Exact solution of the quartic matrix model and application to 4D noncommutative QFT

Raimar Wulkenhaar

Mathematisches Institut, Westfälische Wilhelms-Universität Münster

joint work with Harald Grosse (Vienna)

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Introduction

- Quantum field theory (QFT) is the theory that describes Nature at very high energy density.
- One famous such experiment measures the magnetic moment g of the electron: $\frac{g_{experiment}}{2} = 1.001\,159\,652\,180\,7$
- QFT predicts that number in terms of the electron charge e measured to $e^{-2} = 137.035\,999\,084$:

$$\begin{aligned}\frac{g_{QFT}}{2} &= 1 + \frac{1}{2}\frac{e^2}{\pi} + \left\{\frac{197}{144} + \left(\frac{1}{2} - 3\log 2\right)\zeta(2) + \frac{3}{4}\zeta(3)\right\}\left(\frac{e^2}{\pi}\right)^2 \\ &+ \left\{\frac{28259}{5184} + \left(\frac{17101}{235} - \frac{596}{2}\cdot\log 2\right)\zeta(2) + \frac{139}{18}\zeta(3) + \frac{100}{3}\mathrm{Li}_4\left(\frac{1}{2}\right)\right. \\ &\quad \left. + \frac{25}{18}\left(\log^4 2 - \pi^2\log^2 2\right) - \frac{239\zeta(4) - 166\zeta(2)\cdot\zeta(3) + 215\zeta(5)}{24}\right\}\left(\frac{e^2}{\pi}\right)^3 \\ &+ \left\{\ldots\right\}\left(\frac{e^2}{\pi}\right)^4 \\ &= 1.001\,159\,652\,153\,5\end{aligned}$$

- The discipline that achieves this spectacular agreement with experiment is called perturbative quantum field theory.
- It starts from a classical field theory model and computes quantum corrections as series in quantities like $\frac{e^2}{\pi}$.
- Contributions to $(\frac{e^2}{\pi})^n$ are sums of integrals encoded by n-loop Feynman graphs.
- The problem is: The radius of convergence of the perturbation series is zero!

The success of perturbative QFT lacks a mathematical understanding.

Contents

1. I start with some background material from statistical physics and Euclidean quantum field theory.

   Some rigorously constructed models will be mentionend.

2. Matrix models arise in a limit of quantum chromodynamics for infinitely many colours, from 2D quantum gravity and from string theory.

   I introduce a formulation close to NCG and show how the presence of the $U(\infty)$-group allows one to exactly solve the matrix model with quartic interaction.

3. The ϕ^4-model on noncommutative $\mathbb{R}^4$ (Moyal space) is of this type. The solution involves singular integral equations and the Schauder fixed point theorem.

Statistical Physics

consider spin maps $\sigma : \mathbb{Z}^d \to M$ (discrete space)

- assign probability $p[\sigma] = \frac{1}{\mathcal{Z}} \exp(-\beta H[\sigma])$ to Hamiltonian $H[\sigma]$, where $\mathcal{Z} := \sum_\sigma \exp(-\beta H[\sigma])$ is partition function
- gives rise to expectation value of observables $\mathcal{O}[\sigma]$ by $\langle \mathcal{O} \rangle = \sum_\sigma p[\sigma]\mathcal{O}[\sigma] = \frac{1}{\mathcal{Z}} \sum_\sigma \mathcal{O}[\sigma] \exp(-\beta H[\sigma])$
- often critical behaviour at certain inverse temperature β_c, i.e. the correlation length diverges
- power-law behaviour of physical quantities near the critical point ($\to$ critical exponents)
- relations between and universality of the critical exponents expained by renormalisation group [Wilson, 1971]

Solvable statistical physics models

- Ising model (1925), spins $\sigma : \mathbb{Z}^d \to \pm 1$
  2D critical model solved by [Onsager, 1944]
- Potts model (1952), spins $\sigma : \mathbb{Z}^d \to \{e^{\frac{2\pi i k}{q}}\}$,
  in 2D solvable for $q = 3$ and $q = \infty$ (XY-model)
- 6-vertex model (or ice-model), solved by [Lieb, 1967]
- 8-vertex model, solved by [Baxter, 1971]
- Hard hexagon model, solved by [Baxter, 1980]
- many 1D-models (1D Potts model, Heisenberg model, Hubbard model, Luttinger model, Toda lattice, . . .)

Methods: transfer matrix, Bogoliubov transformation, quantum inverse scattering method, renormalisation group, Yang-Baxter equation, Rogers-Ramanujan identity

Euclidean QFT

make Feynman's path integral rigorous (Feynman-Kac formula) [Symanzik, 1964]

- starting point is action $S[\phi] = \int_{\mathbb{R}^d} dx\,\mathcal{L}[\phi]$ of classical field theory on Euclidean space
- assign measure $p[\phi]\mathcal{D}[\phi] = \dfrac{\exp(-S[\phi])}{\mathcal{Z}}\mathcal{D}[\phi]$, where $\mathcal{Z} := \int \mathcal{D}[\phi]\ \exp(-S[\phi])$ is partition function
- main difficulty is to make sense of the measure $p[\phi]\mathcal{D}[\phi]$ (renormalisation)
- gives candidates for Schwinger functions $\mathcal{S}(x_1,\dots,x_N) = \dfrac{1}{\mathcal{Z}}\int \mathcal{D}[\phi]\ \phi(x_1)\cdots\phi(x_N)\exp(-S[\phi])$

Osterwalder-Schrader reconstruction theorem

Distinguish a vector $v \in \mathbb{R}^d$ (which becomes time) and let $x \mapsto \bar{x}$ be the reflection on the hyperplane $v^\perp$.

Theorem [Osterwalder-Schrader, 1973–1975]

Assume for $\mathcal{S}(x_1, \dots, x_N)$:

1. symmetry and analyticity
2. Euclidean covariance
3. reflection positivity: for each $N \leq N_0$ test function $f_N \in \mathcal{S}^N$,
$$\sum_{M,N} \int dxdy\; \mathcal{S}(x_1, \dots, x_N, y_1, \dots y_M)\overline{f_N(\bar{x}_1, \dots, \bar{x}_N)} f_M(y_1, \dots, y_M) \geq 0$$
4. cluster property

Then the analytical continuation of $\mathcal{S}(x_1, \dots, x_N)$ provides Wightman functions of a true relativistic QFT.

This is the main road to construct non-trivial QFT models!

Constructed quantum field theories

- 2D ϕ^4-model [Glimm-Jaffe, 1968–1972]
- $P(\phi)_2$, i.e. 2D scalar field with polynomial interaction [Glimm-Jaffe-Spencer, 1974], [Simon, 1974]
- 3D ϕ^4-model [Glimm-Jaffe, 1973], [Feldman-Osterwalder, 1976]
- Gross-Neveu model (1974), constructed in 2D by [Gawędzki-Kupiainen, 1985] and [Feldman-Magnen-Rivasseau-Séneor, 1986]
- non-example: ϕ^4 in $D = 4+\epsilon$ is trivial [Aizenman, 1981], [Fröhlich, 1982]

Methods: random walks, cluster expansion, correlation inequalities, Borel resummation

Models in 2D conformal field theory

- Infinitesimal 2D conformal transformations form the infinite-dimensional Witt algebra.
- Most conformal quantum field theories have a conformal anomaly that leads to a central extension of the Witt algebra.
- The resulting Virasoro algebra has generators L_n, $n \in \mathbb{Z}$,
$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$
- Highest-weight represesentations of the Virasoro algebra define a family of solvable "Minimal Models" at central charge
$$c = 1 - \frac{6}{m(m+1)}, \quad m = 3, 4, 5, \ldots$$
[Belavin-Polyakov-Zamolodchikov, Friedan-Qiu-Schenker, 1984]
$m = 3$ is critical Ising model, $m = 5$ is 3-state Potts model,...
- Solvable model of different type: Wess-Zumino-Witten model. The solution is realised by affine Kac-Moody algebras.

Matrix models

1. 2D quantum gravity is the enumeration of random triangulations of surfaces.
   - Its asymptotic behaviour is captured by the matrix model partition function

$$\mathcal{Z} = \int dM \; \exp\Big(-\mathcal{N}\sum_n t_n \operatorname{tr}(M^n)\Big) \;, \qquad M = M^* \in M_{\mathcal{N}}(\mathbb{C})$$

   - For $\mathcal{N} \to \infty$, this series in (t_n) is evaluated in terms of the τ-function for the Korteweg-de Vries (KdV) hierarchy.
2. 2D topological quantum gravity has correlation functions which are intersection numbers of complex curves.
   - They can be arranged into a generating functional with series parameters (t_n).

[Witten, 1990] conjectured that both (t_n)-series are the same.

The Kontsevich model

- [Kontsevich, 1992] computed the intersection numbers in terms of weighted sums over ribbon graphs.
- He proved these graphs to be generated from the Airy function matrix model (Kontsevich model)

$$\mathcal{Z}[E] = \frac{\displaystyle\int dM \; \exp\Big(-\tfrac{1}{2}\mathrm{tr}(EM^2) + \tfrac{\mathrm{i}}{6}\mathrm{tr}(M^3)\Big)}{\displaystyle\int dM \; \exp\Big(-\tfrac{1}{2}\mathrm{tr}(EM^2)\Big)}\,, \qquad M{=}M^*{\in}M_{\mathcal{N}}(\mathbb{C})$$

for $E = E^* > 0$ and $t_n = (2n{-}1)!!\mathrm{tr}(E^{-(2n-1)})$.

- Limit $\mathcal{N} \to \infty$ of $\mathcal{Z}[E]$ gives the KdV evolution equation, thus proving Witten's conjecture.

A matrix model inspired by noncommutative QFT

- The simplest QFT on a 4D noncommutative manifold can be written as a matrix model

$$\mathcal{Z}[E,J,\lambda] = \frac{\displaystyle\int dM\ \exp\big(-\mathrm{tr}(EM^2) + \mathrm{tr}(JM) - \tfrac{\lambda}{4}\mathrm{tr}(M^4)\big)}{\displaystyle\int dM\ \exp\big(-\mathrm{tr}(EM^2) - \tfrac{\lambda}{4}\mathrm{tr}(M^4)\big)},$$

  where $E = E^* \in M_{\mathcal{N}}(\mathbb{C})$ is the 4D Laplacian, $\lambda \geq 0$ and $J \in M_{\mathcal{N}}(\mathbb{C})$ generates correlation functions.
- In joint work with Harald Grosse [arXiv:1205.0465v4] we achieved the exact solution of $\mathcal{Z}[E,J,\lambda]$ for $\mathcal{N} \to \infty$ and after renormalisation of E, λ.
- This defines a QFT toy model in four dimensions, which is non-trivial with coupling constant $0 \leq \lambda \leq \frac{1}{\pi}$.
- It might give another view of 2D quantum gravity.

Field-theoretical matrix models

- classical scalar field $\phi \in \mathcal{C}_0(\mathbb{R}^d) \subset \mathcal{B}(H)$, with $\frac{m}{2}\int_{\mathbb{R}^d} dx\, \phi^2(x)$
- translates to $\mathrm{tr}(\phi^2) < \infty$, i.e. nc scalar field is Hilbert-Schmidt compact operator on Hilbert space $H = L^2(I, \mu)$
- realise as integral kernel operators: $M = (M_{ab}) \in L^2(I \times I, \mu \times \mu)$
  - product: $(MN)_{ab} = \int_I d\mu(c)\, M_{ac}M_{cb}$
  - trace: $\mathrm{tr}(M) = \int_I d\mu(a)\, M_{aa}$
  - adjoint: $(M^*)_{ab} = \overline{M_{ba}}$
- action = non-linear functional S for $\phi = \phi^*$ in volume V:

$$S[\phi] = V\,\mathrm{tr}(E\phi^2 + P[\phi])$$

  E – unbounded positive selfadjoint op. with compact resolvent,
  $P[\phi]$ – polynomial in ϕ with scalar coefficients
- partition function $\mathcal{Z}[J] = \int \mathcal{D}\phi\; \exp(-S[\phi] + V\,\mathrm{tr}(\phi J))$

Topological expansion

- Connected Feynman graphs in matrix models are ribbon graphs.
- Viewed as simplicial complexes, they encode the topology (B, g) of a genus-g Riemann surface with B boundary components (or punctures, marked points, holes, faces).
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1 \dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.
- Expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p_1 \dots p_{N_1}| \dots |q_1 \dots q_{N_B}|} J^{N_1}_{p_1 \dots p_{N_1}} \cdots J^{N_B}_{q_1 \dots q_{N_B}}$ according to the cycle structure.

Ward identity

- Unitary transformation $\phi \mapsto U\phi U^*$ leads to Ward identity

$$0 = \int \mathcal{D}\phi \left[E\phi\phi - \phi\phi E - J\phi + \phi J\right] \exp(-S[\phi] + V\,\mathrm{tr}(\phi J))$$

that describes how E, J break the invariance of the action.

…choose E (but not J) diagonal, use $\phi_{ab} = \frac{\partial}{V\partial J_{ba}}$:

Proposition [Disertori-Gurau-Magnen-Rivasseau, 2006]

The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities

$$0 = \sum_{n \in I} \left(\frac{(E_a - E_p)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \right)$$

For E of compact resolvent we can always assume that $m \mapsto E_m > 0$ is injective!

We turn the Ward identity for E injective into formula for $\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}$. The J-cycle structure in $\log \mathcal{Z}$ creates

- singular contributions $\sim \delta_{ap}$
- regular contributions present for all a, p

Theorem (Ward identity for injective E)

$$\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} = \delta_{ap}\Big\{ V \sum_{(K)} \frac{J_{P_1}\cdots J_{P_K}}{S_K} \Big(\sum_{n\in I} G_{|an|P_1|\dots|P_K|} + G_{|a|a|P_1|\dots|P_K|} + \sum_{r\geq 1} \sum_{q_1,\dots,q_r\in I} G_{|q_1 a q_1 \dots q_r|P_1|\dots|P_K|} J^r_{q_1\dots q_r} \Big) + V^2 \sum_{(K),(K')} \frac{J_{P_1}\cdots J_{P_K} J_{Q_1}\cdots J_{Q_{K'}}}{S_K S_{K'}} G_{|a|P_1|\dots|P_K|} G_{|a|Q_1|\dots|Q_{K'}|} \Big\} \mathcal{Z}[J] + \frac{V}{E_p - E_a} \sum_{n\in I} \Big(J_{pn} \frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}[J]}{\partial J_{np}} \Big)$$

How to use the Ward identity

Write $S = \frac{V}{2}\sum_{a,b}(E_a+E_b)\phi_{ab}\phi_{ba} + VS_{int}[\phi]$.
Functional integration yields, up to irrelevant constant,

$$\mathcal{Z}[J] = e^{-VS_{int}[\frac{\partial}{V\partial J}]}e^{\frac{V}{2}\langle J,J\rangle_E}\,, \qquad \langle J,J\rangle_E := \sum_{m,n\in I}\frac{J_{mn}J_{nm}}{E_m+E_n}$$

Example: $G_{|ab|}$ (for $a \neq b$)

$$\begin{aligned} G_{|ab|} &= \frac{1}{V\mathcal{Z}[0]}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{ba}\partial J_{ab}}\Big|_{J=0} \\ &= \frac{1}{V\mathcal{Z}[0]}\Big\{\frac{\partial}{\partial J_{ba}}e^{-VS_{int}\left[\frac{\partial}{V\partial J}\right]}\frac{\partial}{\partial J_{ab}}e^{\frac{V}{2}\langle J,J\rangle_E}\Big\}_{J=0} \\ &= \frac{1}{E_a+E_b} + \frac{1}{(E_a+E_b)\mathcal{Z}[0]}\Big\{\Big(\phi_{ab}\frac{\partial(-VS_{int})}{\partial\phi_{ab}}\Big)\Big[\frac{\partial}{V\partial J}\Big]\Big\}\mathcal{Z}[J]\Big|_{J=0} \end{aligned}$$

$\frac{\partial(-VS_{int})}{\partial\phi_{ab}}$ contains, for any $P[\phi]$, the derivative $\sum_n \frac{\partial^2}{\partial J_{an}\partial J_{np}}$

Schwinger-Dyson equations (for $S_{int}[\phi] = \frac{\lambda}{4}\mathrm{tr}(\phi^4)$)

The previous formula lets the usually infinite tower of Schwinger-Dyson equations collapse:

after genus expansion $G_{\dots} = \sum_{g=0}^{\infty} V^{-2g} G^{(g)}_{\dots}$:

1. A closed non-linear equation for $G^{(0)}_{ab}$ (planar+regular):

$$G^{(0)}_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{V(E_a+E_b)} \sum_{p\in I} \Big(G^{(0)}_{|ab|} G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|} - G^{(0)}_{|ab|}}{E_p - E_a} \Big)$$

2. For every other $G^{(g)}_{a_1\dots a_N}$ an equation which only depends on

- $G^{(g)}_{a_1\dots a_k}$ for $k \leq N$,
- $G^{(h)}_{a_1\dots a_k}$ with $h < g$ and $k \leq N+2$;

this dependence is linear in the top degree (N, g)

Some $G_{\dots}$ need renormalisation of E, M, and λ!

Exact solution for $\phi = \phi^*$

Reality implies invariance under orientation reversal

$$G_{|p_0^1 p_1^1 \dots p_{N_1-1}^1|\dots|p_0^B p_1^B \dots p_{N_B-1}^B|} = G_{|p_0^1 p_{N_1-1}^1 \dots p_1^1|\dots|p_0^B p_{N_B-1}^B \dots p_1^B|}$$

- empty for $G_{|ab|}$
- cancellations in $(E_a + E_{b_1}) G_{ab_1 b_2 \dots b_{N-1}} - (E_a + E_{b_{N_1}}) G_{ab_{N-1} \dots b_2 b_1}$

Theorem (universal algebraic recursion formula)

$$\begin{aligned} &G_{|b_0 b_1 \dots b_{N-1}|} \\ &= (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|} G_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G_{|b_{2l} b_1 \dots b_{2l-1}|} G_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})} \\ &+ \frac{(-\lambda)}{V} \sum_{k=1}^{N-1} \frac{G_{|b_0 b_1 \dots b_{k-1}|b_k b_{k+1} \dots b_{N-1}|} - G_{|b_k b_1 \dots b_{k-1}|b_0 b_{k+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_k})(E_{b_1} - E_{b_{N-1}})} \end{aligned}$$

Last line increases the genus and is absent in $G^{(0)}_{|b_0 b_1 \dots b_{N-1}|}$

Further observations

- Non-planar contributions with genus $g \geq 1$ are suppressed by V^{-2g}. They disappear for $V \to \infty$.
- The non-linear equation

$$G^{(0)}_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{V(E_a + E_b)} \sum_{p \in I} \Big(G^{(0)}_{|ab|} G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|} - G^{(0)}_{|ab|}}{E_p - E_a} \Big)$$

  is not algebraic and to be solved case by case for given E
- The index sums over $p \in I$ will diverge. In important cases these can be renormalised by $E_a \mapsto Z(E_a + \frac{\mu^2}{2} - \frac{\mu^2_{bare}}{2})$ and $\lambda \mapsto Z^2 \lambda$.
- Pattern extends to $B \geq 2$ boundary components: Equation for $(N_1 + \ldots + \ldots N_B)$-point functions $G_{|p^1_1 \ldots p^1_{N_1}|\ldots|p^B_1 \ldots p^B_{N_B}|}$ is
  1. universally algebraic if one $N_i \geq 3$
  2. an affine equation to be solved case by case if all $N_i \leq 2$. The coefficients are known by induction.

Renormalisation theorem

The renormalisation leaves algebraic equations invariant:

Theorem

Given a real scalar matrix model with $S = V\,\mathrm{tr}(E\phi^2 + \frac{\lambda}{4}\phi^4)$ and $m \mapsto E_m$ injective, which determines the set $G_{|p_1^1 \dots p_{N_1}^1|\dots|p_1^B \dots p_{N_B}^B|}$ of $(N_1 + \dots + \dots N_B)$-point functions.

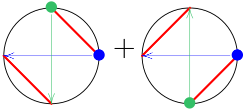
Assume the basic functions with all $N_i \leq 2$ are turned finite by $E_a \mapsto Z(E_a + \frac{\mu^2}{2} - \frac{\mu_{bare}^2}{2})$ and $\lambda \mapsto Z^2\lambda$.

Then all functions with one $N_i \geq 3$

1. are finite without further need of a renormalisation of λ, i.e. all renormalisable quartic matrix models have vanishing β-function.
2. are given by algebraic recursion formulae in terms of renormalised basic functions with $N_i \leq 2$.

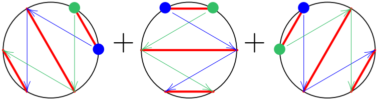
Graphical realisation $(B = 1, g = 0)$

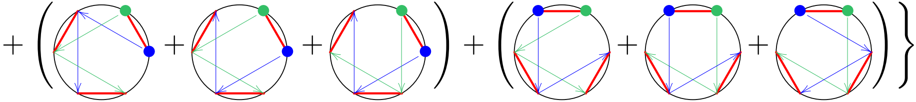
$$G_{b_0b_1b_2b_3} = (-\lambda)\frac{G_{b_0b_1}G_{b_2b_3} - G_{b_0b_3}G_{b_2b_1}}{(E_{b_0}-E_{b_2})(E_{b_1}-E_{b_3})} = -\lambda\Big\{$$



$$\Big\}$$

$$G_{b_0\dots b_5} = \lambda^2\Big\{$$





$$\Big\}$$

b_i b_j $= G_{b_ib_j}$

leads to non-crossing chord diagrams; these are counted by the Catalan number $C_{\frac{N}{2}} = \frac{N!}{(\frac{N}{2}+1)!\,\frac{N}{2}!}$

b_i  b_j $= \frac{1}{E_{b_i} - E_{b_j}}$

leads to rooted trees connecting the even or odd vertices, intersecting the chords only at vertices

Open Problem (Combinatorics): Which trees arise for a given chord diagram?

ϕ^4_4 on Moyal space with harmonic propagation

ϕ^4-theory on 4D-Moyal space w/ harmonic oscillator potential

$$S[\phi] = \int d^4x \Big(\frac{Z}{2}\phi \star \big(-\Delta + \Omega^2 (2\Theta^{-1}x)^2 + \mu_{bare}^2\big)\phi + \frac{\lambda Z^2}{4} \phi\star\phi\star\phi\star\phi \Big)(x)$$

- renormalisable as formal power series in λ [Grosse-W., 2004] (renormalisation of $\mu_{bare}^2, \lambda, Z \in \mathbb{R}_+$ and $\Omega \in [0,1]$) means: well-defined perturbative quantum field theory
- Langmann-Szabo duality (2002): theories at Ω and $\Omega^* = \frac{1}{\Omega}$ are the same; self-dual case $\Omega = 1$ is matrix model
- β-function vanishes to all orders in λ for $\Omega = 1$ [Disertori-Gurau-Magnen-Rivasseau, 2006] means: almost scale-invariant

Is the self-dual (critical) model integrable?

Matrix basis and thermodynamic limit

The Moyal algebra has a matrix basis [Gracia-Bondía+Várilly, 1988] in which the previous action becomes for $\Omega = 1$

$$S[\phi] = V\Bigg(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}} E_{\underline{m}}\,\phi_{\underline{m}\underline{n}}\phi_{\underline{n}\underline{m}} + \frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}} \phi_{\underline{m}\underline{n}}\phi_{\underline{n}\underline{k}}\phi_{\underline{k}\underline{l}}\phi_{\underline{l}\underline{m}}\Bigg)$$

$$E_{\underline{m}} = Z\left(\frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu^2_{bare}}{2}\right), \qquad |\underline{m}| := \underline{m}_1 + \underline{m}_2 \leq \mathcal{N}$$

- $V = \left(\frac{\theta}{4}\right)^2$ is for $\Omega = 1$ the volume of the noncommutative manifold which is sent to ∞ in the thermodynamic limit.
- We do this in the double-scaling limit $\frac{\mathcal{N}}{\sqrt{V}} = \Lambda^2\mu^2 = \mathrm{const}$
- Matrix indices become continuous $\frac{|\underline{p}|}{\sqrt{V}} \mapsto \mu^2 p$ with $p \in [0, \Lambda^2]$.

Integral equations

- Planar 2-point functions $G_{ab} = \mu^2 G^{(0)}_{|ab|}$ depends on $a, b \in [0, \Lambda^2]$
- Difference of eqns for G_{ab} and G_{a0} cancels worst divergence
- Renormalisation $\mu_{bare} \mapsto \mu$ and $Z^{-1} \mapsto (1 + \mathcal{Y})$ by normalisation conditions $G_{00} = 1$ and $\frac{dG_{ab}}{db}\Big|_{a=b=0} = -(1 + \mathcal{Y})$

Integral equation for Hölder-continuous G_{ab} and $\Lambda \to \infty$

$$\left(\frac{b}{a} + \frac{1 + \lambda\pi a \mathcal{H}_a\big[G_{\bullet 0}\big]}{a G_{a0}}\right) D_{ab} - \lambda\pi \mathcal{H}_a\big[D_{\bullet b}\big] = -G_{a0}$$

where

- $D_{ab} := \frac{a}{b}(G_{ab} - G_{a0})$
- $\mathcal{Y} = -\lambda \int_0^\infty \frac{dp}{p} D_{p0}$
- Hilbert transform $\mathcal{H}_a[f(\bullet)] := \frac{1}{\pi} \lim_{\epsilon \to 0} \left(\int_0^{a-\epsilon} + \int_{a+\epsilon}^\infty \right) \frac{f(q)\, dq}{q - a}$

The Carleman equation

Theorem [Carleman 1922, Tricomi 1957]

The singular linear integral equation

$$h(x)y(x) - \lambda\pi\mathcal{H}_x[y] = f(x)\,, \qquad x \in [-1,1]$$

is for $h(x)$ continuous + Hölder near ± 1 and $f \in L^p$ solved by

$$y(x) = \frac{\sin(\theta(x))}{\lambda\pi}\Big(f(x)\cos(\theta(x)) + e^{\mathcal{H}_x[\theta]}\mathcal{H}_x\Big[e^{-\mathcal{H}_\bullet[\theta]}f(\bullet)\sin(\theta(\bullet))\Big] + \frac{Ce^{\mathcal{H}_x[\theta]}}{1-x}\Big)$$

$$\theta(x) = \underset{[0,\pi]}{\arctan}\Big(\frac{\lambda\pi}{h(x)}\Big)\,, \quad \sin(\theta(x)) = \frac{|\lambda\pi|}{\sqrt{(h(x))^2 + (\lambda\pi)^2}}$$

where C is an arbitrary constant.

Assumption: $C = 0$

Solution

- angle $\theta_b(a) := \underset{[0,\pi]}{\arctan}\left(\dfrac{\lambda\pi a}{b + \frac{1+\lambda\pi a \mathcal{H}_a[G_{\bullet 0}]}{G_{a0}}}\right)$
- yields $\lambda\pi a \cot\theta_b(a) = b + \lambda\pi a \cot\theta_0(a)$
- G_{a0} is solved for $\theta_0(a)$: $G_{a0} = \frac{\sin(\theta_0(a))}{|\lambda|\pi a} e^{\mathcal{H}_a[\theta_0(\bullet)] - \mathcal{H}_0[\theta_0(\bullet)]}$
- Addition theorems and Tricomi's identity $e^{-\mathcal{H}_a[\theta_b]}\cos(\theta_b(a)) + \mathcal{H}_a\Big[e^{-\mathcal{H}_\bullet[\theta_b]}\sin(\theta_b(\bullet))\Big] = 1$ give:

Theorem

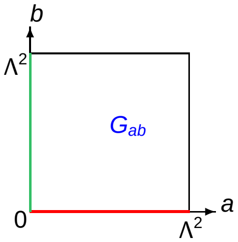
$$G_{ab} = \frac{\sin(\theta_b(a))}{|\lambda|\pi a} e^{\mathcal{H}_a[\theta_b(\bullet)] - \mathcal{H}_0[\theta_0(\bullet)]}$$

- Consequence: $G_{ab} \geq 0$!
- $\mathcal{Y} = \lambda \displaystyle\int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(\frac{1+\lambda\pi p \mathcal{H}_p[G_{\bullet 0}]}{G_{p0}}\right)^2}$

The self-consistency equation

Given boundary value G_{a0},
Carleman computes G_{ab},
in particular G_{0b}

symmetry forces $G_{b0} = G_{0b}$



Master equation

The theory is completely determined by the solution of the fixed point equation $G = TG$

$$G_{b0} = \frac{1}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}_p[G_{\bullet 0}]}{G_{p0}}\right)^2} \right)$$

Existence proof

The operator T satisfies assumptions of Schauder fixed point theorem. Define

$$\mathcal{K}_\lambda := \Big\{ f \in \mathcal{C}^1_0(\mathbb{R}_+) \;:\; f(0)=1\,,\quad 0 < f(b) \leq \frac{1}{1+b}\,,\\ 0 \leq -f'(b) \leq \Big(\frac{1}{1+b} + C_\lambda\Big) f(b) \Big\}$$

with C_λ from $2\lambda P_\lambda^2 (1+C_\lambda) e^{C_\lambda P_\lambda} = 1$ at $P_\lambda = \frac{\exp(-\frac{1}{\lambda\pi^2})}{\sqrt{1+4\lambda}}$. Then:

1. $\mathcal{K}_\lambda$ convex
2. $\overline{T\mathcal{K}_\lambda} \subset \mathcal{K}_\lambda$
3. $(Tf)''(b) \leq \big(\frac{23}{4} + \frac{2}{\pi} + \frac{7+8\pi}{2}\frac{1}{(\lambda\pi^2 P_\lambda)^2}\big)(Tf)(b)$ for any $f \in \mathcal{K}_\lambda$.

   $\Rightarrow$ $T\mathcal{K}_\lambda$ is relatively compact in $\mathcal{K}_\lambda$ by variant of Arzelá-Ascoli
4. $T : \mathcal{K}_\lambda \to \mathcal{K}_\lambda$ is continuous

This provides exact solution of ϕ^4-QFT on 4D Moyal space at $\theta \to \infty$

Next steps

(Analysis): The homogeneous Carleman equation has non-trivial solutions not taken into account. They arise from a winding number and seem to be relevant for $\lambda > \frac{1}{\pi}$.

The (important!) uniqueness proof needs prior clarification of this freedom.

(2D model): Carrying these methods and results over to 2D Moyal space is easy. But the master equation has a singularity at $a = 0$ (infrared) so that the Schauder existence proof does not work in the same way.

(Physics): We currently translate matrix results to position space. Only a restricted sector survives with

$$G\Big(\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}\cdots\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1}\Big|\dots\Big|\underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}\cdots\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}\Big), \qquad \text{all } N_i \text{ even}$$

as momentum space correlation functions

- The most interesting sector is all $N_i = 2$ describing propagation and interaction of B particles. This interaction leaves momenta conserved.
- This is similar to free particles, but additional contribution from $(N_1+\dots+N_B)$-point functions violates clustering. The theory has non-trivial topological sectors.
- Analytic continuation to Minkowski space and Osterwalder-Schader reflection positivity would follow (at least for 2-point function) if $a \mapsto G_{aa}$ is a Stieltjes function.
- This seems to be the case only for wrong sign $\lambda < 0$.

Future steps

(2D quantum gravity) A large class of surfaces admits quadrangulations.

If partition function of 2D quantum gravity localises on such surfaces, then it could equivalently be described by cubic and quartic matrix model.

Quartic model shows positivity and boundedness from below. They admit techniques from constructive QFT (loop vertex expansion) not possible in cubic model. This might be useful in 2D quantum gravity and algebraic geometry.

(Coloured tensor models) extend these methods to quantum gravity in $D \geq 3$. They have Schwinger-Dyson equations and action of $U(\infty)$ group. Our method might generalise to this class.