mathbb{C}^{4(N-1)}$)
- Restriction of W_N to real subspace of Euclidean points (minus diagonals) defines Schwinger functions
- Schwinger functions inherit properties such as real analyticity, Euclidean invariance and complete symmetry
- Hence, moments of probability distributions provide candidate Schwinger functions (link to statistical physics)

Theorem (Osterwalder-Schrader, 1974)

One additional requirement, reflection positivity, leads back to Wightman theory

From matrix model to Schwinger functions on $\mathbb{R}^4$

reverting harmonic oscillator basis, $1+\mathcal{Y} := -\frac{dG_{0b}}{db}\big|_{b=0} \dots$

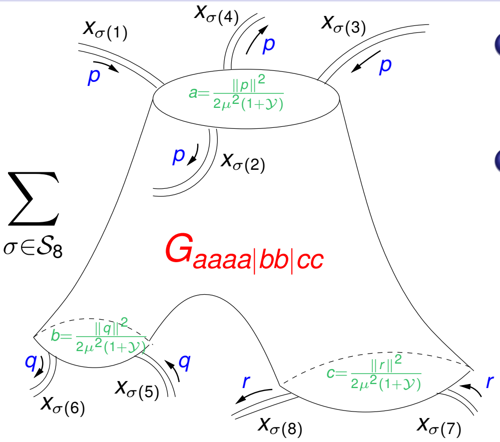
Theorem (2013): *connected* Schwinger functions

$$S_c(\mu x_1,\dots,\mu x_N) = \frac{1}{64\pi^2} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \Big(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4 p_\beta}{4\pi^2\mu^4} e^{\mathrm{i}\big\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta} (-1)^{i-1} \mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)}\big\rangle}\Big) \times G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1}\Big|\dots\Big|\underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}}$$

Confinement of noncommutativity: have internal interaction of matrices; commutative subsector propagates to outside world

- Schwinger functions are symmetric and invariant under full Euclidean group (completely unexpected for NCQFT!)
- remains: reflection positivity
- finally: Is it non-trivial?

Connected (4+2+2)-point function



1. individual Euclidean symmetry in every boundary component (no clustering)
2. particle scattering without momentum exchange
   - in 4D a sign of triviality (mind assumptions!)
   - familiar in 2D models with factorising S-matrix
   - a consequence of integrability [Moser, 1975] & [Kulish, 1976]

Is there a precise link between exact solution of our 4D model and traditional integrability known from 2D?

Osterwalder-Schrader reflection positivity

Proposition (2013)

$S(x_1, x_2)$ is reflection positive iff $a \mapsto G_{aa}$ is a Stieltjes function,

$$G_{aa} = \int_0^\infty \frac{d(\rho(t))}{a+t}, \qquad \rho \text{ – positive measure.}$$

Excluded for any $\lambda > 0$ (unless rescued by winding number)

- naïve anomalous dimension η positive for $\lambda > 0$, but η diverges to $+\infty$
- renormalisation oversubtracts: η_{ren}, λ of opposite sign
- model-independent: p-space 2-point function $\frac{1}{(p^2+m^2)^{1-\eta/2}}$
  Positivity and convergence contradict each other!
- Need (analytical?) continuation between
  – one regime where existence can be proved and
  – another regime where positivity holds.

Dismiss the partition function

Partition function is merely a guiding principle to identify the quantum equations of motion (and combinatorics).

These Schwinger-Dyson equations, and not the partition function, define the field theory!

Non-triviality (proposed by J. Schlemmer)

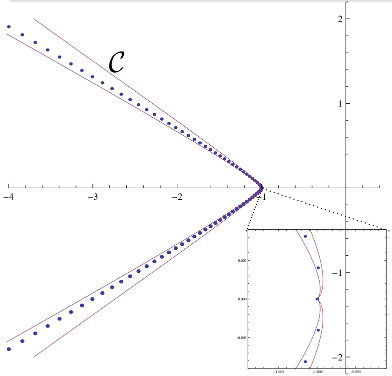
- *If* our Schwinger functions do *not* arise from a Euclidean measure, there is no reason they satisfy Nelson-Symanzik positivity.
- Since the free field is Nelson-Symanzik positive, our Schwinger functions cannot be trivial.

Analyticity of the 2-point function

Proposition

Fixed point equation $\Leftrightarrow$ $\displaystyle \frac{d}{dz}\log G_{0z} = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(t)\,dt}{t-z}$,

where $\mathcal{C}$, f depend on G_{0b} with $b \in \mathbb{R}^+$



Theorem

$z \mapsto G_{0z}$ holomorphic and anti-Herglotz right of $\mathcal{C}$

remaining steps:

- analytic continuation beyond $\mathcal{C}$
- prove same results for $z \mapsto G_{zz}$

Reflection positivity simplifies the problem

If G_{x0} is Stieltjes, then Hilbert transform can be avoided:

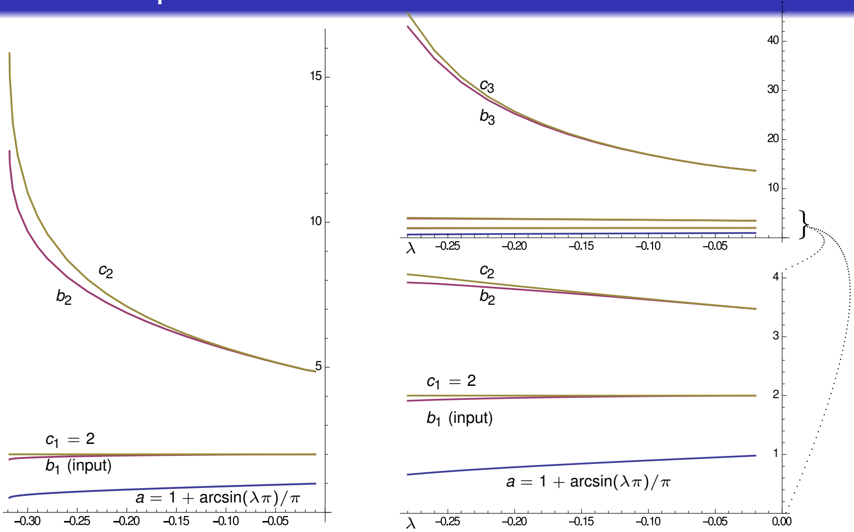
$$\frac{G_{xy}}{G_{x0}} = \exp\left(-\frac{1}{\pi}\int_1^\infty \frac{dt}{t+x} \underset{[0,\pi]}{\arctan}\left(\frac{y\,\mathrm{Im}(G_{-(t+\mathrm{i}\epsilon),0})}{1-\lambda t\int_0^\infty ds\,\frac{G_{s0}}{t+s} + y\,\mathrm{Re}(G_{-(t+\mathrm{i}\epsilon),0})}\right)\right)$$

Which class of functions has desired analyticity+holomorphicity and manageable integral transforms?

hypergeometric functions $G_{x0} = {}_nF_{n-1}\left(\begin{smallmatrix} a, b_1, \dots b_{n-1} \\ c_1, \dots, c_{n-1}\end{smallmatrix}\middle| -x\right)$ if $a \in [0,1]$ and $c_i > b_i > a$

- holomorphicity at $y > 0$: determine a, b_i, c_i by $G^{(k)}_{0y} = G^{(k)}_{y0}$
- find: $a = 1 + \frac{1}{\pi}\arcsin(\lambda\pi)$, $\qquad \prod_{i=1}^n \frac{c_i - 1}{b_i - 1} = \frac{\arcsin(\lambda\pi)}{\lambda\pi}$
- critical coupling constant is $\lambda_c = -\frac{1}{\pi} = -0.3183\dots$
- also take $c_1 = 2$ (nicer Stieltjes and Hilbert transforms)

Parameter spectrum



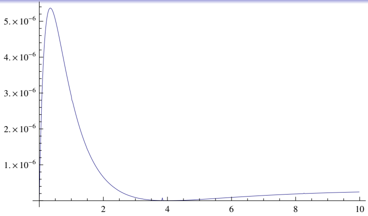
for $G(x,0) = {}_3F_2\left(\begin{smallmatrix} a,b_1,b_2 \\ c_1,c_2 \end{smallmatrix}\middle| -x\right)$ for $G(x,0) = {}_4F_3\left(\begin{smallmatrix} a,b_1,b_2,b_3 \\ c_1,c_2,c_3 \end{smallmatrix}\middle| -x\right)$

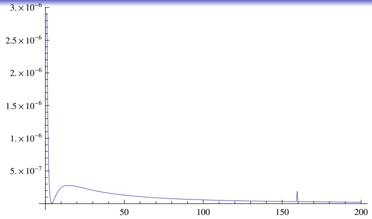
Convergence of the method

${}_3F_2$

at $\lambda = -0.1$

$G^{(0..1)}_{4,0} = G^{(0..1)}_{0,4}$

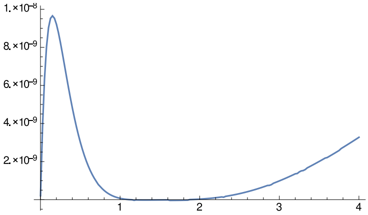


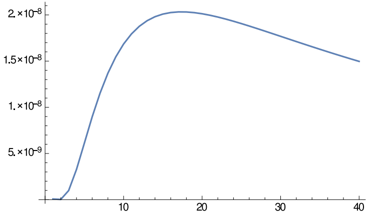


${}_4F_3$

at $\lambda = -0.1$

$G^{(0..3)}_{\frac{3}{2},0} = G^{(0..3)}_{0,\frac{3}{2}}$

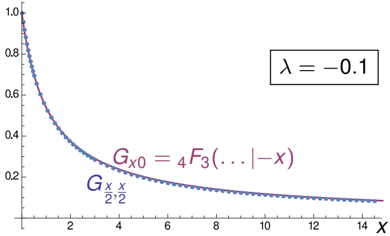


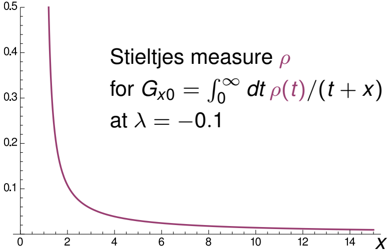


- improvement by factor ~ 250
- ready to compute the Schwinger 4-point function!

Källén-Lehmann spectrum

- these results make is completely clear (but don't prove) that G_{x0} is Stieltjes
- reflection positivity equivalent to G_{xx} a Stieltjes function
- the shape makes this plausible:





- measure for G_{x0} has mass gap $[0,1[$, but no further gap (remnant of UV/IR-mixing)
- absence of the second gap (usually $]1,4[$) circumvents triviality theorems

Summary

1. $\lambda\phi^4_4$ on nc Moyal space is, at infinite noncommutativity, exactly solvable in terms of a fixed point problem
   - theory defined by quantum equations of motion (= Schwinger-Dyson equations), not by a measure
   - existence proved within small region
   - phase transitions and critical phenomena
2. Projection to Schwinger functions for scalar field on $\mathbb{R}^4$ = confinement of noncommutativity
   - full Euclidean symmetry (completely unexpected)
   - no momentum exchange (close to triviality), possibly a consequence of integrability
   - numerical approach with tiny error: leaves no doubt that Schwinger 2-point function is reflection positive for $-\frac{1}{\pi}<\lambda\leq 0$
3. ready to embark on higher Schwinger functions