Construction of a 4D (noncommutative) quantum field theory

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(based on joint work with Harald Grosse,
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Introduction

- Quantum field theory (QFT) is the theory that describes Nature at very high energy density.
- One famous such experiment measures the magnetic moment g of the electron: $\frac{g_{experiment}}{2} = 1.001\,159\,652\,180\,7$
- QFT predicts that number in terms of the electron charge e measured to $e^{-2} = 137.035\,999\,084$:

$$\frac{g_{QFT}}{2} = 1 + \frac{1}{2}\frac{e^2}{\pi} + \Big\{\frac{197}{144} + \Big(\frac{1}{2} - 3\log 2\Big)\zeta(2) + \frac{3}{4}\zeta(3)\Big\}\Big(\frac{e^2}{\pi}\Big)^2$$
$$+ \Big\{\frac{28259}{5184} + \Big(\frac{17101}{235} - \frac{596}{2}\cdot\log 2\Big)\zeta(2) + \frac{139}{18}\zeta(3) + \frac{100}{3}\mathrm{Li}_4\Big(\frac{1}{2}\Big)$$
$$+ \frac{25}{18}\Big(\log^4 2 - \pi^2\log^2 2\Big) - \frac{239\zeta(4) - 166\zeta(2)\cdot\zeta(3) + 215\zeta(5)}{24}\Big\}\Big(\frac{e^2}{\pi}\Big)^3$$
$$+ \Big\{\dots\Big\}\Big(\frac{e^2}{\pi}\Big)^4$$
$$= 1.001\,159\,652\,153\,5$$

- The discipline that achieves this spectacular agreement with experiment is called perturbative quantum field theory.
- It starts from a classical field theory model and computes quantum corrections as series in quantities like $\frac{e^2}{\pi}$.
- Contributions to $(\frac{e^2}{\pi})^n$ are sums of integrals encoded by n-loop Feynman graphs.
- The problem is: The radius of convergence of the perturbation series is zero!

  (Another mystery is confinement in QCD)

The success of perturbative QFT lacks a mathematical understanding.

- Refined summation techniques (e.g. Borel) may establish reasonable domains of analyticity.
- Unfortunately, this also fails for QED due to the Landau ghost problem.

  It is expected to work for non-Abelian gauge theories because of asymptotic freedom.
- But these theories are too complicated (millenium prize problem), and simpler 4D-models (such as ϕ^4_4) cannot be constructed.

A main difficulty is the non-linearity of field equations.

A toy model

Non-linear problems sometimes solvable by fixed point methods

- We propose a toy model of a 4D Euclidean QFT inspired by noncommutative geometry.
- This is essentially a matrix model. The action of the $U(\infty)$ group induces an infinite number of Ward identities.
- The Ward identities turn the Schwinger-Dyson equations into a fixed point problem which is solvable in the infinite volume limit.
- We find numerical evidence for a phase structure, phase transitions and critical phenomena.
- The infinite volume limit restores Euclidean covariance and symmetry. In one of the phases we find numerical evidence for reflection positivity of the 2-point function.

Field-theoretical matrix models

- A matrix is for us a compact (Hilbert-Schmidt) operator on Hilbert space $H = L^2(I,\mu)$.
- realise as integral kernel operators: $M = (M_{ab}) \in L^2(I\times I, \mu\times\mu)$
  - product: $(MN)_{ab} = \int_I d\mu(c)\; M_{ac}M_{cb}$
  - trace: $\mathrm{tr}(M) = \int_I d\mu(a)\; M_{aa}$
  - adjoint: $(M^*)_{ab} = \overline{M_{ba}}$
- action = non-linear functional S for $\phi = \phi^*$ in volume V:

$$S[\phi] = V\,\mathrm{tr}(E\phi^2 + P[\phi])$$

  E – unbounded positive selfadjoint op. with compact resolvent, $P[\phi]$ – polynomial in ϕ with scalar coefficients

- partition function $\mathcal{Z}[J] = \int \mathcal{D}[\phi]\; \exp(-S[\phi] + V\,\mathrm{tr}(\phi J))$
- For $P[\phi] \equiv 0$, $\mathcal{D}[\phi]e^{-V\mathrm{tr}(E\phi^2)}/\mathcal{Z}[0]$ is Gaußian measure (of covariance determined by E) on random selfadjoint matrices.

Topological expansion

- Connected Feynman graphs in matrix models are ribbon graphs.
- Viewed as simplicial complexes, they encode the topology (B,g) of a genus-g Riemann surface with B boundary components (or punctures, marked points, holes, faces).
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1\dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.
- Expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p_1\dots p_{N_1}|\dots|q_1\dots q_{N_B}|} J^{N_1}_{p_1\dots p_{N_1}} \cdots J^{N_B}_{q_1\dots q_{N_B}}$ according to the cycle structure.
- QFT of matrix models determines the weights of Riemann surfaces with $\mathbb{N}$-decorated punctures compatible with gluing.

Ward identity

- Unitary transformation $\phi \mapsto U\phi U^*$ leads to Ward identity

$$0 = \int \mathcal{D}\phi \left[E\phi\phi - \phi\phi E - J\phi + \phi J\right] \exp(-S[\phi] + V\,\mathrm{tr}(\phi J))$$

that describes how E, J break the invariance of the action.

…choose E (but not J) diagonal, use $\phi_{ab} = \frac{\partial}{V\partial J_{ba}}$:

Proposition [Disertori-Gurau-Magnen-Rivasseau, 2006]

The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities

$$0 = \sum_{n\in I} \left(\frac{(E_a - E_p)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \right)$$

For E of compact resolvent we can always assume that $m \mapsto E_m > 0$ is injective!

We turn the Ward identity for E injective into formula for $\sum_{n\in I}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}$. The J-cycle structure in $\log\mathcal{Z}$ creates

- singular contributions $\sim \delta_{ap}$
- regular contributions present for all a, p

Theorem (Ward identity for injective E)

$$\sum_{n\in I}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} = \delta_{ap}\Big\{V^2\sum_{(K)}\frac{J_{P_1}\cdots J_{P_K}}{S_K}\Big(\sum_{n\in I}\frac{G_{|an|P_1|\dots|P_K|}}{V^{|K|+1}}+\frac{G_{|a|a|P_1|\dots|P_K|}}{V^{|K|+2}}$$
$$+\sum_{r\geq 1}\sum_{q_1,\dots,q_r\in I}\frac{G_{|q_1aq_1\dots q_r|P_1|\dots|P_K|}J^r_{q_1\dots q_r}}{V^{|K|+1}}\Big)$$
$$+V^4\sum_{(K),(K')}\frac{J_{P_1}\cdots J_{P_K}J_{Q_1}\cdots J_{Q_{K'}}}{S_K S_{K'}}\frac{G_{|a|P_1|\dots|P_K|}}{V^{|K|+1}}\frac{G_{|a|Q_1|\dots|Q_{K'}|}}{V^{|K'|+1}}\Big\}\mathcal{Z}[J]$$
$$+\frac{V}{E_p-E_a}\sum_{n\in I}\Big(J_{pn}\frac{\partial\mathcal{Z}[J]}{\partial J_{an}}-J_{na}\frac{\partial\mathcal{Z}[J]}{\partial J_{np}}\Big)$$

Schwinger-Dyson equations

Write $S = \frac{V}{2}\sum_{a,b}(E_a + E_b)\phi_{ab}\phi_{ba} + VS_{int}[\phi]$.
Functional integration yields, up to irrelevant constant,

$$\mathcal{Z}[J] = e^{-VS_{int}[\frac{\partial}{V\partial J}]} e^{\frac{V}{2}\langle J,J\rangle_E} , \qquad \langle J,J\rangle_E := \sum_{m,n\in I} \frac{J_{mn}J_{nm}}{E_m + E_n}$$

- Much better than the perturbative expansion of $e^{-VS_{int}[\frac{\partial}{V\partial J}]}$ is to apply J-derivatives to $\mathcal{Z}[J]$.
  $\rightarrow$ Choose them to give $G_{\dots}$ on the rhs.
- These external derivatives combine with internal derivatives from $S_{int}[\frac{\partial}{V\partial J}]$ to certain identities for $G_{\dots}$.

These Schwinger-Dyson equations are often of little use because they express an N-point function in terms of $(N+2)$-point functions.

Schwinger-Dyson equations (for $S_{int}[\phi] = \frac{\lambda}{4}\mathrm{tr}(\phi^4)$)

$\sum_n \frac{\partial^2 \mathcal{Z}}{\partial J_{an}\partial J_{np}}$ and reality $\mathcal{Z} = \overline{\mathcal{Z}}$ let the usually infinite tower of Schwinger-Dyson equations collapse:

after genus expansion $G_{\dots} = \sum_{g=0}^{\infty} V^{-2g} G^{(g)}_{\dots}$:

1. A closed non-linear equation for $G^{(0)}_{ab}$ (planar+regular)

$$G^{(0)}_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a + E_b)} \frac{1}{V} \sum_{p\in I} \Big(G^{(0)}_{|ab|} G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|} - G^{(0)}_{|ab|}}{E_p - E_a} \Big)$$

2. For $N \geq 3$ a universal algebraic recursion formula

$$G^{(0)}_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G^{(0)}_{|b_0 b_1 \dots b_{2l-1}|} G^{(0)}_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G^{(0)}_{|b_{2l} b_1 \dots b_{2l-1}|} G^{(0)}_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

Restriction to genus $g = 0$ is exact for $V \to \infty$ and $\frac{1}{V}\sum_{p\in I}$ finite

Renormalisation theorem

The renormalisation leaves algebraic equations invariant:

Theorem

Given a real scalar matrix model with $S = V\,\mathrm{tr}(E\phi^2 + \frac{\lambda}{4}\phi^4)$ and $m \mapsto E_m$ injective, which determines the set $G_{|p^1_1 \dots p^1_{N_1}|\dots|p^B_1 \dots p^B_{N_B}|}$ of $(N_1 + \dots + N_B)$-point functions.

Assume the basic functions with all $N_i \leq 2$ are turned finite by $E_a \mapsto Z(E_a + \frac{\mu^2}{2} - \frac{\mu^2_{bare}}{2})$ and $\lambda \mapsto Z^2\lambda$.

Then all functions with one $N_i \geq 3$

1. are finite without further need of a renormalisation of λ, i.e. all renormalisable quartic matrix models have vanishing β-function. The observation $\beta = 0$ for Moyal is generic!
2. are given by algebraic recursion formulae in terms of renormalised basic functions with $N_i \leq 2$.

ϕ^4_4 on Moyal space with harmonic propagation

$$S[\phi]=64\pi^2\int_{\mathbb{R}^4} dx \Big(\frac{Z}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2_{bare})\phi+\frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with Moyal product $(f\star g)(x)=\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dy\,dk}{(2\pi)^4}f(x+\frac{1}{2}\Theta k)\,g(x+y)\,e^{\mathrm{i}\langle k,y\rangle}$

takes at $\Omega=1$ in matrix basis $f_{\underline{mn}}(x)=f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\frac{2}{\theta}}y\Big)^{n-m}L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

due to $f_{\underline{mn}}\star f_{\underline{kl}}=\delta_{\underline{nk}}f_{\underline{ml}}$ and $\int dx\,f_{\underline{mn}}(x)=64\pi^2V\delta_{\underline{mn}}$ the form

$$S[\phi]=V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}E_{\underline{m}}\,\phi_{\underline{mn}}\phi_{\underline{nm}}+\frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}}\phi_{\underline{mn}}\phi_{\underline{nk}}\phi_{\underline{kl}}\phi_{\underline{lm}}\Big)$$

$$E_{\underline{m}}=Z\Big(\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu^2_{bare}}{2}\Big)\,,\qquad |\underline{m}|:=\underline{m}_1+\underline{m}_2\leq\mathcal{N}$$

- $V=\left(\frac{\theta}{4}\right)^2$ is for $\Omega=1$ the volume of the noncommutative manifold which is sent to ∞ in the thermodynamic limit.

Scaling limit $\mathcal{N}, \mu^4 V \to \infty$ with $\frac{\mathcal{N}}{\sqrt{\mu^4 V}} = \Lambda^2$ fixed

- Matrix indices become continuous $\frac{|p|}{\sqrt{V}} \mapsto \mu^2 p$ with $p \in [0, \Lambda^2]$
- Difference of eqns for $G^{(0)}_{|ab|}$ & $G^{(0)}_{|a0|}$ cancels worst divergence
- $\Lambda \to \infty$: Renormalisation $\mu_{bare} \mapsto \mu$ and $Z^{-1} \mapsto (1+\mathcal{Y})$ by normalisation $G^{(0)}_{|00|} = \mu^{-2}$ and $\frac{dG^{(0)}_{|ab|}}{db}\Big|_{a=b=0} = -\mu^{-2}(1+\mathcal{Y})$

Integral equation for Hölder-continuous $G_{ab} = \mu^2 G^{(0)}_{|ab|}$ and $\Lambda \to \infty$

$$\left(\frac{b}{a} + \frac{1 + \lambda\pi a \mathcal{H}_a\big[G_{\bullet 0}\big]}{a G_{a0}}\right) D_{ab} - \lambda\pi \mathcal{H}_a\big[D_{\bullet b}\big] = -G_{a0}$$

where

- $D_{ab} := \frac{a}{b}(G_{ab} - G_{a0})$, $\qquad \mathcal{Y} = -\lambda \int_0^\infty \frac{dp}{p}\, D_{p0}$
- Hilbert transform $\quad \mathcal{H}_a[f(\bullet)] := \frac{1}{\pi} \lim_{\epsilon \to 0} \left(\int_0^{a-\epsilon} + \int_{a+\epsilon}^\infty \right) \frac{f(q)\, dq}{q-a}$

- $\lambda < 0$: $\mathcal{Z}[E, J]$ undefined, but eqns. for $G_{\dots}$ extend to $\lambda < 0$!

The Carleman equation

Theorem [Carleman 1922, Tricomi 1957]

The singular linear integral equation

$$h(a)y(a) - \lambda\pi\mathcal{H}_a[y] = f(a)\;, \qquad a \in]0,\Lambda^2[$$

is for $h(a)$ continuous + Hölder near $0,\Lambda^2$ and $f \in L^p$ solved by

$$\begin{aligned} y(a) &= \frac{\sin(\vartheta(a))e^{-\mathcal{H}_a[\pi-\vartheta]}}{\lambda\pi a}\Big(a\,f(a)e^{\mathcal{H}_a[\pi-\vartheta]}\cos(\vartheta(a)) \\ &\qquad + \mathcal{H}_a\Big[e^{\mathcal{H}_\bullet[\pi-\vartheta]}\bullet f(\bullet)\sin(\vartheta(\bullet))\Big] + C\Big) \\ &\overset{*}{=} \frac{\sin(\vartheta(a))e^{\mathcal{H}_a[\vartheta]}}{\lambda\pi}\Big(f(a)e^{-\mathcal{H}_a[\vartheta]}\cos(\vartheta(a)) \\ &\qquad + \mathcal{H}_a\Big[e^{-\mathcal{H}_\bullet[\vartheta]}f(\bullet)\sin(\vartheta(\bullet))\Big] + \frac{C'}{\Lambda^2 - a}\Big) \end{aligned}$$

$$\vartheta(a) = \underset{[0,\pi]}{\arctan}\Big(\frac{\lambda\pi}{h(a)}\Big)\;, \quad \sin(\vartheta(a)) = \frac{|\lambda\pi|}{\sqrt{(h(a))^2 + (\lambda\pi)^2}}$$

where C, C' are arbitrary constants.

Normalisation conditions may select one of the solutions.

Solution

- angle $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\dfrac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a\mathcal{H}_a[G_{\bullet 0}]}{G_{a0}}}\right)$
- reversal:
$$G_{a0} = \frac{\sin(\tau_0(a))}{|\lambda|\pi a} e^{\mathrm{sign}(\lambda)(\mathcal{H}_0[\tau_0(\bullet)] - \mathcal{H}_a[\tau_0(\bullet)])} \begin{cases} 1 & \text{for } \lambda<0 \\ \left(1+\frac{Ca}{\Lambda^2-a}\right) & \text{for } \lambda>0 \end{cases}$$
- Addition theorems and Tricomi's identity $e^{-\mathcal{H}_a[\tau]}\cos(\tau(a)) + \mathcal{H}_a\big[e^{-\mathcal{H}_\bullet[\tau]}\sin(\tau(\bullet))\big] = 1$ give:

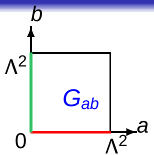
Theorem

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\mathrm{sign}(\lambda)(\mathcal{H}_0[\tau_0(\bullet)] - \mathcal{H}_a[\tau_b(\bullet)])} \begin{cases} 1 & \text{for } \lambda<0 \\ \left(1+\frac{Ca+bF(b)}{\Lambda^2-a}\right) & \text{for } \lambda>0 \end{cases}$$

- Consequence: $G_{ab} \geq 0$ (at least for $\lambda < 0$)!

The self-consistency equation

Given boundary value G_{a0},
Carleman computes G_{ab}, in particular G_{0b}.
Symmetry forces $G_{b0} = G_{0b}$:



Master equation

The theory is completely determined by the solution of the fixed point equation $G = TG$

$$G_{b0} = \frac{\begin{Bmatrix} 1 & \text{for } \lambda<0 \\ 1+bF(b) & \text{for } \lambda>0 \end{Bmatrix}}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

Theorem

For $\lambda > 0$ but $F(b) = 0$, the master equation has a solution $G_{\bullet 0} \in \mathcal{C}^1_0(\mathbb{R}_+)$ by the Schauder fixed point theorem. The solution is automatically smooth and monotonously decrasing.

Translation to 4D Euclidean QFT model

infinite volume limit $\mu^4 V \to \infty$ requires densities

Schwinger functions

$$\mu^N S_c(\mu x_1, \dots, \mu x_N)$$

$$:= \lim_{V\mu^4\to\infty} \sum_{\underline{m}_1,\underline{n}_1,\dots,\underline{m}_N,\underline{n}_N\in\mathbb{N}^2} f_{\underline{m}_1\underline{n}_1}(x_1)\cdots f_{\underline{m}_N\underline{n}_N}(x_N) \frac{\mu^{4N}\partial^N \mathcal{F}[J]}{\partial J_{\underline{m}_1\underline{n}_1}\dots\partial J_{\underline{m}_N\underline{n}_N}}\Big|_{J=0}$$

$$\mathcal{F}[J] := \frac{1}{64\pi^2 V^2\mu^8} \log\left(\frac{\int \mathcal{D}[\phi]\, e^{-S[\phi]+V\sum_{\underline{a},\underline{b}\in\mathbb{N}^2}\phi_{\underline{ab}}J_{\underline{ba}}}}{\int \mathcal{D}[\phi]\, e^{-S[\phi]}}\right)_{\substack{Z\mu^2_{bare}\mapsto\mu^2 \\ Z\mapsto(1+\mathcal{Y})}}$$

- J-cycle structure in $\mathcal{F}$ produces $f_{\underline{mn}}$-cycles for every face: $\sum_{\underline{m}_1,\dots,\underline{m}_j} f_{\underline{m}_1\underline{m}_2}\cdots f_{\underline{m}_{j-1}\underline{m}_j} f_{\underline{m}_j\underline{m}_1} G_{|\dots|\underline{m}_1\dots\underline{m}_j|\dots|}$
- Write $G_{|\dots|\underline{m}_1\dots\underline{m}_j|\dots|}$ for every face as Laplace transform in $\frac{|\underline{m}_1|+\dots+|\underline{m}_j|}{\sqrt{V}}$ and Fourier transform in $\frac{|\underline{m}_{i+1}|-|\underline{m}_i|}{\sqrt{V}}$

Lemma

(with $J+i \equiv i, \quad |z_i|<1$)

$$\sum_{m_1,\dots,m_J=0}^{\infty} \prod_{i=1}^{J} z_i^{m_i} L_{m_i}^{m_{i+1}-m_i}(r_i) = \frac{\exp\left(-\dfrac{\sum_{i,k=1}^{J} r_i (z_{k+i}\cdots z_{J+i})}{1-(z_1\cdots z_J)}\right)}{1-(z_1\cdots z_J)}$$

- $1-(z_1\cdots z_J) \stackrel{V\to\infty}{\longrightarrow} \begin{cases} 2 & (J \text{ odd}) \\ \frac{t}{\sqrt{V}} & (J \text{ even}) \end{cases}$ (t – Laplace par., $r \propto \frac{x^2}{\sqrt{V}}$)
- gives factor $V^{\#(\text{even faces})}$, and G gives factor $V^{-\#(\text{all faces})}$

Proposition

$$S_c(\mu x_1,\dots,\mu x_N) = \frac{1}{64\pi^2} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \left(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4 p_\beta}{4\pi^2\mu^4} e^{\mathrm{i}\left\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta} (-1)^{i-1} \mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)} \right\rangle} \right)$$

$$\times\, G\Big(\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1} \Big| \dots \Big| \underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B} \Big)$$

Results

- Only a restricted sector of the matrix model contributes to position space: All faces have common matrix indices.
- Schwinger functions are symmetric and invariant under the full Euclidean group (this is limit $\theta \to \infty$!)
- Most interesting sector: every face has $N_\beta = 2$ indices. It is described by the functions $G_{\left|\frac{p_1^2}{2(1+\mathcal{Y})}\frac{p_1^2}{2(1+\mathcal{Y})}\right|\dots\left|\frac{p_B^2}{2(1+\mathcal{Y})}\frac{p_B^2}{2(1+\mathcal{Y})}\right|}$
- This sector describes propagation and interaction of B particles without any momentum exchange. This is usually accepted in 2D, but not in 4D! . . . Triviality?
- The pattern is not compatible with free particles because $(N_1 + \dots + N_B)$-point functions violate clustering. The corresponding decomposition reflects non-trivial topological sectors.

Analytic continuation to Minkowski space

- Requires Schwinger functions to satisfy axioms of [Osterwalder-Schrader, 1974], in particular reflection positivity.
- For the 2-point function, analyticity + reflection positivity boil down to require that $a \mapsto G_{aa}$ is a Stieltjes function (consequence of Källén-Lehmann spectral representation)
- $f : \mathbb{R}_+ \to \mathbb{R}$ is Stieltjes iff $f(x) = c + \int_0^\infty \frac{d(\rho(t))}{x+t}$ with $c \geq 0$ and ρ positive and non-decreasing.

Theorem [Widder, 1938]

$f : \mathbb{R}_+ \to \mathbb{R}$ is Stieltjes iff smooth and $L_{k,t}[f(\bullet)] \geq 0$ $\forall t \in \mathbb{R}_+$ and $k \in \mathbb{N}$, where $L_{0,t}[f(\bullet)] = f(t)$, $L_{1,t}[f(\bullet)] = \frac{d}{dt}(tf(t))$ and

$$L_{k,t}[f(\bullet)] := \frac{(-t)^{k-1}}{k!(k-2)!} \frac{d^{2k-1}}{dt^{2k-1}}(t^k f(t)) , \qquad k \geq 2$$

In this case $\rho'(t) = \lim_{k\to\infty} L_{k,t}[f(\bullet)]$ (weakly and a.e.).

Perturbative argument: $a \mapsto G_{aa}$ cannot be Stieltjes for $\lambda > 0$!

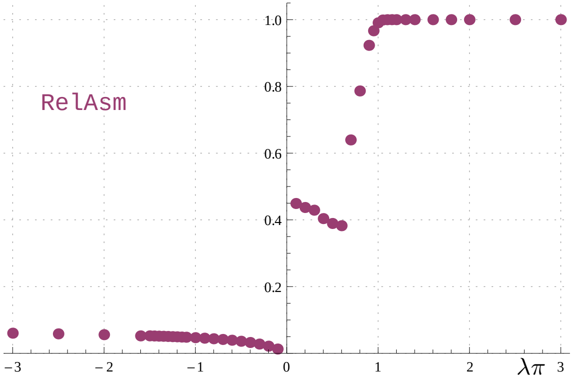
Computer simulations

- We implement G_{a0} for $a \in [0, \Lambda^2]$ as piecewise-linear function with edges arranged as geometric progression.
- We find numerically that the operator T in the fixed point equation $G = TG$ (with $F(b) = 0$) satisfies the assumptions of the Banach fixed point theorem in Lipschitz space.
- The sequence $G^{n+1}_{a0} = (TG^n)_{a0}$ converges for any λ.

  There is no discontinuity of $G_{a0}(\lambda)$ at $\lambda = 0$.
- The required symmetry $G_{ab} = G_{ba}$ is numerically (for $F(b) = C = 0$)
  - accurately realised for any $\lambda < 0$
  - badly violated for any $\lambda > 0$

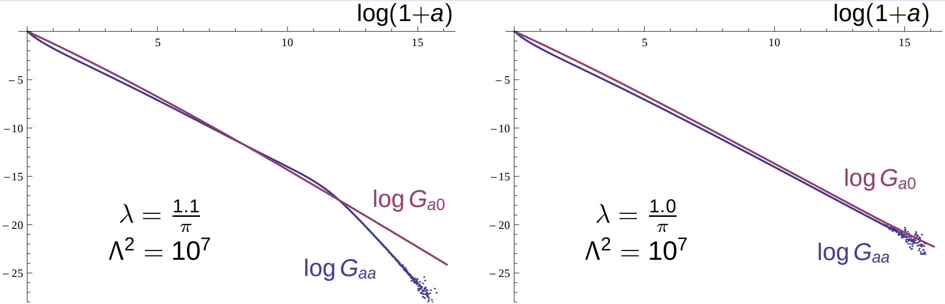
  This is clear evidence that $C \neq 0$ and $F(b) \neq 0$ for $\lambda > 0$. (further investigation currently out of reach)

Relative asymmetry $\sup_{a,b} \frac{|G_{ab}-G_{ba}|}{G_{ab}+G_{ba}}$



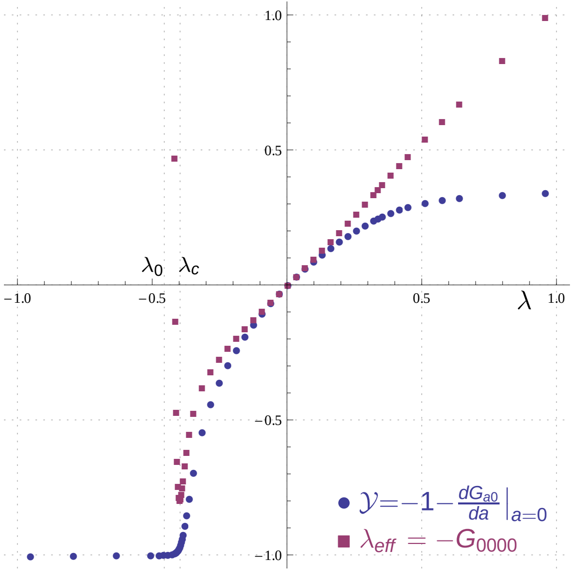
- results for $\Lambda^2 = 10^7$ with 2000 sample points
- $\approx$ 5% asymmetry for $\lambda < 0$, traced back to discretisation
- $\approx$ 40% asymmetry for $\lambda > 0$, increases to 100% at $\lambda\pi \approx 0.6$
- Neglection of C, F for $\lambda > 0$ not justified!

$\log G_{a0}$ and $\log G_{aa}$ as function of $\log(1+a)$



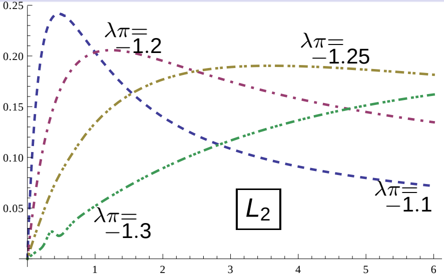
- For $\lambda \geq \frac{1.1}{\pi}$ the function G_{aa} suddenly bends and increases the (negative) slope by 1.
- As $\lambda \searrow 0$, this bend moves to larger a, possibly to $a \gg \Lambda^2$. This might explain the jump of asymmetry.
- For $\lambda > 0$ we have $G_{aa} \approx \frac{C}{(1+a)^{1+\eta}}$ with $\eta > 0$. Such functions are not Stieltjes (negative anomalous dimension).

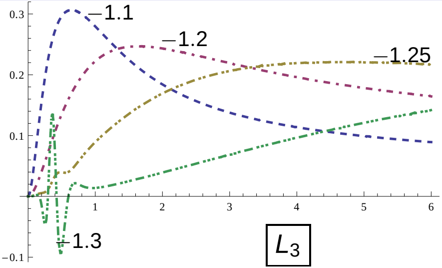
Main result: evidence for phase transition

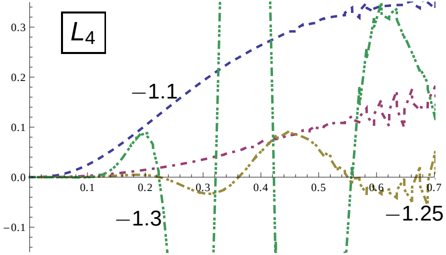


- results for $\Lambda^2 = 10^7$ with 2000 sample points
- $\mathcal{Y}'$ discontinuous at $\lambda_c = -0.396$
- Excellent agreement with $\lambda_s = -0.392$ where Stieltjes property is lost
- λ_{eff} singular at $\lambda_0 = -0.455$ where $\mathcal{Y} = -1$
- Nothing particular at pole $\lambda_b = -\frac{1}{72} = 0.014$ of Borel resummation

Widder's criteria $L_{k,a}[G_{\bullet\bullet}] := \frac{(-a)^{k-1}}{k!(k-2)!}\frac{d^{2k-1}}{da^{2k-1}}(a^k G_{aa}) \geq 0$



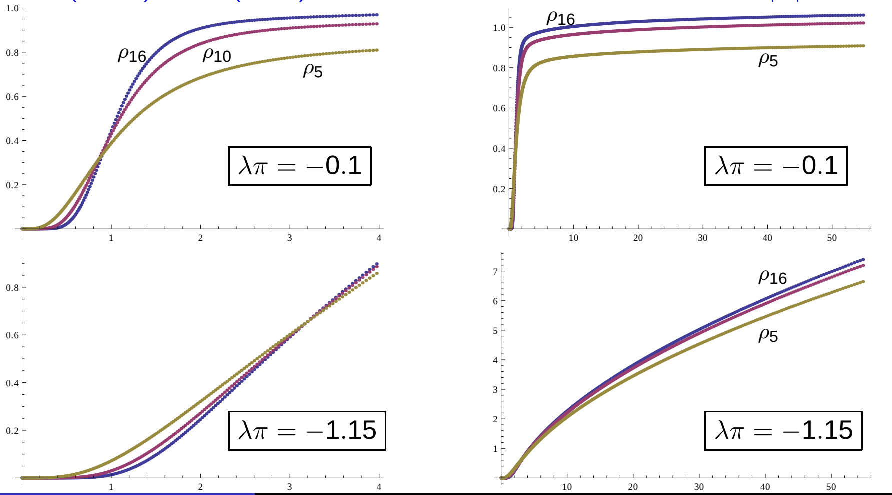




- based on interpolation of discrete data, noisy for $k \geq 4$
- Stieltjes clearly violated for $\lambda < \lambda_c$

Integrated "mass densities" $\rho_k(m^2) = \int_0^{m^2} dt\; L_{k,t}[G_{\bullet 0}]$

$$\frac{(\log G_{a0})^{(\ell)}}{(\ell-1)!} = \frac{(-1)^\ell}{(1+a)^\ell} + (-1)^\ell \operatorname{sign}(\lambda)\,\mathcal{H}^\Lambda_0\Big[\sin\big(\ell\tau_a(\bullet)\big)\Big(\frac{\sin\tau_a(\bullet)}{|\lambda|\pi\bullet}\Big)^\ell\Big]$$



Summary

1. The quartic matrix model $\mathcal{Z} = \int dM \exp(\mathrm{tr}(JM - EM^2 - \frac{\lambda}{4}M^4))$ is exactly solvable in terms of solution of a non-linear equation.
2. ϕ^4_4-theory on Moyal space is of that type. The non-linear equation is reduced to a fixed-point problem.
3. The ϕ^4_4-Moyal matrix model has a unique non-perturbative and non-trivial solution for $\lambda \leq 0$.
4. The corresponding Euclidean quantum field theory is only sensitive to diagonal matrix elements.
5. The model might satisfy Osterwalder-Schrader axioms for $\lambda \in [-0.394 \pm 0.003, 0]$, definitely not outside this interval.
6. If true, this would be a Wightman QFT in 4 dimensions. Question: Can absence of gap $[m^2, 4m^2]$ circumvent triviality?
7. It describes interacting particles without momentum transfer. $G_{aa|bb} \neq 0$ implies presence of non-trivial vacuum sectors.