

Construction of a quantum field theory in four dimensions

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Introduction

axiomatic settings for rigorous quantum field theories by

- ① Wightman [1956]
- ② Haag-Kastler [1964]
- ③ Osterwalder-Schrader [1974]

today: numerous examples in dimension 1,2,3;

not a single non-trivial example in 4 dimensions

We have got a candidate:

- Construction of 4D Euclidean QFT is achieved (2012/13).
Find phase transitions and critical phenomena.
- Osterwalder-Schrader axioms are under investigation.
So far everything is OK.
- Non-triviality is open, but not impossible.
Ideally, we can get the 4D-analogue of factorising S-matrices [Iagolnitzer, 1978]; [Zamolodchikov-Zamolodchikov, 1979]

Historical notes on 4D QFT

- 1 Perturbative argument that **QED cannot exist as 4D QFT** [Landau-Abrikosov-Khalatnikov, 1954]
(this almost killed renormalisation theory)
- 2 Same argument (sign of β -function) for $\lambda\phi_4^4$.
 $\lambda\phi_{4+\epsilon}^4$ is trivial: [Aizenman, 1981]; [Fröhlich, 1982]
- 3 **Asymptotic freedom in QCD**
[Gross-Wilczek, 1973]; [Politzer, 1973]
- 4 **Construction of Yang-Mills theory is Millennium Prize problem.**

Having one example of a rigorously constructed 4D QFT, even with $S = e^{i\alpha} \mathbf{1}$, would be something. . .

Regularisation & renormalisation

- 1 We follow the **Euclidean track**, starting from a **partition function**.
- 2 To make this rigorous we need two regulators:
finite volume and **finite energy density**.
- 3 Pass to quantities (**densities** and with certain **normalised functions**) which have infinite volume & energy limits.

Symmetry

- The regulated theory **usually has less symmetry**. Proving that symmetry is restored in the end is part of the game.
- We propose another strategy:
Search for a regulator which has **more (or very different) symmetry**, **so constraining that it completely solves the model**.

With some luck, **a limit procedure** gives a **constructive QFT** on standard \mathbb{R}^4 . With even more luck, it satisfies OS.

A regularisation of ϕ_4^4

$$S[\phi] = 64\pi^2 \int_{\mathbb{R}^4} dx \left(\frac{Z}{2} \phi \star (-\Delta + \Omega^2 (2\Theta^{-1}x)^2 + \mu_{bare}^2) \phi + \frac{\lambda Z^2}{4} \phi \star \phi \star \phi \star \phi \right) (x)$$

with **Moyal product** $(f \star g)(x) = \int_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{dy dk}{(2\pi)^4} f(x + \frac{1}{2}\Theta k) g(x+y) e^{i\langle k, y \rangle}$

takes at $\Omega = 1$ in matrix basis $f_{\underline{mn}}(x) = f_{m_1 n_1}(x^0, x^1) f_{m_2 n_2}(x^3, x^4)$

$$f_{mn}(y^0, y^1) = 2(-1)^m \sqrt{\frac{m!}{n!}} \left(\sqrt{\frac{2}{\theta}} y \right)^{n-m} L_m^{n-m} \left(\frac{2|y|^2}{\theta} \right) e^{-\frac{|y|^2}{\theta}}$$

due to $f_{\underline{mn}} \star f_{\underline{kl}} = \delta_{\underline{nk}} f_{\underline{ml}}$ and $\int dx f_{\underline{mn}}(x) = 64\pi^2 V \delta_{\underline{mn}}$ the form

$$S[\Phi] = V \left(\sum_{\underline{m}, \underline{n} \in \mathbb{N}_{\mathcal{N}}^2} E_{\underline{m}} \Phi_{\underline{mn}} \Phi_{\underline{nm}} + \frac{Z^2 \lambda}{4} \sum_{\underline{m}, \underline{n}, \underline{k}, \underline{l} \in \mathbb{N}_{\mathcal{N}}^2} \Phi_{\underline{mn}} \Phi_{\underline{nk}} \Phi_{\underline{kl}} \Phi_{\underline{lm}} \right)$$

$$E_{\underline{m}} = Z \left(\frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu_{bare}^2}{2} \right), \quad |\underline{m}| := m_1 + m_2 \leq \mathcal{N}$$

- $V = \left(\frac{\theta}{4}\right)^2$ is for $\Omega = 1$ the **volume** of the nc manifold.

More generally: field-theoretical matrix models

Euclidean quantum field theory

- action $S[\Phi] = V \operatorname{tr}(E\Phi^2 + P[\Phi])$
for unbounded positive selfadjoint operator E with compact resolvent, and $P[\Phi]$ a polynomial
- partition function $\mathcal{Z}[J] = \int \mathcal{D}[\Phi] \exp(-S[\Phi] + V \operatorname{tr}(\Phi J))$

Observe: \mathcal{Z} is **covariant**, but **not invariant** under $\Phi \mapsto U\Phi U^*$:

$$0 = \int \mathcal{D}\Phi \left[E\Phi\Phi - \Phi\Phi E - J\Phi + \Phi J \right] \exp(-S[\Phi] + V \operatorname{tr}(\Phi J))$$

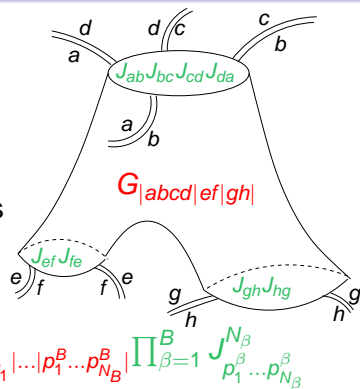
... choose E (but not J) diagonal, use $\Phi_{ab} = \frac{\partial}{V \partial J_{ba}}$:

Ward identity [Disertori-Gurau-Magnen-Rivasseau, 2007]

$$0 = \sum_{n \in I} \left(\frac{(E_a - E_p)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \right)$$

Topological expansion

- Feynman graphs in matrix models are **ribbon graphs**.
- Encode **genus- g** Riemann surface with **B boundary components**
- The k^{th} boundary component carries a **cycle** $J_{p_1 \dots p_{N_k}}^{N_k} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.



- Expand $\log \mathcal{Z}[\mathcal{J}] = \sum \frac{1}{\mathfrak{S}} V^{2-B} G_{|p_1^1 \dots p_{N_1}^1| \dots |p_1^B \dots p_{N_B}^B|} \prod_{\beta=1}^B J_{p_1^\beta \dots p_{N_\beta}^\beta}^{N_\beta}$ according to the cycle structure.
- The $G_{|p_1^1 \dots p_{N_1}^1| \dots |p_1^B \dots p_{N_B}^B|}$ become (smeared) **Schwinger functions**.
- QFT of matrix models determines the **weights of Riemann surfaces** with **decorated boundary components** compatible with (1) gluing and (2) symmetry.

For E of compact resolvent, the kernel of $E_p - E_a$ can be determined from the J -cycle structure in $\log \mathcal{Z}$:

Theorem (2012): Ward identity for E of compact resolvent

$$\begin{aligned} \sum_{n \in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an} \partial J_{np}} &= \delta_{ap} \left\{ V^2 \sum_{(K)} \frac{J_{P_1} \cdots J_{P_K}}{S_K} \left(\sum_{n \in I} \frac{G_{|a|n|P_1| \dots |P_K|}}{V^{|K|+1}} + \frac{G_{|a|a|P_1| \dots |P_K|}}{V^{|K|+2}} \right) \right. \\ &\quad \left. + \sum_{r \geq 1} \sum_{q_1, \dots, q_r \in I} \frac{G_{|q_1 a q_1 \dots q_r |P_1| \dots |P_K|} J_{q_1 \dots q_r}^r}{V^{|K|+1}} \right) \\ &\quad + V^4 \sum_{(K), (K')} \frac{J_{P_1} \cdots J_{P_K} J_{Q_1} \cdots J_{Q_{K'}}}{S_K S_{K'}} \frac{G_{|a|P_1| \dots |P_K|}}{V^{|K|+1}} \frac{G_{|a|Q_1| \dots |Q_{K'}|}}{V^{|K'|+1}} \left. \right\} \mathcal{Z}[J] \\ &\quad + \frac{V}{E_p - E_a} \sum_{n \in I} \left(J_{pn} \frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}[J]}{\partial J_{np}} \right) \end{aligned}$$

- J -derivatives of $\mathcal{Z}[J] = e^{-V S_{\text{int}}[\frac{\partial}{\partial J}]} e^{\frac{V}{2} \langle J, J \rangle_E}$, where $\langle J, J \rangle_E := \sum_{m, n \in I} \frac{J_{mn} J_{nm}}{E_m + E_n}$, lead to **Schwinger-Dyson equations**.
- The Theorem lets the usually infinite tower collapse:

Schwinger-Dyson equations (for $S_{int}[\phi] = \frac{\lambda}{4}\text{tr}(\phi^4)$)

In a scaling limit $V \rightarrow \infty$ and $\frac{1}{V} \sum_{p \in I}$ finite, we have

1. A closed non-linear equation for $G_{|ab|}$

$$G_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a + E_b)} \frac{1}{V} \sum_{p \in I} \left(G_{|ab|} G_{|ap|} - \frac{G_{|pb|} - G_{|ab|}}{E_p - E_a} \right)$$

2. For $N \geq 4$ a universal algebraic recursion formula

$$G_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|} G_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G_{|b_{2l} b_1 \dots b_{2l-1}|} G_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

- 2. uses **reality** $\mathcal{Z} = \overline{\mathcal{Z}}$
- scaling limit corresponds to restriction to genus $g = 0$
- similar formulae for $B \geq 2$
- no index summation in $G_{|abcd|} \Rightarrow$ **β -function zero!**

Back to $\lambda\phi_4^4$ on Moyal space

- Infinite volume limit (i.e. $\theta \rightarrow \infty$) turns discrete matrix indices into continuous variables $a, b, \dots \in \mathbb{R}_+$ and sums into integrals
- Need energy cutoff $a, b, \dots \in [0, \Lambda^2]$ and normalisation of lowest Taylor terms of two-point function $G_{|nm|} \mapsto G_{ab}$
- **Carleman-type singular integral equation** for $G_{ab} - G_{a0}$

Theorem (2012/13) (for $\lambda < 0$, using $G_{b0} = G_{0b}$)

Let $\mathcal{H}_a^\Lambda(f) = \frac{1}{\pi} \mathcal{P} \int_0^{\Lambda^2} \frac{f(p) dp}{p-a}$ be the *finite Hilbert transform*. Then

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\text{sign}(\lambda)(\mathcal{H}_0^\Lambda[\tau_0(\bullet)] - \mathcal{H}_a^\Lambda[\tau_b(\bullet)])}$$

where $\tau_b(a) := \arctan_{[0, \pi]} \left(\frac{|\lambda|\pi a}{b + \frac{1 + \lambda\pi a \mathcal{H}_a^\Lambda[G_{0\bullet}]}{G_{0a}}} \right)$ and G_{0b} solution of

$$G_{0b} = \frac{1}{1+b} \exp \left(-\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1 + \lambda\pi p \mathcal{H}_p^\Lambda[G_{0\bullet}]}{G_{0p}} \right)^2} \right)$$

Discussion

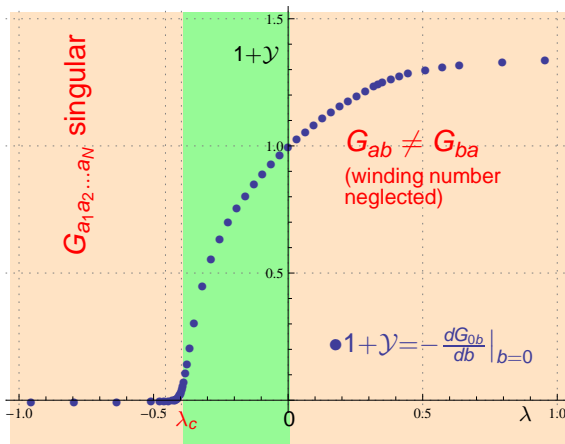
Together with explicit (but **complicated** for $G_{ab|cd}$, $G_{ab|cd|ef}$, \dots) formulae for higher correlation functions, we have **exact solution** of $\lambda\phi_4^4$ on **extreme Moyal space** in terms of

$$G_{0b} = \frac{1}{1+b} \exp \left(-\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p^2 t_p^\Lambda [G_{0\bullet}]}{G_{0p}} \right)^2} \right)$$

- ① For $\lambda > 0$ solution exists by **Schauder fixed point theorem** (but ambiguity due to winding number)
- ② For $\lambda < 0$ and $\Lambda^2 \rightarrow \infty$ one **exact solution is $G_{0b} = 1$**
- ③ Formula can be put on a computer and solved by iteration.
- ④ Shows that **$G_{0b} = 1$ is unstable**, but **attractive solution G_{0b} exists** for all $\lambda \in \mathbb{R}$.

Computer simulation: evidence for phase transitions

piecewise linear approximation of G_{0b} , G_{ab} for $\Lambda^2=10^7$ and 2000 sample points. Consider $1+\mathcal{Y} := -\frac{dG_{0b}}{db} \Big|_{b=0}$



- $(1 + \mathcal{Y})'(\lambda)$ discontinuous at $\lambda_c = -0.39$
- order parameter $b_\lambda = \sup\{b : G_{0b}=1\}$ non-zero for $\lambda < \lambda_c$
- Nothing particular at pole $\lambda_b = -\frac{1}{72} = 0.014$ of Borel resummation
- A key property for Schwinger functions is realised in subinterval of $[\lambda_c, 0]$, not outside!

Osterwalder-Schrader reconstruction theorem (1974)

Assume for Schwinger functions $S(x_1, \dots, x_N)$:

- §0 **growth rate:** $\left| \int dx f(x_1, \dots, x_N) S(x_1, \dots, x_N) \right| \leq c_1 (N!)^{c_2} |f|_{Nc_3}$
- §1 **Euclidean invariance:** $S(x_1, \dots, x_N) = S(Rx_1 + a, \dots, Rx_N + a)$
- §2 **reflection positivity:** for each (f_0, \dots, f_K) with $f_N \in \mathcal{S}(\mathbb{R}^{Nd})$,

$$\sum_{M, N=0}^K \int dx dy S(x_N, \dots, x_1, y_1, \dots, y_M) \overline{f_N(x_1, \dots, x_N)} f_M(y_1, \dots, y_M) \geq 0$$
 where $r(x^0, x^1, \dots, x^{d-1}) := (-x^0, x^1, \dots, x^{d-1})$
- §3 **permutation symmetry:** $S(x_1, \dots, x_N) = S(x_{\sigma(1)}, \dots, x_{\sigma(N)})$

Then the $S(\xi_1, \dots, \xi_{N-1})|_{\xi_i^0 > 0}$, with $\xi_i = x_i - x_{i+1}$, are **inverse Laplace-Fourier transforms** of FT $\hat{W}(q_1, \dots, q_{N-1})$ of Wightman distributions in a relativistic QFT.

If in addition the $S(x_1, \dots, x_N)$ satisfy

- §4 **clustering**

then the Wightman QFT has a unique vacuum state

From matrix model to Schwinger functions on \mathbb{R}^4

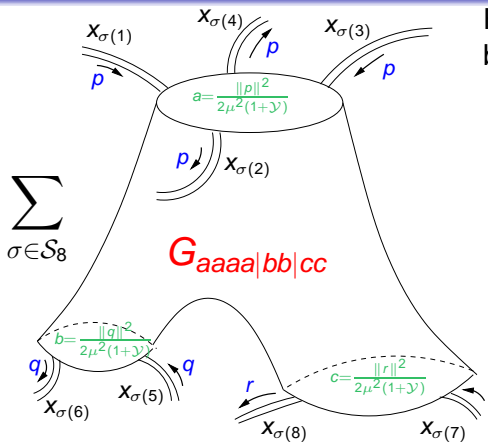
reverting harmonic oscillator basis, $1 + \mathcal{Y} := -\frac{dG_{0b}}{db} \Big|_{b=0} \dots$

Theorem (2013): *connected* Schwinger functions

$$\begin{aligned}
 & S_C(\mu X_1, \dots, \mu X_N) \\
 &= \frac{1}{64\pi^2} \sum_{\substack{N_1 + \dots + N_B = N \\ N_\beta \text{ even}}} \sum_{\sigma \in S_N} \left(\prod_{\beta=1}^B \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4 p_\beta}{4\pi^2 \mu^4} e^{i \langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta} (-1)^{i-1} \mu X_{\sigma(N_1 + \dots + N_{\beta-1} + i)} \rangle} \right) \\
 & \quad \times \mathbf{G} \underbrace{\left(\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})} \right)}_{N_1} \dots \underbrace{\left(\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})} \right)}_{N_B}
 \end{aligned}$$

- Schwinger functions are symmetric $\textcircled{S3}$ and **invariant under full Euclidean group** $\textcircled{S1}$ (completely unexpected for NCQFT)
- growth conditions $\textcircled{S0}$ established
- **clustering** $\textcircled{S4}$ is violated: The $(N_1 + \dots + N_B)$ -point functions are insensitive to the distance of different boundaries.
- remains: **reflection positivity** $\textcircled{S2}$

Connected (4+2+2)-point function



For each σ and every boundary component:

- 1 individual Euclidean symmetry
 - 2 identical momentum (cyclically opposite direction)
- translation of x_1, \dots, x_{2r} kills some, but not all σ (no clustering)
 - particle scattering without momentum exchange (close to triviality)

$G_{aaaa|bb|cc}$ merely has **external indices** put on-shell. Internally all **non-diagonal degrees of freedom contribute!** Non-triviality?

Osterwalder-Schrader reflection positivity

- Reflection positivity S2 gives spectrum condition which guarantees representation as Laplace transform in ξ^0 , hence **analyticity in $\text{Re}(\xi^0) > 0$** .

Proposition (2013)

$S(x_1, x_2)$ is reflection positive iff $a \mapsto G_{aa}$ is a **Stieltjes function**,

$$G_{aa} = \int_0^\infty \frac{d(\rho(t))}{a+t}$$

with ρ **positive and non-decreasing**. Proof: Källén-Lehmann

- **Excluded for any $\lambda > 0$** (due to renormalisation)!
- The Stieltjes property is a **particularly strong positivity** in mathematics.

Is **positivity in quantum field theory**

(Hilbert space scalar product and spectrum condition)

exactly the same as strong positivity in mathematics?

Classes of positive definite functions

$f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is **positive definite** if for any $x_1, \dots, x_n \in \mathbb{R}_+$ the matrix $F = (f(x_i + x_j))_{ij}$ is positive (semi-)definite. These are:

- 1 \mathcal{C} = completely monotonic functions: $(-1)^n f^{(n)} \geq 0$
 - implies rep'n as Laplace transform $f(z) = \int_0^\infty d\mu(t) e^{-tz}$
 - related to Bernstein and Pick/Nevalinna functions and Hausdorff moment problem

- 2 $\mathcal{L} \subset \mathcal{C}$ = logarithmically completely monotonic functions: $(-1)^n (\log f)^{(n)} \geq 0$

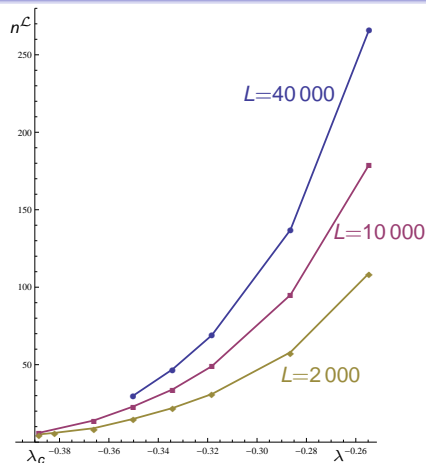
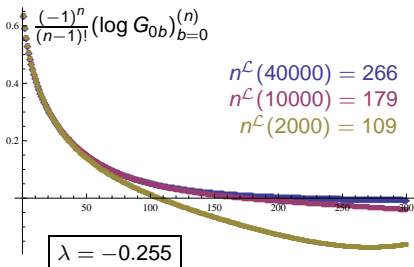
- 3 $\mathcal{S} \subset \mathcal{L} \subset \mathcal{C}$ Stieltjes functions: $L_{k,t}[f(\bullet)] \geq 0$ where [Widder, 1938]

$$L_{k,t}[f(\bullet)] := \frac{(-t)^{k-1}}{c_k} \frac{d^{2k-1}}{dt^{2k-1}} (t^k f(t)), \quad c_1 = 1, \quad c_{k>1} = k!(k-2)!$$

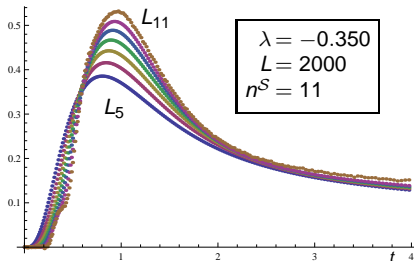
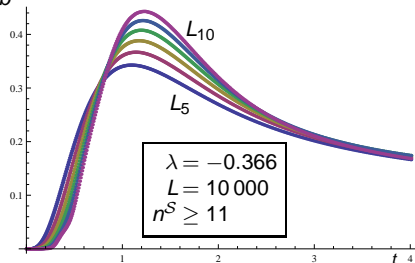
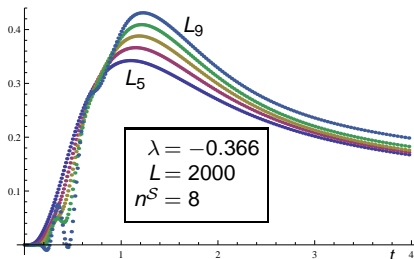
- imply analyticity in cut plane $\mathbb{C} \setminus]-\infty, 0]$ with $\text{Im}(f(z)) < 0$ for $\text{Im}(z) > 0$ (anti-Herglotz function)
- measure recovered from $\rho'(t) = \lim_{k \rightarrow \infty} L_{k,t}[f(\bullet)]$

Positivity of **approximated** boundary function G_{0b}

λ	L	$n^{\mathcal{L}}$	$n^{\mathcal{C}}$	$n^{\mathcal{S}}$
-0.255	2000	109		
-0.255	10000	179		
-0.255	40000	266		
-0.318	2000	31	35	37
-0.318	10000	49	55	
-0.350	2000	15	17	18
-0.350	10000	23	25	26
-0.388	2000	5	5	6
-0.388	10000	6	7	8



- improvement of $n^{\mathcal{L}}$ with $\uparrow L$ slows down precisely at λ_c !
- Stieltjes failure $n^{\mathcal{S}} > n^{\mathcal{L}}$!

Positivity of approximated G_{ab} : Widder's $L_{k,t}[G_{\bullet\bullet}]$ key step: integral formula for $\frac{\partial^{n+\ell} G_{ab}}{\partial^n a \partial^\ell b}$ 

- improvement of n^S with $\uparrow L$ and $\downarrow |\lambda|$
- convergence of $\int_0^{m^2} dt L_{k,t}[G_{\bullet\bullet}]$ to mass spectrum $\rho(m^2)$
- mass gap $\rho|_{[0, m_0^2]} = 0$, but no further gap!

Summary

- ① $\lambda\phi_4^4$ on nc Moyal space is, at infinite noncommutativity, **exactly solvable** in terms of a fixed point solution
 - **stable non-perturbative solution for $\lambda < 0$**
→ planar wrong-sign $\lambda\phi_4^4$ -model [t'Hooft; Rivasseau, 1983]
 - **phase transitions and critical phenomena**, hence interesting statistical physics model
 - non-trivial as a matrix model
- ② Projection to **Schwinger functions for scalar field on \mathbb{R}^4** :
 - \mathbb{S}_3 automatic, **full Euclidean symmetry** \mathbb{S}_1 , control about \mathbb{S}_0
 - no clustering \mathbb{S}_4
 - **no momentum exchange** (close to triviality), but scattering remnants from NCG substructure
- ③ Reflection positivity \mathbb{S}_2 does not fail immediately. Why? Needs verification and extension to higher correlation functions

(Non)-triviality?

Projection to diagonal matrices brings the non-trivial intermediate matrix model **close to triviality**. This is more subtle:

- suppose we can prove $\textcircled{S2}$, then reconstruct Hilbert space H , field operators $\varphi(f)$, unitaries $U(a, L)$ and **some vacuum Ω**
- uniqueness of Ω cannot be proved without clustering $\textcircled{S4}$
- main problem: **characterise set of Poincaré-invariant unit vectors of H , and find its extremal points Ω_e**
- each **restricted Hilbert space H_e** , generated by its cyclic vector Ω_e , **admits collision states** (Haag-Ruelle theory) and (if asymptotically complete) an **S-matrix**
- involves new Wightman distributions

$$W_e(x_1, \dots, x_N) = \langle \Omega_e, \varphi(x_1) \cdots \varphi(x_N) \Omega_e \rangle$$

expected to differ from $W(x_1, \dots, x_N) = \langle \Omega, \varphi(x_1) \cdots \varphi(x_N) \Omega \rangle$

Consequently, a **non-trivial $S \neq \mathbf{1}$ is not impossible**.