Construction of a quantum field theory in four dimensions

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Introduction

axiomatic settings for rigorous quantum field theories by

1. Wightman [1956]
2. Haag-Kastler [1964]
3. Osterwalder-Schrader [1974]

today: numerous examples in dimension 1,2,3;
not a single non-trivial example in 4 dimensions

this talk:

- We construct a toy model for 4D QFT.

  It involves noncommutative geometry at an intermediate step, but later lives on standard $\mathbb{R}^4$.

- There are good chances that the model satisfies Osterwalder-Schrader.

Regularisation & renormalisation

1. We follow the Euclidean track, starting from a partition function.
2. To make this rigorous we need two regulators: finite volume and finite energy density.
3. Pass to quantities (densities and with certain normalised functions) which have infinite volume & energy limits.

Symmetry

- The regulated theory usually has less symmetry. Proving that symmetry is restored in the end is part of the game.
- We propose another strategy:

  Search for a regulator which has more (or very different) symmetry, so constraining that it completely solves the model.

With some luck, a limit procedure gives a constructive QFT on standard $\mathbb{R}^4$. With even more luck, it satisfies OS.

A toy model for 4D non-linear QFT

- Regularise ϕ^4_4 on noncommutative Moyal space $\mathcal{M}_\theta$ with critical oscillator potential.
- This is essentially a matrix model, with infinite number of Ward identities from action of $U(\infty)$ group.
- These Ward id's, and the theory of singular integral equations, turn the Schwinger-Dyson eq's into a fixed point problem.

1. For $\theta \to \infty$ we prove existence of a solution.
2. Find numerical evidence for phase structure, phase transitions and critical phenomena.
3. Surprisingly, $\theta \to \infty$ describes Schwinger functions on $\mathbb{R}^4$!
4. They satisfy (OS0) boundedness, (OS1) invariance and (OS3) symmetry.
5. Provide numerical evidence for (OS2) reflection positivity of the 2-point function in one of the phases.

ϕ^4_4 on Moyal space with harmonic propagation

$$S[\phi]=64\pi^2\int_{\mathbb{R}^4} dx \Big(\frac{Z}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2_{bare})\phi+\frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with Moyal product $(f\star g)(x)=\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dy\,dk}{(2\pi)^4}f(x+\frac{1}{2}\Theta k)\,g(x+y)\,e^{i\langle k,y\rangle}$

takes at $\Omega=1$ in matrix basis $f_{\underline{mn}}(x)=f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\tfrac{2}{\theta}}y\Big)^{n-m}L_m^{n-m}\Big(\frac{2|y|^2)}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

due to $f_{\underline{mn}}\star f_{\underline{kl}}=\delta_{\underline{nk}}f_{\underline{ml}}$ and $\int dx\, f_{\underline{mn}}(x)=64\pi^2V\delta_{\underline{mn}}$ the form

$$S[\Phi]=V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}E_{\underline{m}}\Phi_{\underline{mn}}\Phi_{\underline{nm}}+\frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Big)$$

$$E_{\underline{m}}=Z\Big(\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu^2_{bare}}{2}\Big), \qquad |\underline{m}|:=m_1+m_2\le\mathcal{N}$$

- $V=\big(\frac{\theta}{4}\big)^2$ is for $\Omega=1$ the volume of the nc manifold.

More generally: field-theoretical matrix models

Euclidean quantum field theory

- action $S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + P[\Phi])$ for unbounded positive selfadjoint operator E with compact resolvent, and $P[\Phi]$ a polynomial
- partition function $\mathcal{Z}[J] = \int \mathcal{D}[\Phi]\ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))$

Observe: $\mathcal{Z}$ is covariant, but not invariant under $\Phi \mapsto U\Phi U^*$:

$$0 = \int \mathcal{D}\Phi \left[E\Phi\Phi - \Phi\Phi E - J\Phi + \Phi J\right] \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))$$

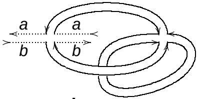
…choose E (but not J) diagonal, use $\Phi_{ab} = \frac{\partial}{V\partial J_{ba}}$:

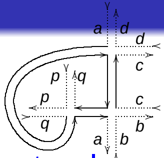
Ward identity [Disertori-Gurau-Magnen-Rivasseau, 2007]

$$0 = \sum_{n\in I}\left(\frac{(E_a - E_p)}{V}\frac{\partial^2\mathcal{Z}}{\partial J_{an}\partial J_{np}} + J_{pn}\frac{\partial\mathcal{Z}}{\partial J_{an}} - J_{na}\frac{\partial\mathcal{Z}}{\partial J_{np}}\right)$$

Topological expansion

- Feynman graphs in matrix models are ribbon graphs.





- Viewed as simplicial complexes, they encode the topology (B, g) of a genus-g Riemann surface with B boundary components (or punctures, marked points, holes, faces).
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1\dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.
- Expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p^1_1\dots p^1_{N_1}|\dots|p^B_1\dots p^B_{N_B}|} \prod_{\beta=1}^{B} J^{N_\beta}_{p^\beta_1\dots p^\beta_{N_\beta}}$ according to the cycle structure.
- The $G_{|p^1_1\dots p^1_{N_1}|\dots|p^B_1\dots p^B_{N_B}|}$ become (smeared) Schwinger functions.
- QFT of matrix models determines the weights of Riemann surfaces with decorated boundary components compatible with (1) gluing and (2) symmetry.

For E of compact resolvent, the kernel of $E_p - E_a$ can be determined from the J-cycle structure in $\log \mathcal{Z}$:

Theorem (2012): Ward identity for E of compact resolvent

$$\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} = \delta_{ap}\Big\{ V^2 \sum_{(K)} \frac{J_{P_1}\cdots J_{P_K}}{S_K}\Big(\sum_{n\in I}\frac{G_{|an|P_1|\dots|P_K|}}{V^{|K|+1}} + \frac{G_{|a|a|P_1|\dots|P_K|}}{V^{|K|+2}} + \sum_{r\geq 1}\sum_{q_1,\dots,q_r\in I}\frac{G_{|q_1aq_1\dots q_r|P_1|\dots|P_K|}J^r_{q_1\dots q_r}}{V^{|K|+1}}\Big) + V^4\sum_{(K),(K')}\frac{J_{P_1}\cdots J_{P_K}J_{Q_1}\cdots J_{Q_{K'}}}{S_K S_{K'}}\frac{G_{|a|P_1|\dots|P_K|}}{V^{|K|+1}}\frac{G_{|a|Q_1|\dots|Q_{K'}|}}{V^{|K'|+1}}\Big\}\mathcal{Z}[J] + \frac{V}{E_p - E_a}\sum_{n\in I}\Big(J_{pn}\frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}[J]}{\partial J_{np}}\Big)$$

- J-derivatives of $\mathcal{Z}[J] = e^{-VS_{int}[\frac{\partial}{V\partial J}]}e^{\frac{V}{2}\langle J,J\rangle_E}$, where $\langle J,J\rangle_E := \sum_{m,n\in I}\frac{J_{mn}J_{nm}}{E_m+E_n}$, lead to Schwinger-Dyson equations.
- The Theorem lets the usually infinite tower collape:

Schwinger-Dyson equations (for $S_{int}[\phi] = \frac{\lambda}{4}\mathrm{tr}(\phi^4)$)

In a scaling limit $V \to \infty$ and $\frac{1}{V}\sum_{p\in I}$ finite, we have

1. A closed non-linear equation for $G^{(0)}_{|ab|}$

$$G^{(0)}_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a + E_b)} \frac{1}{V} \sum_{p\in I} \Big(G^{(0)}_{|ab|} G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|} - G^{(0)}_{|ab|}}{E_p - E_a} \Big)$$

2. For $N \geq 4$ a universal algebraic recursion formula

$$G^{(0)}_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G^{(0)}_{|b_0 b_1 \dots b_{2l-1}|} G^{(0)}_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G^{(0)}_{|b_{2l} b_1 \dots b_{2l-1}|} G^{(0)}_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

- 2. uses reality $\mathcal{Z} = \overline{\mathcal{Z}}$
- scaling limit corresponds to restriction to genus $g = 0$
- similar formulae for $B \geq 2$

Renormalisation theorem

Theorem (2013)

Given a real quartic matrix model $S = V\,\mathrm{tr}(E\Phi^2 + \frac{\lambda}{4}\Phi^4)$ with E of compact resolvent.

Assume that the selfconsistency equation for $G^{(0)}_{|ab|}$ has a finite solution after affine renormalisation $E \mapsto Z(E + C\mathbb{1})$ and $\lambda \mapsto Z^2\lambda$. Then

- All higher functions $G^{(0)}_{|b_0 \dots b_{N-1}|}$ with $N \geq 4$ are automatically finite without further need of a renormalisation of λ.
- All quartic matrix models (with renormalisable $G^{(0)}_{|ab|}$) have vanishing β-function (i.e. are almost scale-invariant).
- The perturbative observation $\beta = 0$ for Moyal [Disertori-Gurau-Magnen-Rivasseau, 2007] is generic!

(Similar statements hold for $B \geq 2$)

Back to $\lambda\phi^4_4$ on Moyal space

- Infinite volume limit (i.e. $\theta \to \infty$) turns discrete matrix indices into continuous variables $a, b, \dots \in \mathbb{R}_+$ and sums into integrals
- Need energy cutoff $a, b, \dots \in [0, \Lambda^2]$ and normalisation of lowest Taylor terms of two-point function $G_{|nm|} \mapsto G_{ab}$
- Carleman-type singular integral equation for $G_{ab} - G_{a0}$

Theorem (2012/13) (for $\lambda < 0$, using $G_{b0} = G_{0b}$)

Let $\mathcal{H}^\Lambda_a(f) = \frac{1}{\pi}\mathcal{P}\int_0^{\Lambda^2} \frac{f(p)\,dp}{p-a}$ be the *finite Hilbert transform*. Then

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\operatorname{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)] - \mathcal{H}^\Lambda_a[\tau_b(\bullet)])}$$

where $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a\mathcal{H}^\Lambda_a[G_{0\bullet}]}{G_{0a}}}\right)$ and G_{0b} solution of

$$G_{0b} = \frac{1}{1+b}\exp\left(-\lambda\int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p\mathcal{H}^\Lambda_p[G_{0\bullet}]}{G_{0p}}\right)^2}\right)$$

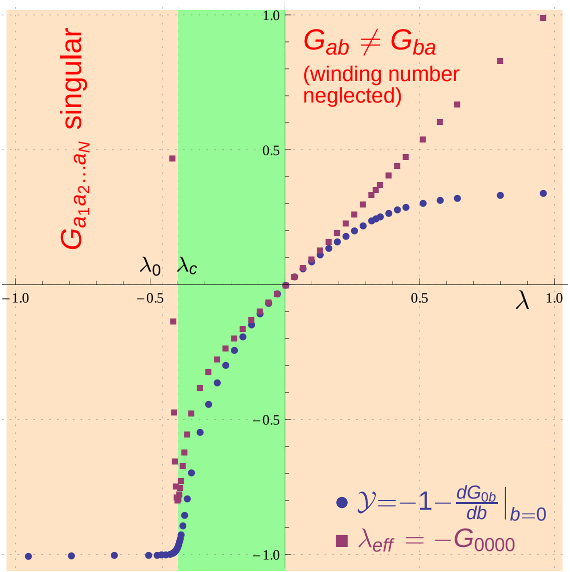
Discussion

Together with explicit (but complicated for $G_{ab|cd}$, $G_{ab|cd|ef}$, $\dots$) formulae for higher correlation functions, we have exact solution of $\lambda\phi^4_4$ on extreme Moyal space in terms of

$$G_{0b}=\frac{1}{1+b}\exp\left(-\lambda\int_0^b dt\int_0^{\Lambda^2}\frac{dp}{(\lambda\pi p)^2+\left(t+\frac{1+\lambda\pi p\mathcal{H}^\Lambda_p[G_{0\bullet}]}{G_{0p}}\right)^2}\right)$$

1. For $\lambda>0$ solution exists by Schauder fixed point theorem (but ambiguity due to winding number)
2. For $\lambda<0$ and $\Lambda^2\to\infty$ one exact solution is $G_{0b}=1$
3. Formula can be put on a computer and solved by iteration.
4. Shows that $G_{0b}=1$ is unstable, but attractive solution G_{0b} exists for all $\lambda\in\mathbb{R}$.

Computer simulation: evidence for phase transitions



- G_{ab} for $\Lambda^2 = 10^7$ with 2000 sample points
- $G''_{0b}(\lambda)$ discontinuous at $\lambda_c = -0.39$
- λ_{eff} singular at $\lambda_0 = -0.455$ where $G'_{0b}(\lambda) = 0$
- Nothing particular at pole $\lambda_b = -\frac{1}{72} = 0.014$ of Borel resummation
- A key property for Schwinger functions is realised in subinterval of $[\lambda_c, 0]$, not outside!

Osterwalder-Schrader reconstruction theorem (1974)

Assume for Schwinger functions $S(x_1,\dots,x_N)$:

- S0 growth rate: $\Big|\int dx\, f(x_1,\dots,x_N)S(x_1,\dots,x_N)\Big| \le c_1(N!)^{c_2}|f|_{Nc_3}$
- S1 Euclidean invariance: $S(x_1,\dots,x_N) = S(Rx_1{+}a,\dots,Rx_N{+}a)$
- S2 reflection positivity: for each $(f_0,\dots,f_K)$ with $f_N \in \mathcal{S}(\mathbb{R}^{Nd})$,

$$\sum_{M,N=0}^{K} \int dxdy\; S(x_N,\dots,x_1,y_1,\dots y_M)\overline{f_N({}^r x_1,\dots,{}^r x_N)}f_M(y_1,\dots,y_M) \ge 0$$

where ${}^r(x^0,x^1,\dots x^{d-1}) := (-x^0,x^1,\dots x^{d-1})$

- S3 permutation symmetry: $S(x_1,\dots,x_N) = S(x_{\sigma(1)},\dots,x_{\sigma(N)})$

Then the $S(\xi_1,\dots,\xi_{N-1})\big|_{\xi_i^0>0}$, with $\xi_i = x_i - x_{i+1}$, are inverse Laplace-Fourier transforms of FT $\hat{W}(q_1,\dots,q_{N-1})$ of Wightman distributions in a relativistic QFT.

If in addition the $S(x_1,\dots,x_N)$ satisfy

- S4 clustering

then the Wightman QFT has a unique vacuum state

From matrix model to Schwinger functions on $\mathbb{R}^4$

folding $G_{\dots}$ with eigenbasis of 4D harmonic oscillator:

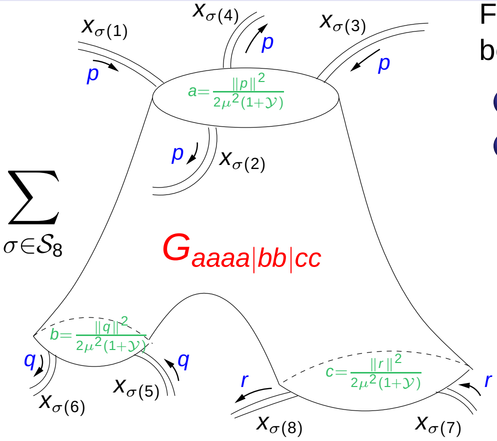
Theorem (2013): *connected* Schwinger functions

$$S_c(\mu x_1,\dots,\mu x_N) = \frac{1}{64\pi^2} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \Bigg(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4p_\beta}{4\pi^2\mu^4} e^{\mathrm{i}\left\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta}(-1)^{i-1}\mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)}\right\rangle}\Bigg)$$

$$\times G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1}\Big|\dots\Big|\underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}}$$

- Schwinger functions are symmetric (S3) and invariant under full Euclidean group (S1) (completely unexpected for NCQFT)
- growth conditions (S0) established
- clustering (S4) is violated: The $(N_1+\dots+N_B)$-point functions are insensitive to the distance of different boundaries.
- remains: reflection positivity (S2)

Connected (4+2+2)-point function



For each σ and every boundary component:

1. individual Euclidean symmetry
2. identical momentum (cyclically opposite direction)

- translation of $x_1, \dots, x_{2r}$ kills some, but not all σ (no clustering)
- particle scattering without momentum exchange (close to triviality)

$G_{aaaa|bb|cc}$ merely has external indices put on-shell. Internally all non-diagonal degrees of freedom contribute! Non-triviality?

Osterwalder-Schrader reflection positivity

- Reflection positivity S2 gives spectrum condition which guarantees representation as Laplace transform in ξ^0, hence analyticity in $\mathrm{Re}(\xi^0) > 0$.

Proposition (2013)

$S(x_1, x_2)$ is reflection positive iff $a \mapsto G_{aa}$ is a Stieltjes function,

$$G_{aa} = \int_0^\infty \frac{d(\rho(t))}{a+t}$$

with ρ positive and non-decreasing. Proof: Källén-Lehmann

- Excluded for any $\lambda > 0$ (due to renormalisation)!
- The Stieltjes property is a particularly strong positivity in mathematics.

Is positivity in quantum field theory
(Hilbert space scalar product and spectrum condition)
exactly the same as strong positivity in mathematics?

Classes of strongly positive smooth functions

1. $\mathcal{C}$ = completely monotonic functions: $(-1)^n f^{(n)} \geq 0$
   - implies rep'n as Laplace transform $f(z) = \int_0^\infty d\mu(t)\, e^{-tz}$
   - related to Bernstein and Pick/Nevanlinna functions and Hausdorff moment problem

2. $\mathcal{L} \subset \mathcal{C}$ = logarithmically completely monotonic functions: $(-1)^n (\log f)^{(n)} \geq 0$

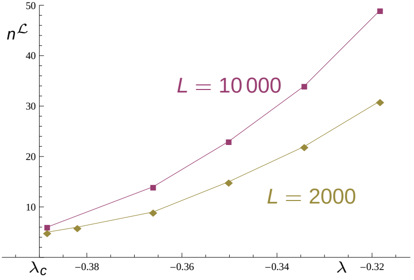
3. $\mathcal{S} \subset \mathcal{L} \subset \mathcal{C}$ Stieltjes functions: $L_{k,t}[f(\bullet)] \geq 0$ where[Widder, 1938]

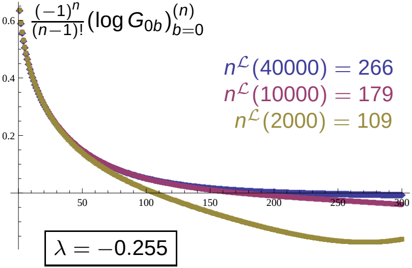
$$L_{k,t}[f(\bullet)] := \frac{(-t)^{k-1}}{c_k} \frac{d^{2k-1}}{dt^{2k-1}}\big(t^k f(t)\big)\,, \qquad c_1 = 1,\; c_{k>1} = k!(k-2)!$$

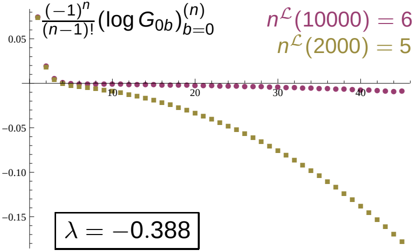
   - imply analyticity in cut plane $\mathbb{C} \setminus]-\infty, 0]$ with $\mathrm{Im}(f(z)) < 0$ for $\mathrm{Im}(z) > 0$ (anti-Herglotz function)
   - measure recoved from $\rho'(t) = \lim_{k\to\infty} L_{k,t}[f(\bullet)]$

Positivity of the boundary two-point function G_{0b}

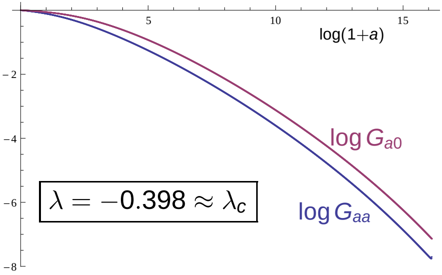
λ	L	$n^{\mathcal{L}}$	$n^{\mathcal{C}}$	$n^{\mathcal{S}}$
-0.255	2000	109		
-0.255	10000	179		
-0.255	40000	266		
-0.318	2000	31	35	37
-0.318	10000	49	55	
-0.350	2000	15	17	18
-0.350	10000	23	25	26
-0.388	2000	5	5	6
-0.388	10000	6	7	8

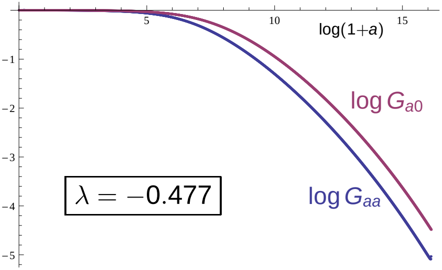


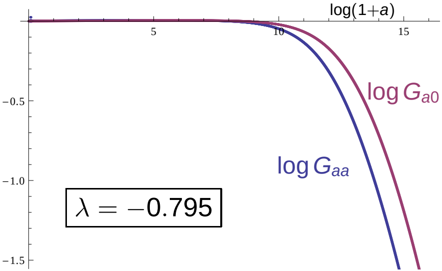




An order parameter



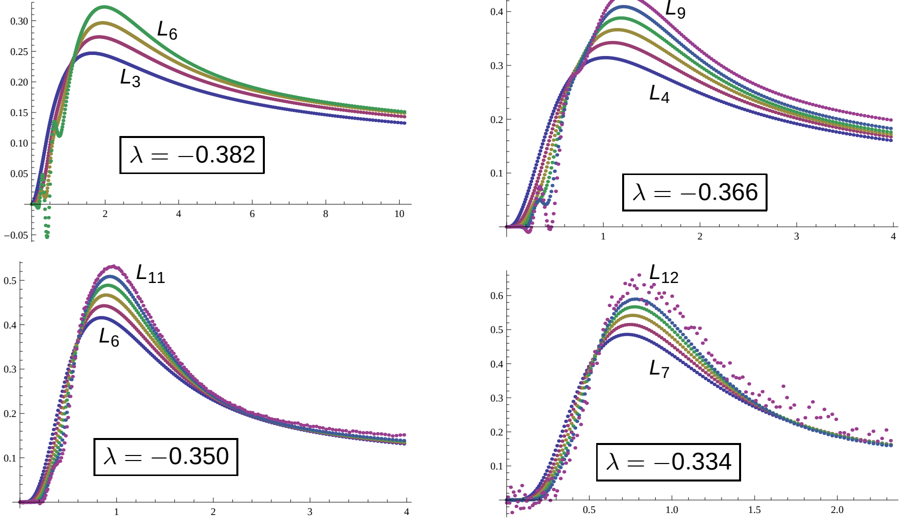




- Solution $G_{0b} = 1$ becomes stable for $\lambda < \lambda_c$
- $A := \sup\{b \;:\; G_{0b} = 1\}$ is order parameter; $A = 0$ for $\lambda > \lambda_c$ and $A > 0$ for $\lambda < \lambda_c$.
- Higher correlation function ill-defined for matrix indices $\leq A$

Positivity of G_{ab}: Widder's $L_{k,t}[G_{\bullet\bullet}]$

key step: integral formula for $\frac{\partial^{n+\ell} G_{ab}}{\partial^n a \partial^\ell b}$



Summary

1. $\lambda\phi^4_4$ on nc Moyal space is, at infinite noncommutativity, exactly solvable in terms of a fixed point solution
   - stable non-perturbative solution for $\lambda < 0$
     $\rightarrow$ planar wrong-sign $\lambda\phi^4_4$-model [t'Hooft; Rivasseau, 1983]
   - phase transitions and critical phenomena, hence interesting statistical physics model
   - non-trivial as a matrix model
2. Projection to Schwinger functions for scalar field on $\mathbb{R}^4$:
   - S3 automatic, full Euclidean symmetry S1, control about S0
   - no clustering S4
   - no momentum exchange (close to triviality), but scattering remnants from NCG substructure
3. Reflection positivity S2 does not fail immediately. Why? Needs verification and extension to higher correlation functions

(Non)-triviality?

Projection to diagonal matrices brings the non-trivial intermediate matrix model close to triviality. This is more subtle:

- suppose we can prove S2, then reconstruct Hilbert space H, field operators $\varphi(f)$, unitaries $U(a,L)$ and some vacuum Ω
- uniqueness of Ω cannot be proved without clustering S4
- main problem: characterise set of Poincaré-invariant unit vectors of H, and find its extremal points Ω_e
- each restricted Hilbert space H_e, generated by its cyclic vector Ω_e, admits collision states (Haag-Ruelle theory) and (if asymptotically complete) an S-matrix
- involves new Wightman distributions
$$W_e(x_1,\dots,x_N) = \langle \Omega_e, \varphi(x_1)\cdots\varphi(x_N)\Omega_e\rangle$$
expected to differ from $W(x_1,\dots,x_N) = \langle \Omega, \varphi(x_1)\cdots\varphi(x_N)\Omega\rangle$

Consequently, a non-trivial $S \neq \mathbb{1}$ is not impossible.