Exact solution of a four-dimensional field theory

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(based on joint work with Harald Grosse,
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Introduction

- Quantum field theory (QFT) is the theory that describes Nature at very high energy density.
- One famous such experiment measures the magnetic moment g of the electron: $\frac{g_{experiment}}{2} = 1.001\,159\,652\,180\,7$
- QFT predicts that number in terms of the electron charge e measured to $e^{-2} = 137.035\,999\,084$:

$$
\begin{aligned}
\frac{g_{QFT}}{2} &= 1 + \frac{1}{2}\frac{e^2}{\pi} + \Big\{\frac{197}{144} + \Big(\frac{1}{2} - 3\log 2\Big)\zeta(2) + \frac{3}{4}\zeta(3)\Big\}\Big(\frac{e^2}{\pi}\Big)^2 \\
&+ \Big\{\frac{28259}{5184} + \Big(\frac{17101}{235} - \frac{596}{2}\cdot\log 2\Big)\zeta(2) + \frac{139}{18}\zeta(3) + \frac{100}{3}\mathrm{Li}_4\Big(\frac{1}{2}\Big) \\
&+ \frac{25}{3}\Big(\frac{\log^4 2}{6} - \zeta(2)\log^2 2\Big) - \frac{239\zeta(4) - 166\zeta(2)\cdot\zeta(3) + 215\zeta(5)}{24}\Big\}\Big(\frac{e^2}{\pi}\Big)^3 \\
&+ \Big\{\dots\Big\}\Big(\frac{e^2}{\pi}\Big)^4 \\
&= 1.001\,159\,652\,153\,5
\end{aligned}
$$

Problems

Clay Mathematics Institute Millennium Prize Problem (2000)

5. Yang-Mills Existence and Mass Gap

Prove that for any compact simple gauge group G, a non-trivial quantum Yang-Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$. Existence includes establishing axiomatic properties at least as strong as those of [Wightman, Osterwalder-Schrader].

- This problem is out of reach for our century.
- There is no solution for a far more modest problem:

Prove that a non-trivial toy model for a quantum field theory on $\mathbb{R}^4$ exists and satisfies [Wightman, Osterwalder-Schrader].

- We have got a candidate for such a toy model.
  Its construction involves several areas of mathematics.

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4}\frac{dx}{64\pi^2}\Big(\frac{1}{2}\phi\;\big(-\Delta\qquad\qquad\qquad +\mu^2\quad\big)\phi+\frac{\lambda}{4}\phi\;\;\phi\;\;\phi\;\;\phi\Big)(x)$$

Regularisation of $\lambda\phi^4_4$ on noncommutative space

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Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4}\frac{dx}{64\pi^2}\Big(\frac{1}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2\quad)\phi+\frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with Moyal product $(f\star g)(x)=\displaystyle\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dy\,dk}{(2\pi)^4}f(x+\tfrac{1}{2}\Theta k)\,g(x+y)\,e^{\mathrm{i}\langle k,y\rangle}$

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4}\frac{dx}{64\pi^2}\Big(\frac{Z}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2_{bare})\phi+\frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with Moyal product $(f\star g)(x)=\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dy\,dk}{(2\pi)^4}f(x+\tfrac{1}{2}\Theta k)\,g(x+y)\,e^{\mathrm{i}\langle k,y\rangle}$

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matrix basis $f_{\underline{m}\underline{n}}(x)=f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\tfrac{2}{\theta}}y\Big)^{n-m}L^{n-m}_m\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

due to $f_{\underline{m}\underline{n}}\star f_{\underline{k}\underline{l}}=\delta_{\underline{n}\underline{k}}f_{\underline{m}\underline{l}}$ and $\int dx\,f_{\underline{m}\underline{n}}(x)=64\pi^2V\delta_{\underline{m}\underline{n}}$

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4}\frac{dx}{64\pi^2}\Big(\frac{Z}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2_{bare})\phi+\frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with Moyal product $(f\star g)(x)=\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dy\,dk}{(2\pi)^4}f(x+\tfrac{1}{2}\Theta k)\,g(x+y)\,e^{\mathrm{i}\langle k,y\rangle}$

takes at $\Omega=1$ in matrix basis $f_{\underline{mn}}(x)=f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\tfrac{2}{\theta}}y\Big)^{n-m}L^{n-m}_m\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

due to $f_{\underline{mn}}\star f_{\underline{kl}}=\delta_{\underline{nk}}f_{\underline{ml}}$ and $\int dx\,f_{\underline{mn}}(x)=64\pi^2V\delta_{\underline{mn}}$ the form

$$S[\Phi]=V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}E_{\underline{m}}\Phi_{\underline{mn}}\Phi_{\underline{nm}}+\frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Big)$$

$$E_{\underline{m}}=Z\Big(\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu^2_{bare}}{2}\Big)\,,\qquad |\underline{m}|:=m_1+m_2\leq\mathcal{N}$$

- $V=\big(\frac{\theta}{4}\big)^2$ is for $\Omega=1$ the volume of the nc manifold.

More generally: field-theoretical matrix models

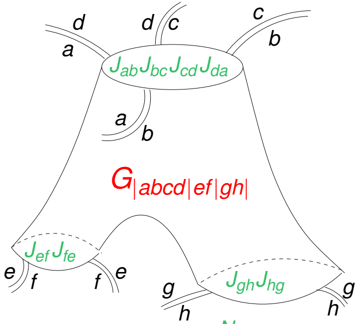
Euclidean quantum field theory

- action $S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + P[\Phi])$
  for unbounded positive selfadjoint operator E with compact resolvent, and $P[\Phi]$ a polynomial
- partition function $\mathcal{Z}[J] = \int \mathcal{D}[\Phi]\ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))$

- For $P[\Phi] = \frac{\mathrm{i}}{6}\Phi^3$ this is the Kontsevich model which computes the intersection theory on the moduli space of complex curves. We choose $P[\Phi] = \frac{\lambda}{4}\Phi^4$.
- Perturbative expansion $e^{V\,\mathrm{tr}(P[\Phi])} = \sum_{n=0}^{\infty} \frac{1}{n!}(V\,\mathrm{tr}(P[\Phi])^n$ leads to ribbon graphs. They encode genus-g Riemann surface with B boundary components.
- We avoid the expansion, but keep the topological structure:

Topological expansion

- Choosing $E = \mathrm{diag}(E_a)$, matrix index conserved along every strand.
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1\dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.



- Expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p^1_1\dots p^1_{N_1}|\dots|p^B_1\dots p^B_{N_B}|} \prod_{\beta=1}^{B} J^{N_\beta}_{p^\beta_1\dots p^\beta_{N_\beta}}$ according to the cycle structure.
- QFT of matrix models determines the weights of Riemann surfaces with decorated boundary components compatible with
  1. gluing (of fringes, not boundaries!)
  2. covariance (under $\Phi \mapsto U^* \Phi U$, which is not a symmetry!)

Consistency conditions (for $S_{int}[\Phi] = \frac{\lambda}{4}\mathrm{tr}(\Phi^4)$)

In a scaling limit $V \to \infty$ and $\frac{1}{V}\sum_{p\in I}$ finite, we have

1. A closed non-linear equation for $G_{|ab|}$

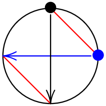
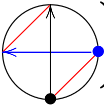
$$G_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a + E_b)} \frac{1}{V} \sum_{p\in I} \Big(G_{|ab|} G_{|ap|} - \frac{G_{|pb|} - G_{|ab|}}{E_p - E_a} \Big)$$

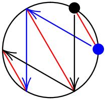
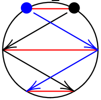
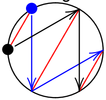
2. For $N \geq 4$ a universal algebraic recursion formula

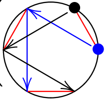
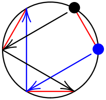
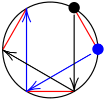
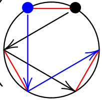
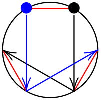
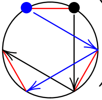
$$G_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|} G_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G_{|b_{2l} b_1 \dots b_{2l-1}|} G_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

- scaling limit corresponds to restriction to genus $g = 0$
- similar formulae for $B \geq 2$
- no index summation in $G_{|abcd|} \quad \Rightarrow \quad \beta$-function zero!

Graphical realisation for $N \geq 4$ (and $B = 1$)

$$G_{|b_0 b_1 b_2 b_3|} = (-\lambda)\frac{G_{|b_0 b_1|}G_{|b_2 b_3|} - G_{|b_0 b_3|}G_{|b_2 b_1|}}{(E_{b_0} - E_{b_2})(E_{b_1} - E_{b_3})} = -\lambda \Big\{$$  $$+$$  $$\Big\}$$

$$G_{|b_0 \dots b_5|} = \lambda^2 \Big\{$$  $$+$$  $$+$$ 

$$+ \Big($$  $$+$$  $$+$$  $$\Big) + \Big($$  $$+$$  $$+$$  $$\Big)\Big\}$$

b_i b_j $= G_{|b_i b_j|}$ leads to non-crossing chord diagrams ; these are counted by the Catalan number $C_{\frac{N}{2}} = \frac{N!}{(\frac{N}{2}+1)!\frac{N}{2}!}$

b_i  b_j $= \frac{1}{E_{b_i} - E_{b_j}}$ leads to rooted trees connecting the even or odd vertices, intersecting the chords only at vertices

Open Problem: Which trees arise for a given chord diagram?

not unique: $\frac{1}{(E_a - E_b)(E_b - E_c)} + \frac{1}{(E_b - E_c)(E_c - E_a)} + \frac{1}{(E_c - E_a)(E_a - E_b)} = 0$

Back to $\lambda\Phi^4_4$ on Moyal space

- Infinite volume limit (i.e. $\theta \to \infty$) turns discrete matrix indices into continuous variables $a, b, \dots \in \mathbb{R}_+$ and sums into integrals
- Need energy cutoff $a, b, \dots \in [0, \Lambda^2]$ and normalisation of lowest Taylor terms of two-point function $G_{|nm|} \mapsto G_{ab}$
- Carleman-type singular integral equation for $G_{ab} - G_{a0}$

Theorem (2012/13) (for $\lambda < 0$, using $G_{b0} = G_{0b}$)

Let $\mathcal{H}^\Lambda_a(f) = \frac{1}{\pi}\mathcal{P}\int_0^{\Lambda^2} \frac{f(p)\,dp}{p-a}$ be the *finite Hilbert transform*. Then

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\mathrm{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)] - \mathcal{H}^\Lambda_a[\tau_b(\bullet)])}$$

where $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a\mathcal{H}^\Lambda_a[G_{\bullet 0}]}{G_{a0}}}\right)$ and G_{0b} solution of

$$G_{b0} = G_{0b} = \frac{1}{1+b}\exp\left(-\lambda\int_0^b dt\int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p\mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

Discussion

Together with explicit (but complicated for $G_{ab|cd}, G_{ab|cd|ef}, \dots$) formulae for higher correlation functions, we have exact solution of $\lambda\phi^4_4$ on extreme Moyal space in terms of

$$G_{b0} = G_{0b} = \frac{1}{1+b}\exp\left(-\lambda\int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

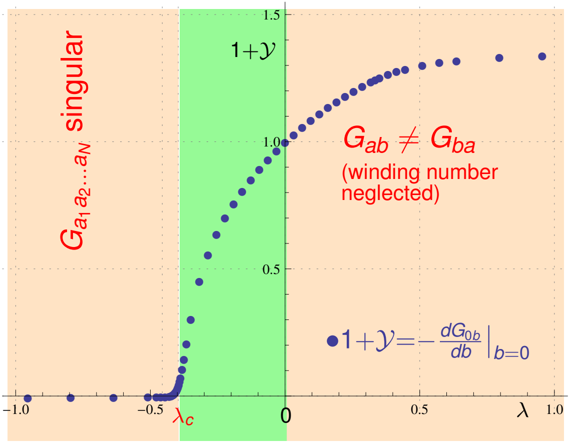
Theorem (2015)

Solution exists by Schauder fixed point theorem.

- Perturbative solution reproduces all Feynman graphs. Polylogarithms and ζ-functions are generated.
- Perturbation series does not converge: τ_b maps $[0,\Lambda^2]$ to $[0,\pi]$ for $\lambda > 0$, but to $[0,\epsilon]$ for $-\delta \le \lambda < 0$
- Formula can be put on a computer and solved by iteration.

Computer simulation: evidence for phase transitions

piecewise linear approximation of G_{0b}, G_{ab} for $\Lambda^2 = 10^7$ and 2000 sample points. Consider $1+\mathcal{Y} := -\frac{dG_{0b}}{db}\Big|_{b=0}$



- $(1+\mathcal{Y})'(\lambda)$ discontinuous at $\lambda_c = -0.39$
- order parameter $b_\lambda = \sup\{b : G_{0b} = 1\}$ non-zero for $\lambda < \lambda_c$
- A key property for Schwinger functions is realised in $]\lambda_c, 0]$, not outside!
  The critical couplings coincide!

Osterwalder-Schrader reconstruction theorem (1974)

Assume for Schwinger functions $S(x_1,\dots,x_N)$:

- S0 growth rate: $\Big|\int dx\, f(x_1,\dots,x_N)S(x_1,\dots,x_N)\Big| \leq c_1(N!)^{c_2}|f|_{Nc_3}$
- S1 Euclidean invariance: $S(x_1,\dots,x_N) = S(Rx_1+a,\dots,Rx_N+a)$
- S2 reflection positivity: for each $(f_0,\dots,f_K)$ with $f_N \in \mathcal{S}(\mathbb{R}^{Nd})$,

$$\sum_{M,N=0}^{K}\int dxdy\; S(x_N,\dots,x_1,y_1,\dots y_M)\overline{f_N({}^r x_1,\dots,{}^r x_N)}f_M(y_1,\dots,y_M) \geq 0$$

where ${}^r(x^0,x^1,\dots x^{d-1}) := (-x^0,x^1,\dots x^{d-1})$

- S3 permutation symmetry: $S(x_1,\dots,x_N) = S(x_{\sigma(1)},\dots,x_{\sigma(N)})$

Then the $S(\xi_1,\dots,\xi_{N-1})\big|_{\xi_i^0>0}$, with $\xi_i = x_i - x_{i+1}$, are inverse Laplace-Fourier transforms of FT $\hat{W}(q_1,\dots,q_{N-1})$ of Wightman distributions in a relativistic QFT.

If in addition the $S(x_1,\dots,x_N)$ satisfy

- S4 clustering

then the Wightman QFT has a unique vacuum state

From matrix model to Schwinger functions on $\mathbb{R}^4$

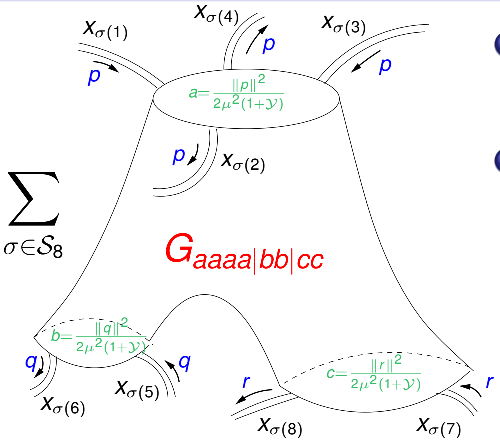
reverting harmonic oscillator basis, $1+\mathcal{Y} := -\frac{dG_{0b}}{db}\Big|_{b=0} \dots$

Theorem (2013): *connected* **Schwinger functions**

$$S_c(\mu x_1,\dots,\mu x_N) = \frac{1}{64\pi^2} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \Big(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4 p_\beta}{4\pi^2\mu^4} e^{\mathrm{i}\big\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta} (-1)^{i-1} \mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)}\big\rangle}\Big)$$
$$\times\, G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1}\Big|\dots\Big|\underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}}$$

- Schwinger functions are symmetric S3 and invariant under full Euclidean group S1 (completely unexpected for NCQFT)
- growth conditions S0 established
- clustering S4 is violated: The $(N_1+\dots+N_B)$-point functions are insensitive to the distance of different boundaries.
- remains: reflection positivity S2

Connected (4+2+2)-point function



1. individual Euclidean symmetry in every boundary component (no clustering)
2. particle scattering without momentum exchange
   - in 4D a sign of triviality (mind assumptions!)
   - familiar in 2D models with factorising S-matrix
   - a consequence of integrability [Moser, 1975] & [Kulish, 1976]

Is there a precise link between exact solution of our 4D model and traditional integrability in 2D? What about Yang-Baxter?

Osterwalder-Schrader reflection positivity

Proposition (2013)

$S(x_1, x_2)$ is reflection positive iff $a \mapsto G_{aa}$ is a Stieltjes function,

$$G_{aa} = \int_0^\infty \frac{d(\rho(t))}{a+t}, \qquad \rho - \text{positive measure}.$$

- Excluded for any $\lambda > 0$ (due to renormalisation)!
- Purely real conditions [Widder, 1938]:

$$L_{k,t}[f(\bullet)] := \frac{(-t)^{k-1}}{c_k} \frac{d^{2k-1}}{dt^{2k-1}}\big(t^k f(t)\big) \geq 0 \;, \qquad \begin{matrix} c_1 = 1 \\ c_{k>1} = k!(k-2)! \end{matrix}$$

$\rightarrow$ lowest orders verified numerically

- Complex conditions [Krein]:
  1. $f(t) \geq 0$ for $t > 0$
  2. $f : \mathbb{C} \setminus]-\infty, 0] \to \mathbb{C}$ holomorphic
  3. $\mathrm{Im}(f(x + \mathrm{i}y)) < 0$ for $y > 0$ (anti-Herglotz)

$\rightarrow$ work in progress

Summary

1. $\lambda\phi^4_4$ on nc Moyal space is, at infinite noncommutativity, exactly solvable in terms of a fixed point solution
   - stable non-perturbative solution for $\lambda < 0$
   - phase transitions and critical phenomena, hence interesting statistical physics model
   - non-trivial as a matrix model
2. Projection to Schwinger functions for scalar field on $\mathbb{R}^4$:
   - S3 automatic, full Euclidean symmetry S1, control about S0
   - no clustering S4
   - no momentum exchange (close to triviality), possibly a consequence of integrability
3. Reflection positivity S2 does not fail immediately. Why? Needs verification and extension to higher correlation functions