

# A solvable quantum field theory in 4 dimensions

Raimar Wolkenhaar

Mathematisches Institut, Westfälische Wilhelms-Universität Münster



(based on joint work with Harald Grosse,  
arXiv: 1205.0465, 1306.2816, 1402.1041, 1406.7755 & 1505.05161)

# Introduction

## Clay Mathematics Institute Millennium Prize Problem (2000)

### 5. Yang-Mills Existence and Mass Gap

Prove that for any compact simple gauge group  $G$ , a **non-trivial quantum Yang-Mills theory exists on  $\mathbb{R}^4$**  and has a **mass gap  $\Delta > 0$** . Existence includes establishing **axiomatic properties** at least as strong as those of **[Wightman, Osterwalder-Schrader]**.

- This problem is out of reach for our century.
- There is no solution for a far more modest problem:

*Prove that a **non-trivial toy model** for a quantum field theory **on  $\mathbb{R}^4$**  exists and satisfies **[Wightman, Osterwalder-Schrader]**.*

# Vanishing $\beta$ -function

It is probably a good idea to try first QFT models with **vanishing  $\beta$ -function**. They are **nice both in UV and IR**.

## Candidates

- ①  **$\mathcal{N} = 4$  super Yang-Mills theory**
    - very active subject with many strong results
    - Wightman axioms are not made for gauge theory, but  **$\mathcal{N} = 4$  SYM might suggest reasonable axioms**
  - ② **self-dual noncommutative  $\phi_4^4$ -theory**
    - perturbatively renormalisable (Grosse-W. 2004)
    - **$\beta$ -function vanishes to all orders in perturbation theory** (Disertori-Gurau-Magnen-Rivasseau, 2006)
    - model is relatively simple
- Can we construct it? ... **YES (as statistical physics model)**  
 To our big surprise, **Wightman axioms seem to be satisfied!**

# Regularisation of $\lambda\phi_4^4$ on noncommutative space

$$S[\phi] = \int_{\mathbb{R}^4} \frac{dx}{64\pi^2} \left( \frac{1}{2} \phi (-\Delta + \mu^2) \phi + \frac{\lambda}{4} \phi \phi \phi \phi \right)(x)$$

# Regularisation of $\lambda\phi_4^4$ on noncommutative space

$$S[\phi] = \int_{\mathbb{R}^4} \frac{dx}{64\pi^2} \left( \frac{Z}{2} \phi \star (-\Delta + \Omega^2 (2\Theta^{-1}x)^2 + \mu_{bare}^2) \phi + \frac{\lambda Z^2}{4} \phi \star \phi \star \phi \star \phi \right) (x)$$

with **Moyal product**  $(f \star g)(x) = \int_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{dy dk}{(2\pi)^4} f(x + \frac{1}{2}\Theta k) g(x+y) e^{i(k,y)}$

takes at  $\Omega = 1$  in matrix basis  $f_{\underline{mn}}(x) = f_{m_1 n_1}(x^0, x^1) f_{m_2 n_2}(x^3, x^4)$

$$f_{mn}(y^0, y^1) = 2(-1)^m \sqrt{\frac{m!}{n!}} \left( \sqrt{\frac{2}{\theta}} y \right)^{n-m} L_m^{n-m} \left( \frac{2|y|^2}{\theta} \right) e^{-\frac{|y|^2}{\theta}}$$

due to  $f_{\underline{mn}} \star f_{\underline{kl}} = \delta_{\underline{nk}} f_{\underline{ml}}$  and  $\int dx f_{\underline{mn}}(x) = 64\pi^2 V \delta_{\underline{mn}}$  the form

$$S[\Phi] = V \left( \sum_{\underline{m}, \underline{n} \in \mathbb{N}_{\mathcal{N}}^2} E_{\underline{m}} \Phi_{\underline{mn}} \Phi_{\underline{nm}} + \frac{Z^2 \lambda}{4} \sum_{\underline{m}, \underline{n}, \underline{k}, \underline{l} \in \mathbb{N}_{\mathcal{N}}^2} \Phi_{\underline{mn}} \Phi_{\underline{nk}} \Phi_{\underline{kl}} \Phi_{\underline{lm}} \right)$$

$$E_{\underline{m}} = Z \left( \frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu_{bare}^2}{2} \right), \quad |\underline{m}| := m_1 + m_2 \leq \mathcal{N}$$

- $V = \left(\frac{\theta}{4}\right)^2$  is for  $\Omega = 1$  the **volume** of the nc manifold.

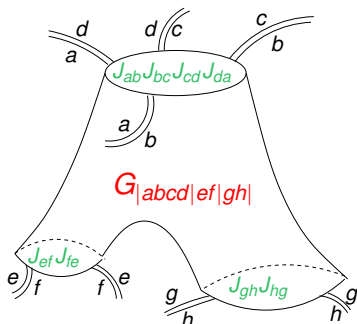
# More generally: field-theoretical matrix models

## Euclidean quantum field theory

- action  $S[\Phi] = V \operatorname{tr}(E\Phi^2 + P[\Phi])$   
for unbounded positive selfadjoint operator  $E$  with compact resolvent, and  $P[\Phi]$  a polynomial
- partition function  $\mathcal{Z}[J] = \int \mathcal{D}[\Phi] \exp(-S[\Phi] + V \operatorname{tr}(\Phi J))$
- For  $P[\Phi] = \frac{i}{6} \Phi^3$  this is the **Kontsevich model** which computes the intersection theory on the moduli space of complex curves. We choose  $P[\Phi] = \frac{\lambda}{4} \Phi^4$ .
- Perturbative expansion  $e^{-V \operatorname{tr}(P[\Phi])} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (V \operatorname{tr}(P[\Phi]))^n$  leads to **ribbon graphs**. They encode **genus- $g$**  Riemann surface with  **$B$  boundary components**.
- We avoid the expansion, but keep the topological structure:

# Topological expansion

- Choosing  $E = \text{diag}(E_a)$ , matrix index conserved along every strand.
- The  $k^{\text{th}}$  boundary component carries a cycle  $J_{p_1 \dots p_{N_k}}^{N_k} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$  of  $N_k$  external sources,  $N_k + 1 \equiv 1$ .



- Expand  $\log \mathcal{Z}[\mathcal{J}] = \sum \frac{1}{S} V^{2-B} G_{|p_1^1 \dots p_{N_1}^1| \dots |p_1^B \dots p_{N_B}^B|} \prod_{\beta=1}^B J_{p_1^\beta \dots p_{N_\beta}^\beta}^{N_\beta}$  according to the cycle structure.
- QFT of matrix models determines the **weights of Riemann surfaces** with **decorated boundary components** compatible with
  - gluing (of fringes, not boundaries!)
  - covariance (under  $\Phi \mapsto U^* \Phi U$ , which is not a symmetry!)

# Schwinger-Dyson equations (for $S_{int}[\Phi] = \frac{\lambda}{4}\text{tr}(\Phi^4)$ )

In a scaling limit  $V \rightarrow \infty$  and  $\frac{1}{V} \sum_{p \in I}$  finite, we have:

1. A closed non-linear equation for  $G_{|ab|}$

$$G_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a + E_b)} \frac{1}{V} \sum_{p \in I} \left( G_{|ab|} G_{|ap|} - \frac{G_{|pb|} - G_{|ab|}}{E_p - E_a} \right)$$

2. For  $N \geq 4$  a universal algebraic recursion formula

$$G_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|} G_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G_{|b_{2l} b_1 \dots b_{2l-1}|} G_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

- scaling limit corresponds to restriction to genus  $g = 0$
- similar formulae for  $B \geq 2$
- no index summation in  $G_{|abcd|} \Rightarrow \beta\text{-function zero!}$



## Back to $\lambda\Phi_4^4$ on Moyal space

- Infinite volume limit (i.e.  $\theta \rightarrow \infty$ ) turns discrete matrix indices into continuous variables  $a, b, \dots \in \mathbb{R}_+$  and sums into integrals
- Need energy cutoff  $a, b, \dots \in [0, \Lambda^2]$  and normalisation of lowest Taylor terms of two-point function  $G_{|nm|} \mapsto G_{ab}$
- **Carleman-type singular integral equation** for  $G_{ab} - G_{a0}$

Theorem (2012/13) (for  $\lambda < 0$ , using  $G_{b0} = G_{0b}$ )

Let  $\mathcal{H}_a^\Lambda(f) = \frac{1}{\pi} \mathcal{P} \int_0^{\Lambda^2} \frac{f(p) dp}{p-a}$  be the *finite Hilbert transform*. Then

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\text{sign}(\lambda)(\mathcal{H}_0^\Lambda[\tau_0(\bullet)] - \mathcal{H}_a^\Lambda[\tau_b(\bullet)])}$$

where  $\tau_b(a) := \arctan_{[0, \pi]} \left( \frac{|\lambda|\pi a}{b + \frac{1 + \lambda\pi a \mathcal{H}_p^\Lambda[G_{\bullet 0}]}{G_{a0}}} \right)$  and  $G_{a0}$  solution of

$$G_{b0} = G_{0b} = \frac{1}{1+b} \exp \left( -\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left( t + \frac{1 + \lambda\pi p \mathcal{H}_p^\Lambda[G_{\bullet 0}]}{G_{p0}} \right)^2} \right)$$

# Discussion

Together with explicit (but complicated for  $G_{ab|cd}$ ,  $G_{ab|cd|ef}$ , ...) formulae for higher correlation functions, we have **exact solution of  $\lambda\phi_4^4$  on extreme Moyal space** in terms of

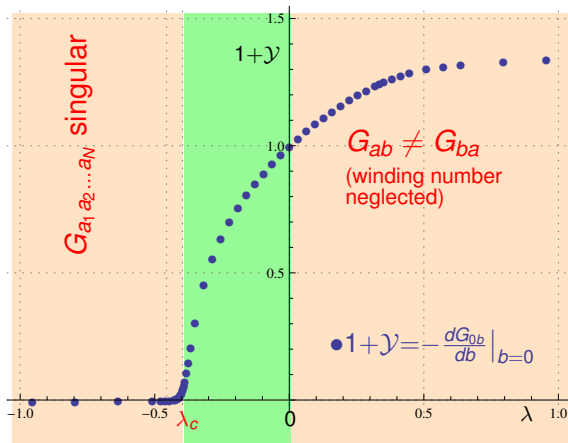
$$G_{b0} = G_{0b} = \frac{1}{1+b} \exp \left( -\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left( t + \frac{1+\lambda\pi p \mathcal{H}_p^\Lambda[G_{\bullet 0}]}{G_{p0}} \right)^2} \right)$$

## Possible treatments

- 1 perturbative solution: reproduces all Feynman graphs, generates polylogarithms and  $\zeta$ -functions
- 2 iterative solution on computer: nicely convergent, find interesting phase structure
- 3 rigorous existence proof of a solution
- 4 work in progress: try to guess the solution; should give uniqueness as by-product

# Computer simulation: evidence for phase transitions

piecewise linear approximation of  $G_{0b}$ ,  $G_{ab}$  for  $\Lambda^2=10^7$  and 2000 sample points. Consider  $1+\mathcal{Y} := -\frac{dG_{0b}}{db} \Big|_{b=0}$



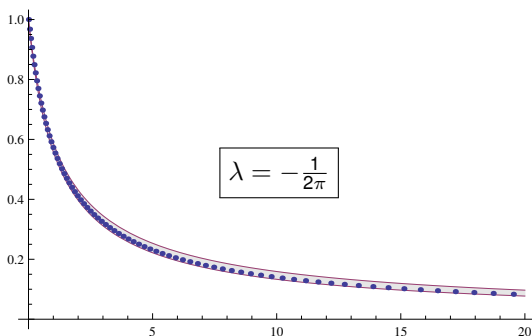
- $(1 + \mathcal{Y})'(\lambda)$  discontinuous at  $\lambda_c = -0.39$
- order parameter  $b_\lambda = \sup\{b : G_{0b}=1\}$  non-zero for  $\lambda < \lambda_c$
- A key property for Schwinger functions is realised in  $]\lambda_c, 0]$ , not outside!  
The critical couplings coincide!

# Fixed point theorem

## Theorem (2015)

Let  $-\frac{1}{6} \leq \lambda \leq 0$ . Then the equation has a  $C_0^1$ -solution

$$\frac{1}{(1+b)^{1-|\lambda|}} \leq G_0 b \leq \frac{1}{(1+b)^{1-\frac{|\lambda|}{1-2|\lambda|}}}$$



Proof via **Schauder fixed point theorem**.

This involves **continuity and compactness** of a certain operator (in norm topology)

# Relativistic and Euclidean quantum field theory

- We view QFT as defined by **Wightman's axioms** for distributions  $\mathcal{W}_N(x_1, \dots, x_N) = W_N(x_1 - x_2, \dots, x_{N-1} - x_N)$
- Theorem: The  $W_N$  are **boundary values of holomorphic functions** (on permuted extended forward tube  $\subset \mathbb{C}^{4(N-1)}$ )
- Restriction of  $W_N$  to real subspace of **Euclidean points** (minus diagonals) defines **Schwinger functions**
- Schwinger functions inherit properties such as real analyticity, Euclidean invariance and **complete symmetry**
- Hence, moments of probability distributions provide candidate Schwinger functions ([link to statistical physics](#))

## Theorem (Osterwalder-Schrader, 1974)

One additional requirement, **reflection positivity**, leads back to **Wightman theory**

# From matrix model to Schwinger functions on $\mathbb{R}^4$

reverting harmonic oscillator basis  $\blacktriangleright$ ,  $1 + \mathcal{Y} := -\frac{dG_{0b}}{db} \Big|_{b=0} \dots$

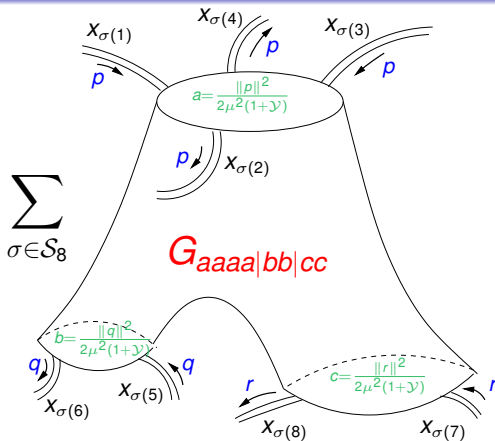
Theorem (2013): *connected* Schwinger functions

$$\begin{aligned}
 & S_C(\mu X_1, \dots, \mu X_N) \\
 &= \frac{1}{64\pi^2} \sum_{\substack{N_1 + \dots + N_B = N \\ N_\beta \text{ even}}} \sum_{\sigma \in S_N} \left( \prod_{\beta=1}^B \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4 p_\beta}{4\pi^2 \mu^4} e^{i \langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta} (-1)^{i-1} \mu X_{\sigma(N_1 + \dots + N_{\beta-1} + i)} \rangle} \right) \\
 & \quad \times \mathbf{G} \underbrace{\left( \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})} \right)}_{N_1} \dots \underbrace{\left( \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})} \right)}_{N_B}
 \end{aligned}$$

**Confinement of noncommutativity:** have internal interaction of matrices; commutative subsector propagates to outside world

- Schwinger functions are symmetric and **invariant under full Euclidean group** (completely unexpected for NCQFT!)
- remains: **reflection positivity**
- finally: Is it **non-trivial?**

# Connected (4+2+2)-point function



- 1 individual Euclidean symmetry in every boundary component (no clustering)
- 2 particle scattering without momentum exchange
  - in 4D a sign of **triviality** (mind assumptions!)
  - familiar in 2D models with **factorising S-matrix**
  - a consequence of **integrability** [Moser, 1975] & [Kulish, 1976]

Is there a precise link between **exact solution of our 4D model** and **traditional integrability** known from 2D?

# Osterwalder-Schrader reflection positivity

## Proposition (2013)

$S(x_1, x_2)$  is reflection positive iff  $a \mapsto G_{aa}$  is a **Stieltjes function**,

$$G_{aa} = \int_0^\infty \frac{d(\rho(t))}{a+t}, \quad \rho - \text{positive measure.}$$

**Excluded for any  $\lambda > 0$**  (unless rescued by winding number)

- naïve anomalous dimension  $\eta$  positive for  $\lambda > 0$ , but  $\eta$  diverges to  $+\infty$
- renormalisation oversubtracts:  $\eta_{ren}, \lambda$  of opposite sign
- **model-independent**:  $p$ -space 2-point function  $\frac{1}{(p^2+m^2)^{1-\eta/2}}$   
**Positivity and convergence contradict each other!**
- Need (analytical?) continuation between
  - one regime where existence can be proved and
  - another regime where positivity holds.



# Dismiss the partition function

Partition function is merely a guiding principle to identify the quantum equations of motion (and combinatorics).

These Schwinger-Dyson equations, and not the partition function, define the field theory!

## Non-triviality (proposed by J. Schlemmer)

- *If* our Schwinger functions do *not* arise from a Euclidean measure, there is no reason they satisfy **Nelson-Symanzik positivity**.
- Since the free field is Nelson-Symanzik positive, our **Schwinger functions cannot be trivial**.

# Reflection positivity simplifies the problem

If  $G_{x0}$  is Stieltjes, then Hilbert transform can be avoided:

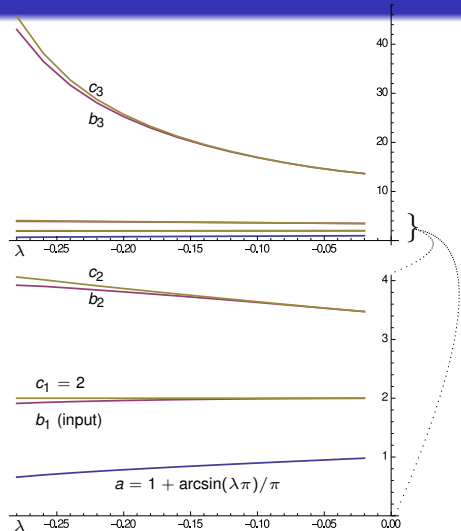
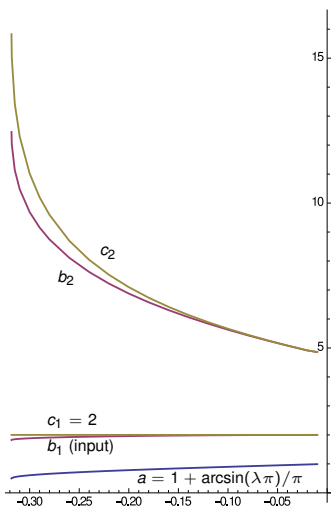
$$\frac{G_{xy}}{G_{x0}} = \exp \left( -\frac{1}{\pi} \int_1^\infty \frac{dt}{t+x} \arctan \left( \frac{y \operatorname{Im}(G_{-(t+i\epsilon),0})}{1 - \lambda t \int_0^\infty ds \frac{G_{s0}}{t+s} + y \operatorname{Re}(G_{-(t+i\epsilon),0})} \right) \right)$$

Which class of functions has desired analyticity+holomorphicity and manageable integral transforms?

hypergeometric functions  $G_{x0} = {}_nF_{n-1} \left( \begin{matrix} a, b_1, \dots, b_{n-1} \\ c_1, \dots, c_{n-1} \end{matrix} \middle| -x \right)$  if  $a \in [0, 1]$  and  $c_i > b_i > a$

- holomorphicity at  $y > 0$ : determine  $a, b_i, c_i$  by  $G_{0y}^{(k)} = G_{y0}^{(k)}$
- find:  $a = 1 + \frac{1}{\pi} \arcsin(\lambda\pi)$ ,  $\prod_{i=1}^n \frac{c_i-1}{b_i-1} = \frac{\arcsin(\lambda\pi)}{\lambda\pi}$
- **critical coupling constant is  $\lambda_c = -\frac{1}{\pi} = -0.3183\dots$**
- also take  $c_1 = 2$  (nicer Stieltjes and Hilbert transforms)

# Parameter spectrum



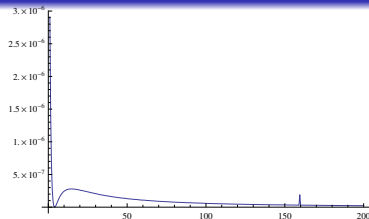
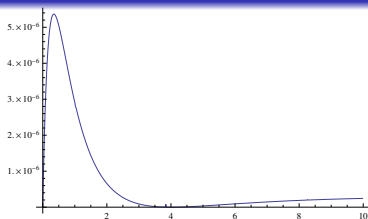
for  $G(x, 0) = {}_3F_2\left(\begin{matrix} a, b_1, b_2 \\ c_1, c_2 \end{matrix} \middle| -x\right)$

for  $G(x, 0) = {}_4F_3\left(\begin{matrix} a, b_1, b_2, b_3 \\ c_1, c_2, c_3 \end{matrix} \middle| -x\right)$

# Convergence of the method

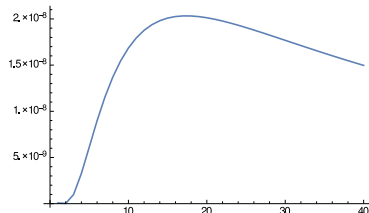
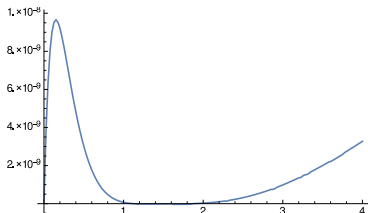
${}_3F_2$   
at  $\lambda = -0.1$

$$G_{4,0}^{(0..1)} = G_{0,4}^{(0..1)}$$



${}_4F_3$   
at  $\lambda = -0.1$

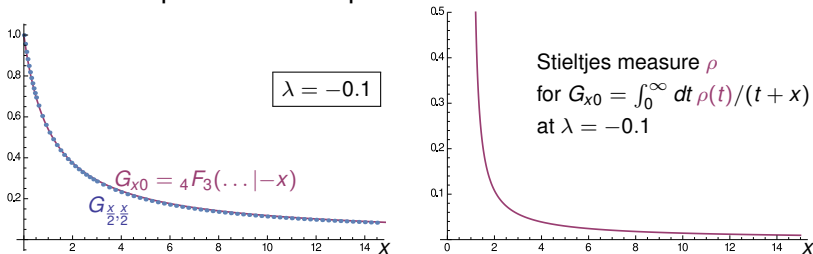
$$G_{\frac{3}{2},0}^{(0..3)} = G_{0,\frac{3}{2}}^{(0..3)}$$



- improvement by factor  $\sim 250$
- ready to compute the Schwinger 4-point function!

# Källén-Lehmann spectrum

- these results make is completely clear (but don't prove) that  $G_{x0}$  is Stieltjes
- reflection positivity equivalent to  $G_{xx}$  a Stieltjes function
- the shape makes this plausible:



- measure for  $G_{x0}$  has mass gap  $[0, 1[$ , but no further gap (remnant of UV/IR-mixing)
- absence of the second gap (usually  $]1, 4[$ ) circumvents triviality theorems

# Summary

- ①  $\lambda\phi_4^4$  on nc Moyal space is, at infinite noncommutativity, **exactly solvable** in terms of a fixed point problem
  - theory defined by quantum equations of motion (= Schwinger-Dyson equations), **not by a measure**
  - **existence proved** within small region
  - **phase transitions and critical phenomena**
- ② Projection to Schwinger functions for scalar field on  $\mathbb{R}^4$  = **confinement of noncommutativity**
  - full Euclidean symmetry (completely unexpected)
  - no momentum exchange (close to triviality), **possibly a consequence of integrability**
  - numerical approach with tiny error: leaves no doubt that **Schwinger 2-point function is reflection positive for  $-\frac{1}{\pi} < \lambda \leq 0$**
- ③ ready to embark on higher Schwinger functions