A non-trivial four-dimensional QFT
Part II

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(based on joint work with Harald Grosse,
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Introduction

axiomatic settings for rigorous quantum field theories by

1. Wightman [1956]
2. Haag-Kastler [1964]
3. Osterwalder-Schrader [1974]

today: numerous examples in dimension 1,2,3;
not a single non-trivial example in 4 dimensions

We have got a candidate:

- Construction of 4D Euclidean QFT is achieved (2012/13). Find phase transitions and critical phenomena.
- Osterwalder-Schrader axioms are under investigation. So far everything works.
- Non-triviality is open, but not impossible.
- This model is not intended for realistic physics, rather for integrability in four dimensions.

A1. Regularisation & renormalisation

1. We follow the Euclidean track, starting from a partition function.
2. To make this rigorous we need two regulators: finite volume and finite energy density.
3. Pass to quantities (densities and with certain normalised functions) which have infinite volume & energy limits.

Symmetry

- The regulated theory usually has less symmetry. Proving that symmetry is restored in the end is part of the game.
- We propose another strategy:

  Search for a regulator which has more (or very different) symmetry, so constraining that it completely solves the model.

With some luck, a limit procedure gives a constructive QFT on standard $\mathbb{R}^4$. With even more luck, it satisfies OS.

A regularisation of $\lambda\phi^4_4$

$$S[\phi] = \int_{\mathbb{R}^4} \frac{dx}{64\pi^2}\Big(\frac{1}{2}\phi\ \big(-\Delta \qquad\qquad + \mu^2 \quad\big)\phi + \frac{\lambda}{4}\phi\ \ \phi\ \ \phi\ \ \phi\Big)(x)$$

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A regularisation of $\lambda\phi^4_4$

$$S[\phi]=\int_{\mathbb{R}^4}\frac{dx}{64\pi^2}\Big(\frac{1}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2\quad)\phi+\frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with Moyal product $(f\star g)(x)=\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dy\,dk}{(2\pi)^4}f(x+\tfrac{1}{2}\Theta k)\,g(x+y)\,e^{\mathrm{i}\langle k,y\rangle}$

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$$S[\phi]=\int_{\mathbb{R}^4}\frac{dx}{64\pi^2}\Big(\frac{Z}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2_{bare})\phi+\frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

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matrix basis $f_{\underline{mn}}(x) = f_{m_1 n_1}(x^0,x^1) f_{m_2 n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1) = 2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\frac{2}{\theta}}y\Big)^{n-m} L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

due to $f_{\underline{mn}}\star f_{\underline{kl}} = \delta_{\underline{nk}} f_{\underline{ml}}$ and $\int dx\, f_{\underline{mn}}(x) = 64\pi^2 V\delta_{\underline{mn}}$

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takes at $\Omega=1$ in matrix basis $f_{\underline{mn}}(x)=f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$

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due to $f_{\underline{mn}}\star f_{\underline{kl}}=\delta_{\underline{nk}}f_{\underline{ml}}$ and $\int dx\,f_{\underline{mn}}(x)=64\pi^2V\delta_{\underline{mn}}$ the form

$$S[\Phi]=V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}E_{\underline{m}}\Phi_{\underline{mn}}\Phi_{\underline{nm}}+\frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Big)$$

$$E_{\underline{m}}=Z\Big(\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu^2_{bare}}{2}\Big),\qquad |\underline{m}|:=m_1+m_2\leq\mathcal{N}$$

- $V=\left(\frac{\theta}{4}\right)^2$ is for $\Omega=1$ the volume of the nc manifold.

A regularisation of $\lambda\phi^4_4$

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- $V=\left(\frac{\theta}{4}\right)^2$ is for $\Omega=1$ the volume of the nc manifold.
- need $V\to\infty$; stringy [Minwalla, van Raamsdonk & Seiberg, 1999]

A2. Field-theoretical matrix models

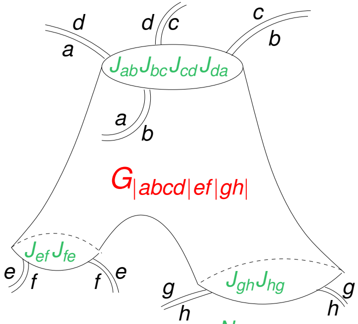
Euclidean quantum field theory

- action $S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + P[\Phi])$ for unbounded positive selfadjoint operator E with compact resolvent, and $P[\Phi]$ a polynomial
- partition function $\mathcal{Z}[J] = \int \mathcal{D}[\Phi]\;\exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))$

- Perturbative expansion $e^{V\,\mathrm{tr}(P[\Phi])} = \sum_{n=0}^{\infty} \frac{1}{n!}(V\,\mathrm{tr}(P[\Phi])^n$ leads to ribbon graphs.
- Encode genus-g Riemann surface with B boundary components.
- We avoid the expansion, but keep the topological structure:

Topological expansion

- Choosing $E = \mathrm{diag}(E_a)$, matrix index conserved along every strand.
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1 \dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.



- Expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p^1_1 \dots p^1_{N_1}| \dots |p^B_1 \dots p^B_{N_B}|} \prod_{\beta=1}^{B} J^{N_\beta}_{p^\beta_1 \dots p^\beta_{N_\beta}}$ according to the cycle structure.
- QFT of matrix models determines the weights of Riemann surfaces with decorated boundary components compatible with
  1. gluing (of fringes, not boundaries!)
  2. covariance (under $\Phi \mapsto U^* \Phi U$, which is not a symmetry!)

Ward identity

Covariance under $\Phi \mapsto U^* \Phi U$:

$$0 = \sum_{n\in I} \Big(\frac{(E_a - E_p)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \Big)$$

Theorem (2012): Ward identity for E of compact resolvent

$$\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an} \partial J_{np}} = \delta_{ap} \Big\{ V^2 \sum_{(K)} \frac{J_{P_1} \cdots J_{P_K}}{S_K} \Big(\sum_{n\in I} \frac{G_{|an|P_1|\dots|P_K|}}{V^{|K|+1}} + \frac{G_{|a|a|P_1|\dots|P_K|}}{V^{|K|+2}}$$
$$+ \sum_{r\geq 1} \sum_{q_1,\dots,q_r \in I} \frac{G_{|q_1 a q_1 \dots q_r|P_1|\dots|P_K|} J^r_{q_1\dots q_r}}{V^{|K|+1}} \Big)$$
$$+ V^4 \sum_{(K),(K')} \frac{J_{P_1} \cdots J_{P_K} J_{Q_1} \cdots J_{Q_{K'}}}{S_K S_{K'}} \frac{G_{|a|P_1|\dots|P_K|}}{V^{|K|+1}} \frac{G_{|a|Q_1|\dots|Q_{K'}|}}{V^{|K'|+1}} \Big\} \mathcal{Z}[J]$$
$$+ \frac{V}{E_p - E_a} \sum_{n\in I} \Big(J_{pn} \frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}[J]}{\partial J_{np}} \Big)$$

- J-derivatives of $\mathcal{Z}[J] = e^{-VS_{int}[\frac{\partial}{V\partial J}]} e^{\frac{V}{2}\langle J,J\rangle_E}$, where $\langle J, J\rangle_E := \sum_{m,n\in I} \frac{J_{mn}J_{nm}}{E_m + E_n}$, lead to Schwinger-Dyson equations.

Schwinger-Dyson equations (for $S_{int}[\Phi] = \frac{\lambda}{4}\mathrm{tr}(\Phi^4)$)

In a scaling limit $V \to \infty$ and $\frac{1}{V}\sum_{p\in I}$ finite, we have:

1. A closed non-linear equation for $G_{|ab|}$

$$G_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a+E_b)} \frac{1}{V}\sum_{p\in I}\Big(G_{|ab|}G_{|ap|} - \frac{G_{|pb|} - G_{|ab|}}{E_p - E_a}\Big)$$

2. For $N \geq 4$ a universal algebraic recursion formula

$$G_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda)\sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|}G_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G_{|b_{2l} b_1 \dots b_{2l-1}|}G_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

- 2. uses reality $\mathcal{Z} = \overline{\mathcal{Z}}$
- scaling limit corresponds to restriction to genus $g = 0$
- similar formulae for $B \geq 2$
- no index summation in $G_{|abcd|}$ $\Rightarrow$ β-function zero!

A3. Solution of $\lambda\phi^4_4$ on extreme Moyal space

Thermodynamic limit $\sqrt{V} = \frac{\theta}{4} \to \infty$ with $\frac{\mathcal{N}}{\sqrt{V\mu^4}}$ fixed:

- ‘continuous’ matrix indices $a, b \in [0, \Lambda^2]$ and sums $\mapsto$ integrals
- finite Hilbert transform arises: $\mathcal{H}^\Lambda_a(f) = \frac{1}{\pi}\mathcal{P}\int_0^{\Lambda^2} \frac{f(p)\,dp}{p-a}$

Theorem (2012/13) (for $\lambda < 0$, using $G_{b0} = G_{0b}$)

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\mathrm{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)] - \mathcal{H}^\Lambda_a[\tau_b(\bullet)])}$$

where $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a \mathcal{H}^\Lambda_a[G_{0\bullet}]}{G_{0a}}}\right)$ and G_{0b} solution of

$$G_{0b} = \frac{1}{1+b}\exp\left(-\lambda\int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}^\Lambda_p[G_{0\bullet}]}{G_{0p}}\right)^2}\right)$$

These and explicit formulae for higher functions (complicated for $G_{ab|cd}, G_{ab|cd|ef}, \ldots$) yield exact solution of $\lambda\phi^4_4$ for extreme Moyal.

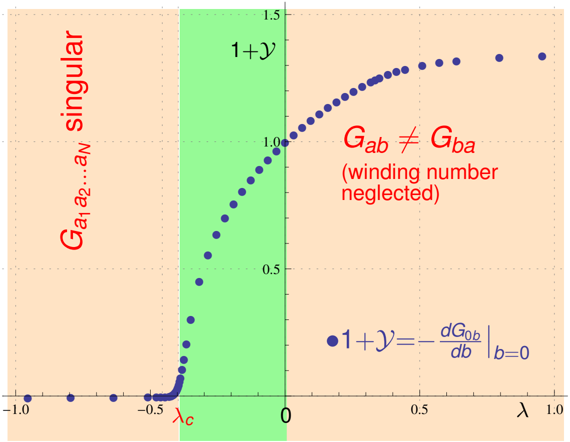
The fixed point equation

$$G_{0b} = \frac{1}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}_p^{\Lambda}[G_{0\bullet}]}{G_{0p}}\right)^2}\right)$$

1. For $\lambda > 0$ solution exists by Schauder fixed point theorem (but ambiguity due to winding number)
2. For $\lambda < 0$ and $\Lambda^2 \to \infty$ one exact solution is $G_{0b} = 1$
3. Perturbative solution reproduces all Feynman graphs. Polylogarithms and ζ-functions are generated.
4. Perturbation series does not converge: τ_b maps $[0, \Lambda^2]$ to $[0, \pi]$ for $\lambda > 0$, but to $[0, \epsilon]$ for $-\delta \leq \lambda < 0$
5. Formula can be put on a computer and solved by iteration.
6. Shows that $G_{0b} = 1$ is unstable, but attractive solution G_{0b} exists for all $\lambda \in \mathbb{R}$.

Computer simulation: evidence for phase transitions

piecewise linear approximation of G_{0b}, G_{ab} for $\Lambda^2=10^7$ and 2000 sample points. Consider $1+\mathcal{Y}:=-\frac{dG_{0b}}{db}\Big|_{b=0}$



- $(1+\mathcal{Y})'(\lambda)$ discontinuous at $\lambda_c = -0.39$
- order parameter $b_\lambda = \sup\{b : G_{0b}=1\}$ non-zero for $\lambda < \lambda_c$
- A key property for Schwinger functions is realised in $]\lambda_c, 0]$, not outside!
  The critical couplings coincide!

A4. From matrix model to Schwinger functions on $\mathbb{R}^4$

reverting harmonic oscillator basis, $1+\mathcal{Y} := -\frac{dG_{0b}}{db}\big|_{b=0} \dots$

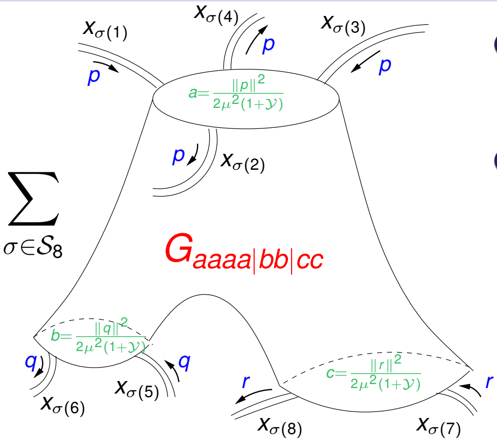
Theorem (2013): *connected* Schwinger functions

$$S_c(\mu x_1, \dots, \mu x_N) = \frac{1}{64\pi^2} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma \in \mathcal{S}_N} \Bigg(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4 p_\beta}{4\pi^2\mu^4} e^{\mathrm{i}\left\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta} (-1)^{i-1} \mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)} \right\rangle} \Bigg)$$

$$\times\, G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1} \Big| \dots \Big| \underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}}$$

- Schwinger functions are symmetric [OS4]
- invariant under full Euclidean group [OS2] (unexpected!!)
- growth conditions [OS1] established
- clustering [OS5] is violated: The $(N_1 + \dots + N_B)$-point functions are insensitive to the distance of different boundaries.
- remains: reflection positivity [OS3]

Connected (4+2+2)-point function



1. individual Euclidean invariance in every boundary component (no clustering)
2. particle scattering without momentum exchange
   - in 4D a sign of triviality (mind assumptions!)
   - familiar in 2D models with factorising S-matrix
   - a consequence of integrability [Moser, 1975] & [Kulish, 1976]

Is there a precise link between exact solution of our 4D model and traditional integrability in 2D? What about Yang-Baxter?

A5. Osterwalder-Schrader reflection positivity

Proposition (2013)

$S(x_1, x_2)$ is reflection positive iff $a \mapsto G_{aa}$ is a Stieltjes function,

$$G_{aa} = \int_0^\infty \frac{d(\rho(t))}{a+t} \qquad \text{with } \rho \text{ positive and non-decreasing.}$$

- Excluded for any $\lambda > 0$ (due to renormalisation)!

Positive definite functions on semigroups

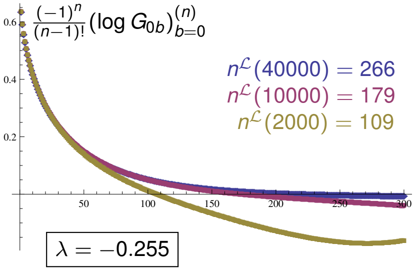
1. $\mathcal{C}$ = completely monotonic functions: $(-1)^n f^{(n)} \geq 0$
   - implies rep'n as Laplace transform $f(z) = \int_0^\infty d\mu(t)\, e^{-tz}$
2. $\mathcal{L} \subset \mathcal{C}$ = logarithmically CM: $(-1)^n (\log f)^{(n)} \geq 0$
3. $\mathcal{S} \subset \mathcal{L} \subset \mathcal{C}$ Stieltjes functions: $L_{k,t}[f(\bullet)] \geq 0$ where[Widder, 1938]

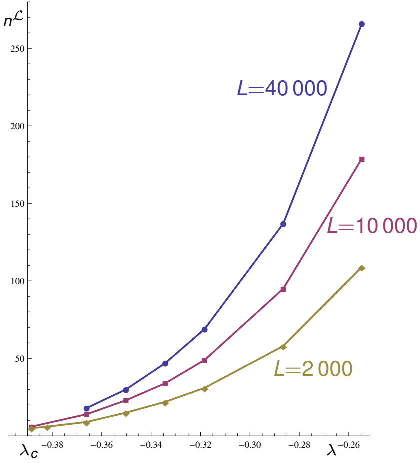
$$L_{k,t}[f(\bullet)] := \frac{(-t)^{k-1}}{c_k} \frac{d^{2k-1}}{dt^{2k-1}} \big(t^k f(t)\big)\,, \qquad c_1 = 1,\; c_{k>1} = k!(k-2)!$$

   - imply analyticity in cut plane $\mathbb{C} \setminus]-\infty, 0]$ with $\mathrm{Im}(f(z)) < 0$ for $\mathrm{Im}(z) > 0$ (anti-Herglotz function)

Positivity of approximated boundary function G_{0b}

λ	L	$n^{\mathcal{L}}$	$n^{\mathcal{C}}$	$n^{\mathcal{S}}$
-0.255	2000	109		
-0.255	10000	179		
-0.255	40000	266		
-0.318	2000	31	35	37
-0.318	10000	49	55	
-0.350	2000	15	17	18
-0.350	10000	23	25	26
-0.388	2000	5	5	6
-0.388	10000	6	7	8

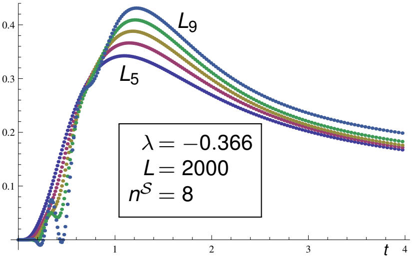


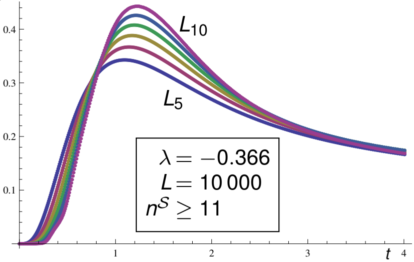


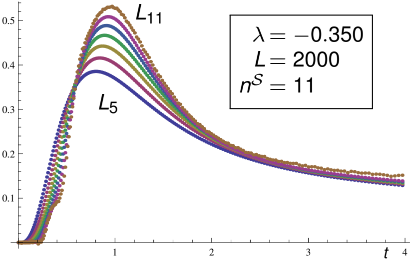
- improvement of $n^{\mathcal{L}}$ with $\uparrow L$ slows down precisely at λ_c!
- Stieltjes failure $n^{\mathcal{S}} > n^{\mathcal{L}}$!

Positivity of approximated G_{ab}: Widder's $L_{k,t}[G_{\bullet\bullet}]$

key step: integral formula for $\frac{\partial^{n+\ell} G_{ab}}{\partial^n a \partial^\ell b}$







- improvement of $n^{\mathcal{S}}$ with $\uparrow L$ and $\downarrow |\lambda|$
- convergence of $\int_0^{m^2} dt\, L_{k,t}[G_{\bullet\bullet}]$ to mass spectrum $\rho(m^2)$
- mass gap $\rho\big|_{[0,m_0^2]} = 0$, but no further gap!

Summary

1. $\lambda\phi^4_4$ on nc Moyal space is, at infinite noncommutativity, exactly solvable in terms of a fixed point solution
   - stable non-perturbative solution for $\lambda < 0$
   - phase transitions and critical phenomena, hence interesting statistical physics model
   - non-trivial as a matrix model
2. Projection to Schwinger functions for scalar field on $\mathbb{R}^4$:
   - Symmetry [OS4] automatic, growth control [OS1]
   - full Euclidean invariance [OS2]!
   - no clustering [OS5]
   - no momentum exchange (close to triviality), possibly a consequence of integrability
3. Reflection positivity [OS3] looks promising!
   Needs verification and extension to higher correlation functions