A solvable quantum field theory in 4 dimensions

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(based on joint work with Harald Grosse,
arXiv: 1205.0465, 1306.2816, 1402.1041, 1406.7755 & 1505.05161)

Introduction

Clay Mathematics Institute Millennium Prize Problem (2000)

5. Yang-Mills Existence and Mass Gap

Prove that for any compact simple gauge group G, a non-trivial quantum Yang-Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$. Existence includes establishing axiomatic properties at least as strong as those of [Wightman, Osterwalder-Schrader].

- This problem is out of reach for our century.
- There is no solution for a far more modest problem:

Prove that a non-trivial toy model for a quantum field theory on $\mathbb{R}^4$ exists and satisfies [Wightman, Osterwalder-Schrader].

Vanishing β-function

It is probably a good idea to try first QFT models with vanishing β-function. They are nice both in UV and IR.

Candidates

1. $\mathcal{N} = 4$ super Yang-Mills theory
   - very active subject with many strong results
   - Wightman axioms are not made for gauge theory, but $\mathcal{N} = 4$ SYM might suggest reasonable axioms
2. self-dual noncommutative ϕ^4_4-theory
   - perturbatively renormalisable (Grosse-W. 2004)
   - β-function vanishes to all orders in perturbation theory (Disertori-Gurau-Magnen-Rivasseau, 2006)
   - model is relatively simple

$\rightarrow$ Can we construct it? ... YES (as statistical physics model)
To our big surprise, Wightman axioms are not impossible

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4}\frac{dx}{64\pi^2}\Big(\frac{1}{2}\phi\;\;(-\Delta\qquad\qquad\qquad\qquad+\mu^2\quad)\phi+\frac{\lambda}{4}\phi\;\;\phi\;\;\phi\;\;\phi\Big)(x)$$

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4}\frac{dx}{64\pi^2}\Big(\frac{Z}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2_{bare})\phi+\frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with Moyal product $(f\star g)(x)=\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dy\,dk}{(2\pi)^4}f(x+\frac{1}{2}\Theta k)\,g(x+y)\,e^{\mathrm{i}\langle k,y\rangle}$

takes at $\Omega=1$ in matrix basis $f_{\underline{mn}}(x)=f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\tfrac{2}{\theta}}y\Big)^{n-m}L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

due to $f_{\underline{mn}}\star f_{\underline{kl}}=\delta_{\underline{nk}}f_{\underline{ml}}$ and $\int dx\,f_{\underline{mn}}(x)=64\pi^2V\delta_{\underline{mn}}$ the form

$$S[\Phi]=V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}E_{\underline{m}}\Phi_{\underline{mn}}\Phi_{\underline{nm}}+\frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Big)$$

$$E_{\underline{m}}=Z\Big(\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu^2_{bare}}{2}\Big),\qquad |\underline{m}|:=m_1+m_2\leq\mathcal{N}$$

- $V=\big(\frac{\theta}{4}\big)^2$ is for $\Omega=1$ the volume of the nc manifold.

More generally: field-theoretical matrix models

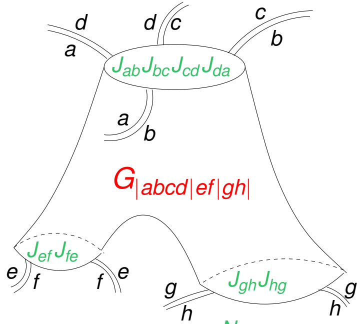
Euclidean quantum field theory

- action $S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + P[\Phi])$ for unbounded positive selfadjoint operator E with compact resolvent, and $P[\Phi]$ a polynomial
- partition function $\mathcal{Z}[J] = \int \mathcal{D}[\Phi]\; \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))$

- For $P[\Phi] = \frac{\mathrm{i}}{6}\Phi^3$ this is the Kontsevich model which computes the intersection theory on the moduli space of complex curves. We choose $P[\Phi] = \frac{\lambda}{4}\Phi^4$.
- Perturbative expansion $e^{V\,\mathrm{tr}(P[\Phi])} = \sum_{n=0}^{\infty} \frac{1}{n!}(V\,\mathrm{tr}(P[\Phi])^n$ leads to ribbon graphs. They encode genus-g Riemann surface with B boundary components.
- We avoid the expansion, but keep the topological structure:

Topological expansion

- Choosing $E = \mathrm{diag}(E_a)$, matrix index conserved along every strand.
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1\dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.



- Expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p^1_1\dots p^1_{N_1}|\dots|p^B_1\dots p^B_{N_B}|} \prod_{\beta=1}^{B} J^{N_\beta}_{p^\beta_1\dots p^\beta_{N_\beta}}$ according to the cycle structure.
- QFT of matrix models determines the weights of Riemann surfaces with decorated boundary components compatible with
  1. gluing (of fringes, not boundaries!)
  2. covariance (under $\Phi \mapsto U^* \Phi U$, which is not a symmetry!)

Ward identity

Proposition [Disertori-Gurau-Magnen-Rivasseau, 2006]

$$0=\sum_{n\in I}\Big(\frac{(E_a-E_p)}{V}\frac{\partial^2\mathcal{Z}}{\partial J_{an}\partial J_{np}}+J_{pn}\frac{\partial\mathcal{Z}}{\partial J_{an}}-J_{na}\frac{\partial\mathcal{Z}}{\partial J_{np}}\Big)$$

Theorem [Grosse-W. 2012]

$$\begin{aligned}\sum_{n\in I}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} &= \delta_{ap}\Big\{V^2\sum_{(K)}\frac{J_{P_1}\cdots J_{P_K}}{S_K}\Big(\sum_{n\in I}\frac{G_{|an|P_1|\dots|P_K|}}{V^{|K|+1}}+\frac{G_{|a|a|P_1|\dots|P_K|}}{V^{|K|+2}}\\ &\qquad +\sum_{r\geq 1}\sum_{q_1,\dots,q_r\in I}\frac{G_{|q_1aq_1\dots q_r|P_1|\dots|P_K|}J^r_{q_1\dots q_r}}{V^{|K|+1}}\Big)\\ &\qquad +V^4\sum_{(K),(K')}\frac{J_{P_1}\cdots J_{P_K}J_{Q_1}\cdots J_{Q_{K'}}}{S_K S_{K'}}\frac{G_{|a|P_1|\dots|P_K|}}{V^{|K|+1}}\frac{G_{|a|Q_1|\dots|Q_{K'}|}}{V^{|K'|+1}}\Big\}\mathcal{Z}[J]\\ &\qquad +\frac{V}{E_p-E_a}\sum_{n\in I}\Big(J_{pn}\frac{\partial\mathcal{Z}[J]}{\partial J_{an}}-J_{na}\frac{\partial\mathcal{Z}[J]}{\partial J_{np}}\Big)\end{aligned}$$

- in Schwinger-Dyson equations resulting from $\frac{\partial^N\mathcal{Z}}{\partial J\dots\partial J}$, two J-derivatives are killed

Schwinger-Dyson equations (for $S_{int}[\Phi] = \frac{\lambda}{4}\mathrm{tr}(\Phi^4)$)

In a scaling limit $V \to \infty$ and $\frac{1}{V}\sum_{p\in I}$ finite, we have:

1. A closed non-linear equation for $G_{|ab|}$

$$G_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a + E_b)} \frac{1}{V} \sum_{p\in I} \Big(G_{|ab|} G_{|ap|} - \frac{G_{|pb|} - G_{|ab|}}{E_p - E_a} \Big)$$

2. For $N \geq 4$ a universal algebraic recursion formula

$$G_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|} G_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G_{|b_{2l} b_1 \dots b_{2l-1}|} G_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

- 2. uses reality $\mathcal{Z} = \overline{\mathcal{Z}}$
- scaling limit corresponds to restriction to genus $g = 0$
- similar formulae for $B \geq 2$
- no index summation in $G_{|abcd|}$ $\Rightarrow$ β-function zero!

Back to $\lambda\Phi^4_4$ on Moyal space

- Infinite volume limit (i.e. $\theta \to \infty$) turns discrete matrix indices into continuous variables $a, b, \dots \in \mathbb{R}_+$ and sums into integrals
- Need energy cutoff $a, b, \dots \in [0, \Lambda^2]$ and normalisation of lowest Taylor terms of two-point function $G_{|nm|} \mapsto G_{ab}$
- Carleman-type singular integral equation for $G_{ab} - G_{a0}$

Theorem (2012/13) (for $\lambda < 0$, using $G_{b0} = G_{0b}$)

Let $\mathcal{H}^\Lambda_a(f) = \frac{1}{\pi}\mathcal{P}\int_0^{\Lambda^2} \frac{f(p)\,dp}{p-a}$ be the *finite Hilbert transform*. Then

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\mathrm{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)] - \mathcal{H}^\Lambda_a[\tau_b(\bullet)])}$$

where $\tau_b(a) := \operatorname*{arctan}_{[0,\pi]} \left(\frac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a \mathcal{H}^\Lambda_a[G_{\bullet 0}]}{G_{a0}}} \right)$ and G_{0b} solution of

$$G_{b0} = G_{0b} = \frac{1}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\right)^2} \right)$$

Discussion

Together with explicit (but complicated for $G_{ab|cd}$, $G_{ab|cd|ef}$, $\dots$) formulae for higher correlation functions, we have exact solution of $\lambda\phi^4_4$ on extreme Moyal space in terms of

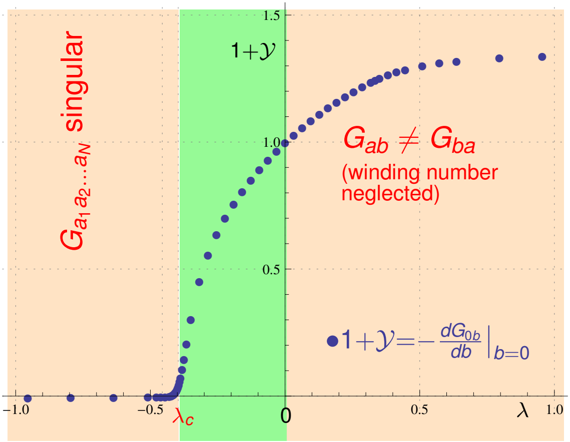
$$G_{b0} = G_{0b} = \frac{1}{1+b}\exp\left(-\lambda\int_0^b dt\int_0^{\Lambda^2}\frac{dp}{(\lambda\pi p)^2 + \left(t+\frac{1+\lambda\pi p\mathcal{H}^{\Lambda}_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

Possible treatments

1. Perturbative solution: reproduces all Feynman graphs, generates polylogarithms and ζ-functions.
   Cannot expect convergence: τ_b not small for $\lambda \searrow 0$
2. Iterative solution on computer: nicely convergent, find interesting phase structure
3. Rigorous existence proof of a solution, uniqueness still open (but obvious for computer)

Computer simulation: evidence for phase transitions

piecewise linear approximation of G_{0b}, G_{ab} for $\Lambda^2{=}10^7$ and 2000 sample points. Consider $1{+}\mathcal{Y}{:=}-\frac{dG_{0b}}{db}\Big|_{b=0}$



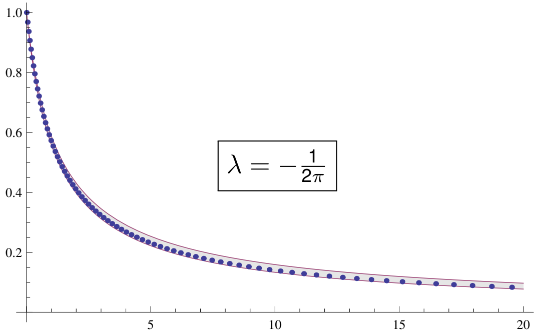
- $(1+\mathcal{Y})'(\lambda)$ discontinuous at $\lambda_c = -0.39$
- order parameter $b_\lambda{=}\sup\{b \;:\; G_{0b}{=}1\}$ non-zero for $\lambda < \lambda_c$
- A key property for Schwinger functions is realised in $]\lambda_c, 0]$, not outside! The critical couplings coincide!

Fixed point theorem

Theorem (2015)

Let $-\frac{1}{6} \leq \lambda \leq 0$. Then the equation has a C^1_0-solution

$$\frac{1}{(1+b)^{1-|\lambda|}} \leq G_{0b} \leq \frac{1}{(1+b)^{1-\frac{|\lambda|}{1-2|\lambda|}}}$$



Proof via Schauder fixed point theorem.

This involves continuity and compactness of a certain operator (in norm topology)

Relativistic and Euclidean quantum field theory

- We view QFT as defined by Wightman's axioms for distributions $\mathcal{W}_N(x_1, \dots, x_N) = W_N(x_1 - x_2, \dots, x_{N-1} - x_N)$
- Theorem: The W_N are boundary values of holomorphic functions (on permuted extended forward tube $\subset \mathbb{C}^{4(N-1)}$)
- Restriction of W_N to real subspace of Euclidean points (minus diagonals) defines Schwinger functions
- Schwinger functions inherit properties such as real analyticity, Euclidean invariance and complete symmetry
- Hence, moments of probability distributions provide candidate Schwinger functions (link to statistical physics)

Theorem (Osterwalder-Schrader, 1974)

One additional requirement, reflection positivity, leads back to Wightman theory

From matrix model to Schwinger functions on $\mathbb{R}^4$

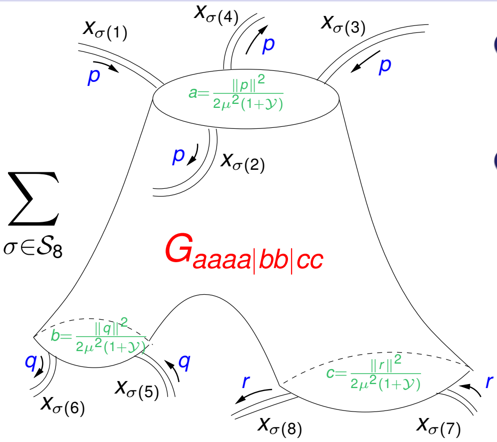
reverting harmonic oscillator basis, $1+\mathcal{Y} := -\frac{dG_{0b}}{db}\Big|_{b=0}\dots$

Theorem (2013): *connected* Schwinger functions

$$S_c(\mu x_1,\dots,\mu x_N) = \frac{1}{64\pi^2} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \Big(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4p_\beta}{4\pi^2\mu^4}\, e^{i\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta}(-1)^{i-1}\mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)}\rangle}\Big) \times G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1}\Big|\dots\Big|\underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}}$$

- Schwinger functions are symmetric and invariant under full Euclidean group (completely unexpected for NCQFT!)
- remains: reflection positivity
- finally: Is it non-trivial?

Connected (4+2+2)-point function



1. individual Euclidean symmetry in every boundary component (no clustering)
2. particle scattering without momentum exchange
   - in 4D a sign of triviality (mind assumptions!)
   - familiar in 2D models with factorising S-matrix
   - a consequence of integrability [Moser, 1975] & [Kulish, 1976]

Is there a precise link between exact solution of our 4D model and traditional integrability in 2D? ... Conjecture: Toda chain

Osterwalder-Schrader reflection positivity

Proposition (2013)

$S(x_1, x_2)$ is reflection positive iff $a \mapsto G_{aa}$ is a Stieltjes function,

$$G_{aa} = \int_0^\infty \frac{d(\rho(t))}{a+t}, \qquad \rho - \text{positive measure}.$$

- Excluded for any $\lambda > 0$ (due to renormalisation)!
- Purely real conditions [Widder, 1938]:

$$L_{k,t}[f(\bullet)] := \frac{(-t)^{k-1}}{c_k} \frac{d^{2k-1}}{dt^{2k-1}}\big(t^k f(t)\big) \geq 0 , \qquad \begin{matrix} c_1 = 1 \\ c_{k>1} = k!(k-2)! \end{matrix}$$

$\rightarrow$ lowest orders verified numerically

- Complex conditions [Krein]:
  1. $f(t) \geq 0$ for $t > 0$
  2. $f : \mathbb{C} \setminus]-\infty, 0] \rightarrow \mathbb{C}$ holomorphic
  3. $\mathrm{Im}(f(x + \mathrm{i}y)) < 0$ for $y > 0$ (anti-Herglotz)

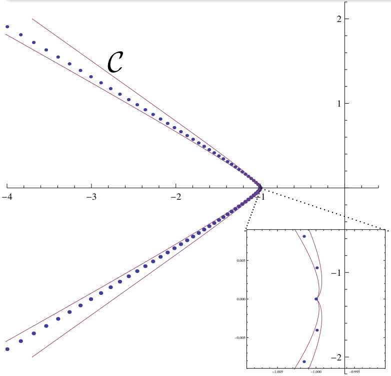
$\rightarrow$ work in progress

Analyticity of the 2-point function

Proposition

Fixed point equation $\quad\Leftrightarrow\quad \dfrac{d}{dz}\log G_{0z} = \dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_{\mathcal{C}} \dfrac{f(t)\,dt}{t-z}$,

where $\mathcal{C}$, f depend on G_{0b} with $b \in \mathbb{R}^+$



Theorem

$z \mapsto G_{0z}$ holomorphic and anti-Herglotz right of $\mathcal{C}$

remaining steps:

- analytic continuation beyond $\mathcal{C}$
- prove same results for $z \mapsto G_{zz}$

Summary

1. $\lambda\phi^4_4$ on nc Moyal space is, at infinite noncommutativity, exactly solvable in terms of a fixed point problem
   - $\lambda < 0$ is the right choice! (gives positive anomalous dimension – a renormalisation effect)
   - theory defined by quantum equations of motion (= Schwinger-Dyson equations), not by a measure
   - existence proved within small region, uniqueness open
   - phase transitions and critical phenomena, hence interesting statistical physics model
2. Projection to Schwinger functions for scalar field on $\mathbb{R}^4$:
   - full Euclidean symmetry (completely unexpected)
   - no clustering
   - no momentum exchange (close to triviality), possibly a consequence of integrability

Summary (II) and Outlook

3. Reflection positivity of 2-point function equivalent to Stieltjes functions
   - real condition holds numerically to low order
   - complex conditions proved in domain right of curve $\mathcal{C}$

Next steps

A. extension to $\mathbb{C} \setminus]-\infty, -1]$
   (true for boundaries of fixed point solution)

B. positivity of $2+\ldots+2$-point functions: not impossible
   (2-point function contains them as subgraphs)
   . . . easy if fixed point solution known explicitly

C. non-triviality: probably true!
   $\lambda < 0$ expected to violate Nelson-Symanzik positivity,
   but the free field has it [due to J. Schlemmer]