Non-trivial Φ^4_4 on Moyal space

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(based on joint work with Harald Grosse,
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Part I

Field-theoretical matrix models

Introduction

- The Standard Model is a perturbatively renormalisable quantum field theory.
- Scattering amplitudes can be computed as formal power series in coupling constants such as $e^2 \approx \frac{1}{137}$.
  The first terms agree to high precision with experiment.
- Perturbative QFT has tremendous problems, e.g.
  1. The radius of convergence in e^2 is zero!
  2. It seems impossible to perturbatively understand confinement in QCD.

- Refined summation techniques (e.g. Borel) may establish reasonable domains of analyticity.
- Unfortunately, this also fails for QED due to the Landau ghost problem.

  It is expected to work for non-Abelian gauge theories because of asymptotic freedom.
- But these theories are too complicated (millenium prize problem), and simpler 4D-models (such as Φ^4_4) cannot be constructed.

QFT's on noncommutative geometries may provide toy models for non-perturbative renormalisation in four dimensions.

These models may have new Ward identities which constrain the renormalisation group flow.

More motivation for QFT on NCG

- Assuming space-time to be a noncommutative geometry entails quantum mechanical uncertainties for position measurements.
- Such localisation uncertainties seem necessary in any framework combining gravity and quantum physics [Wheeler, 1950's], [Doplicher, Fredenhagen & Roberts, 1995]
- We don't know what a physically realistic noncommutative geometry is.

  There is also no mathematically satisfying theory of non-compact Lorentzian noncommutative geometries.
- We thus take simple NCG examples given by deformation or truncation of spaces of functions.

The Moyal space

- A convenient class of deformations of manifolds carrying an $\mathbb{R}^d$ group action was proposed by [Rieffel, 1993].
- For $\mathbb{R}^d$ this yields the Moyal space or noncommutative $\mathbb{R}^d$:
For $f, g \in \mathcal{S}(\mathbb{R}^d)$ smooth rapidly decaying functions, define
$$(f \star g)(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{dy\, dk}{(2\pi)^d} f(x+\tfrac{1}{2}\Theta k)\, g(x+y)\, e^{\mathrm{i}\langle k,y\rangle}$$
- This Moyal product is associative (but noncommutative) for fixed $\Theta = -\Theta^t \in M_d(\mathbb{R})$.
For $\Theta = 0$ the ordinary pointwise product is recovered.
- In extension to larger classes of functions, the coordinate functions x^μ satisfy $[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}\Theta^{\mu\nu}$.
- This relates to quantum mechanics in phase space [Groenewold, 1946], [Moyal, 1949]. The functional analysis of $\star$-product was studied by [Gracia-Bondía & Várilly, 1988]

Field theory on Moyal space

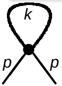
- Euclidean QFT is defined via partition function $Z[J] = \int \mathcal{D}[\phi]\ \exp(-S[\phi] + \langle \phi, J\rangle)$.
- In simplest examples, $\phi \in \mathcal{C}^\infty(M)$ is a function on a manifold M, and classical action $S[\phi]$ is the integral of a polynomial in derivatives of ϕ.
- For Euclidean QFT on Moyal space, simply write $\star$ for the product in the polynomial (integral & derivatives unchanged).

UV/IR-mixing [Minwalla, van Raamsdonk & Seiberg, 1999]

The action $S[\phi] = \int_{\mathbb{R}^4} dx\ \big(\frac{1}{2}\phi \star (-\Delta + m^2)\phi + \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\big)(x)$

does not lead to a renormalisable QFT.

(Similarly for QED, Yang-Mills, . . . on Moyal; SUSY helps)



$$= 2\lambda \int \frac{d^4k}{k^2+m^2} + \lambda \int \frac{d^4k\ e^{\mathrm{i}\langle k,\Theta p\rangle}}{k^2+m^2} = \text{const}_\infty + \frac{4\lambda\pi^2 m}{|\Theta p|} K_1(m|\Theta p|)$$

mass renorm. (← const_∞)

$\sim \frac{1}{|\Theta p|^2}$, IR-divergence in subgraphs

Matrix basis

Moyal algebra has matrix basis [Gracia-Bondía+Várilly, 1988]:

$$\phi(x) = \sum_{\underline{m},\underline{n}\in\mathbb{N}^2} \phi_{\underline{mn}} f_{\underline{mn}}(x), \qquad f_{\underline{mn}}(x) = f_{m_1 n_1}(x^0,x^1) f_{m_2 n_2}(x^3,x^4)$$

$$f_{mn}(y^0,y^1) = 2(-1)^m \sqrt{\frac{m!}{n!}} \Big(\sqrt{\frac{2}{\theta}}y\Big)^{n-m} L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big) e^{-\frac{|y|^2}{\theta}}$$

$$y = y^0 + \mathrm{i}y^1\,, \quad \theta := \Theta_{12} = -\Theta_{21} = \Theta_{34} = -\Theta_{43}\,, \text{ other } \Theta_{ij} \text{ zero}$$

- satisfies $(f_{\underline{kl}} \star f_{\underline{mn}})(x) = \delta_{\underline{ml}} f_{\underline{kn}}(x)$, $\int_{\mathbb{R}^4} dx\, f_{\underline{mn}}(x) = (2\pi\theta)^2 \delta_{\underline{mn}}$
- $\Phi_4^{\star 4}$-interaction becomes matrix product

$$S[\phi] = (2\pi\theta)^2 \sum_{\underline{k},\underline{l},\underline{m},\underline{n}\in\mathbb{N}^2} \Big(\frac{1}{2}\phi_{\underline{kl}}(\Delta_{\underline{kl};\underline{mn}} + m^2 \delta_{\underline{kn}}\delta_{\underline{lm}})\phi_{\underline{mn}} + \frac{\lambda}{4}\phi_{\underline{kl}}\phi_{\underline{lm}}\phi_{\underline{mn}}\phi_{\underline{nk}}\Big)$$

- $\Delta_{\underline{kl};\underline{mn}}$ is local plus nearest neighbour interaction

The harmonic oscillator term

In [Grosse & W., 2004] we studied the renormalisation group flow of the $\Phi^{\star 4}_4$-model in matrix representation.

- We noticed that the local term and the nearest neighbour term in $\Delta_{\underline{kl};\underline{mn}}$ have different flows.
- Necessary is a 4^{th} relevant/marginal operator in the action which corresponds to a harmonic oscillator potential:

$$S[\phi]=64\pi^2\int d^4x\Big(\frac{Z}{2}\phi\star\big(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2_{bare}\big)\phi+\frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

- The corresponding Euclidean QFT is perturbatively renormalisable to all orders [Grosse & W., 2004].

  (renormalisation of $\mu^2_{bare}, \lambda, Z \in \mathbb{R}_+$ and $\Omega \in [0,1]$)
- alternative proofs by [Rivasseau et al, 2005/06]

More results

- $\Omega \neq 0$ breaks translation invariance but achieves covariance under [Langmann & Szabo, 2002] duality transformation:

  $(x \leftrightarrow p, \phi(x) \leftrightarrow \hat{\phi}(p))$ plus Fourier transform
  $\int \phi{\star}\phi{\star}\phi{\star}\phi$ invariant, exchanges $\int \phi(-\Delta)\phi$ and $\int \phi|2\Theta^{-1}x|^2\phi$

- $\Delta^{\Omega=1}_{\underline{kl};\underline{mn}}$ is local and leads to a true matrix model with action

$$S[\phi] = V\Bigg(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}} E_{\underline{m}}\, \phi_{\underline{mn}}\phi_{\underline{nm}} + \frac{Z^2\lambda}{4} \sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}} \phi_{\underline{mn}}\phi_{\underline{nk}}\phi_{\underline{kl}}\phi_{\underline{lm}} \Bigg)$$

$$E_{\underline{m}} = Z\left(\frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu^2_{bare}}{2}\right), \qquad |\underline{m}| := \underline{m}_1 + \underline{m}_2 \leq \mathcal{N}, \quad V = \left(\tfrac{\theta}{4}\right)^2$$

- β-function vanishes to all orders in λ for $\Omega = 1$
  [Disertori, Gurau, Magnen & Rivasseau, 2006]
  means: almost scale-invariant

Is the self-dual (critical) model integrable?

2D quantum gravity and matrix models

1. 2D quantum gravity is the enumeration of random triangulations of surfaces.
   - Its asymptotic behaviour is captured by the matrix model partition function

$$\mathcal{Z} = \int dM \, \exp\Big(-\mathcal{N}\sum_n t_n \operatorname{tr}(M^n)\Big)\,, \qquad M = M^* \in M_{\mathcal{N}}(\mathbb{C})$$

   - For $\mathcal{N} \to \infty$, this series in (t_n) is evaluated in terms of the τ-function for the Korteweg-de Vries (KdV) hierarchy.
2. 2D topological quantum gravity has correlation functions which are intersection numbers of complex curves.
   - They can be arranged into a generating functional with series parameters (t_n).

[Witten, 1990] conjectured that both (t_n)-series are the same.

The Kontsevich model

- [Kontsevich, 1992] computed the intersection numbers in terms of weighted sums over ribbon graphs.
- He proved these graphs to be generated from the Airy function matrix model (Kontsevich model)

$$\mathcal{Z}[E] = \frac{\displaystyle\int dM\ \exp\big(-\tfrac{1}{2}\mathrm{tr}(EM^2) + \tfrac{\mathrm{i}}{6}\mathrm{tr}(M^3)\big)}{\displaystyle\int dM\ \exp\big(-\tfrac{1}{2}\mathrm{tr}(EM^2)\big)}\,, \qquad M{=}M^*{\in}M_{\mathcal{N}}(\mathbb{C})$$

for $E = E^* > 0$ and $t_n = (2n{-}1)!!\mathrm{tr}(E^{-(2n-1)})$.

- Limit $\mathcal{N} \to \infty$ of $\mathcal{Z}[E]$ gives the KdV evolution equation, thus proving Witten's conjecture.

The quartic matrix model

- Φ^4_4-theory on Moyal space with self-dual harmonic oscillator potential is a quartic matrix model

$$\mathcal{Z}[E,J,\lambda] = \frac{\int dM \; \exp\big(-\mathrm{tr}(EM^2) + \mathrm{tr}(JM) - \frac{\lambda}{4}\mathrm{tr}(M^4)\big)}{\int dM \; \exp\big(-\mathrm{tr}(EM^2) - \frac{\lambda}{4}\mathrm{tr}(M^4)\big)} ,$$

  where $E = E^* \in M_{\mathcal{N}}(\mathbb{C})$ is the 4D Laplacian, $\lambda \geq 0$ and $J \in M_{\mathcal{N}}(\mathbb{C})$ generates correlation functions.

- In joint work with Harald Grosse [arXiv:1205.0465v4] we achieved the exact solution of $\mathcal{Z}[E,J,\lambda]$ for $\mathcal{N} \to \infty$ and after renormalisation of E, λ.

- In [arXiv:1306.2816] we studied Schwinger functions for the Euclidean QFT and Osterwalder-Schrader axioms.

- In [arXiv:13??.????] we find a non-trivial phase structure.

Field-theoretical matrix models

- classical scalar field $\phi \in \mathcal{C}_0(\mathbb{R}^d) \subset \mathcal{B}(H)$, with $\frac{m}{2}\int_{\mathbb{R}^d} dx\ \phi^2(x)$
- translates to $\mathrm{tr}(\phi^2) < \infty$, i.e. nc scalar field is Hilbert-Schmidt compact operator on Hilbert space $H = L^2(I, \mu)$
- realise as integral kernel operators: $M = (M_{ab}) \in L^2(I \times I, \mu \times \mu)$
  - product: $(MN)_{ab} = \int_I d\mu(c)\ M_{ac}M_{cb}$
  - trace: $\mathrm{tr}(M) = \int_I d\mu(a)\ M_{aa}$
  - adjoint: $(M^*)_{ab} = \overline{M_{ba}}$
- action = non-linear functional S for $\phi = \phi^*$ in volume V:

$$S[\phi] = V\,\mathrm{tr}(E\phi^2 + P[\phi])$$

  E – unbounded positive selfadjoint op. with compact resolvent,
  $P[\phi]$ – polynomial in ϕ with scalar coefficients
- partition function $\mathcal{Z}[J] = \int \mathcal{D}\phi\ \exp(-S[\phi] + V\,\mathrm{tr}(\phi J))$

Topological expansion

- Connected Feynman graphs in matrix models are ribbon graphs.
- Viewed as simplicial complexes, they encode the topology (B, g) of a genus-g Riemann surface with B boundary components (or punctures, marked points, holes, faces).
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1 \dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.
- Expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p_1 \dots p_{N_1}| \dots |q_1 \dots q_{N_B}|} J^{N_1}_{p_1 \dots p_{N_1}} \cdots J^{N_B}_{q_1 \dots q_{N_B}}$ according to the cycle structure.

Ward identity

- Unitary transformation $\phi \mapsto U\phi U^*$ leads to Ward identity

$$0 = \int \mathcal{D}\phi \left[E\phi\phi - \phi\phi E - J\phi + \phi J\right] \exp(-S[\phi] + V\,\mathrm{tr}(\phi J))$$

that describes how E, J break the invariance of the action.

...choose E (but not J) diagonal, use $\phi_{ab} = \frac{\partial}{V\partial J_{ba}}$:

Proposition [Disertori-Gurau-Magnen-Rivasseau, 2006]

The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities

$$0 = \sum_{n \in I} \left(\frac{(E_a - E_p)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \right)$$

For E of compact resolvent we can always assume that $m \mapsto E_m > 0$ is injective!

We turn the Ward identity for E injective into formula for $\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}$. The J-cycle structure in $\log \mathcal{Z}$ creates

- singular contributions $\sim \delta_{ap}$
- regular contributions present for all a, p

Theorem (Ward identity for injective E)

$$\begin{aligned}
\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} = \delta_{ap}\Big\{ & V \sum_{(K)} \frac{J_{P_1}\cdots J_{P_K}}{S_K} \Big(\sum_{n\in I} G_{|an|P_1|\dots|P_K|} + G_{|a|a|P_1|\dots|P_K|} \\
& \qquad + \sum_{r\geq 1} \sum_{q_1,\dots,q_r\in I} G_{|q_1 a q_1\dots q_r|P_1|\dots|P_K|} J^r_{q_1\dots q_r} \Big) \\
& + V^2 \sum_{(K),(K')} \frac{J_{P_1}\cdots J_{P_K} J_{Q_1}\cdots J_{Q_{K'}}}{S_K S_{K'}} G_{|a|P_1|\dots|P_K|} G_{|a|Q_1|\dots|Q_{K'}|} \Big\} \mathcal{Z}[J] \\
& + \frac{V}{E_p - E_a} \sum_{n\in I} \Big(J_{pn} \frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}[J]}{\partial J_{np}} \Big)
\end{aligned}$$

How to use the Ward identity

Write $S = \frac{V}{2}\sum_{a,b}(E_a + E_b)\phi_{ab}\phi_{ba} + VS_{int}[\phi]$.
Functional integration yields, up to irrelevant constant,

$$\mathcal{Z}[J] = e^{-VS_{int}[\frac{\partial}{V\partial J}]}e^{\frac{V}{2}\langle J,J\rangle_E}\;, \qquad \langle J,J\rangle_E := \sum_{m,n\in I}\frac{J_{mn}J_{nm}}{E_m + E_n}$$

Example: $G_{|ab|}$ (for $a \neq b$)

$$\begin{aligned} G_{|ab|} &= \frac{1}{V\mathcal{Z}[0]}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{ba}\partial J_{ab}}\Big|_{J=0} \\ &= \frac{1}{V\mathcal{Z}[0]}\Big\{\frac{\partial}{\partial J_{ba}}e^{-VS_{int}\left[\frac{\partial}{V\partial J}\right]}\frac{\partial}{\partial J_{ab}}e^{\frac{V}{2}\langle J,J\rangle_E}\Big\}_{J=0} \\ &= \frac{1}{E_a + E_b} + \frac{1}{(E_a + E_b)\mathcal{Z}[0]}\Big\{\Big(\phi_{ab}\frac{\partial(-VS_{int})}{\partial\phi_{ab}}\Big)\Big[\frac{\partial}{V\partial J}\Big]\Big\}\mathcal{Z}[J]\Big|_{J=0} \end{aligned}$$

$\frac{\partial(-VS_{int})}{\partial\phi_{ab}}$ contains, for any $P[\phi]$, the derivative $\sum_n \frac{\partial^2}{\partial J_{an}\partial J_{np}}$

Schwinger-Dyson equations (for $S_{int}[\phi] = \frac{\lambda}{4}\mathrm{tr}(\phi^4)$)

The previous formula lets the usually infinite tower of Schwinger-Dyson equations collapse:

after genus expansion $G_{\dots} = \sum_{g=0}^{\infty} V^{-2g} G^{(g)}_{\dots}$:

1. A closed non-linear equation for $G^{(0)}_{ab}$ (planar+regular):

$$G^{(0)}_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{V(E_a + E_b)} \sum_{p \in I} \Big(G^{(0)}_{|ab|} G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|} - G^{(0)}_{|ab|}}{E_p - E_a} \Big)$$

2. For every other $G^{(g)}_{a_1 \dots a_N}$ an equation which only depends on

- $G^{(g)}_{a_1 \dots a_k}$ for $k \leq N$,
- $G^{(h)}_{a_1 \dots a_k}$ with $h < g$ and $k \leq N + 2$;

this dependence is linear in the top degree (N, g)

Some $G_{\dots}$ need renormalisation of E, M, and λ!

Exact solution for $\phi = \phi^*$

Reality implies invariance under orientation reversal

$$G_{|p_0^1 p_1^1 \ldots p_{N_1-1}^1|\ldots|p_0^B p_1^B \ldots p_{N_B-1}^B|} = G_{|p_0^1 p_{N_1-1}^1 \ldots p_1^1|\ldots|p_0^B p_{N_B-1}^B \ldots p_1^B|}$$

- empty for $G_{|ab|}$
- cancellations in $(E_a+E_{b_1})G_{ab_1b_2\ldots b_{N-1}} - (E_a+E_{b_{N-1}})G_{ab_{N-1}\ldots b_2b_1}$

Theorem (universal algebraic recursion formula)

$$\begin{aligned} G_{|b_0b_1\ldots b_{N-1}|} &= (-\lambda)\sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0b_1\ldots b_{2l-1}|}G_{|b_{2l}b_{2l+1}\ldots b_{N-1}|} - G_{|b_{2l}b_1\ldots b_{2l-1}|}G_{|b_0b_{2l+1}\ldots b_{N-1}|}}{(E_{b_0}-E_{b_{2l}})(E_{b_1}-E_{b_{N-1}})} \\ &+ \frac{(-\lambda)}{V}\sum_{k=1}^{N-1} \frac{G_{|b_0b_1\ldots b_{k-1}|b_kb_{k+1}\ldots b_{N-1}|} - G_{|b_kb_1\ldots b_{k-1}|b_0b_{k+1}\ldots b_{N-1}|}}{(E_{b_0}-E_{b_k})(E_{b_1}-E_{b_{N-1}})} \end{aligned}$$

Last line increases the genus and is absent in $G^{(0)}_{|b_0b_1\ldots b_{N-1}|}$

Further observations

- Non-planar contributions with genus $g \geq 1$ are suppressed by V^{-2g}. In limit $V \to \infty$, full function and its restriction to planar sector satisfy the same equations.
- The non-linear equation

$$G^{(0)}_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{V(E_a + E_b)} \sum_{p \in I} \Big(G^{(0)}_{|ab|} G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|} - G^{(0)}_{|ab|}}{E_p - E_a} \Big)$$

  is not algebraic and to be solved case by case for given E
- Divergent index sums can possibly be renormalised by $E_a \mapsto Z(E_a + \frac{\mu^2}{2} - \frac{\mu^2_{bare}}{2})$ and $\lambda \mapsto Z^2\lambda$.
- Pattern extends to $B \geq 2$ boundary components: Equation for $(N_1 + \ldots + N_B)$-point functions $G_{|p^1_1 \ldots p^1_{N_1}| \ldots |p^B_1 \ldots p^B_{N_B}|}$ is
  1. universally algebraic if one $N_i \geq 3$
  2. an affine equation to be solved case by case if all $N_i \leq 2$. The coefficients are known by induction.

Renormalisation theorem

The renormalisation leaves algebraic equations invariant:

Theorem

Given a real scalar matrix model with $S = V\,\mathrm{tr}(E\phi^2 + \frac{\lambda}{4}\phi^4)$ and $m \mapsto E_m$ injective, which determines the set $G_{|p_1^1\ldots p_{N_1}^1|\ldots|p_1^B\ldots p_{N_B}^B|}$ of $(N_1 + \ldots + N_B)$-point functions.

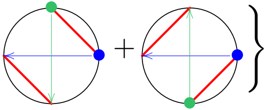
Assume the basic functions with all $N_i \leq 2$ are turned finite by $E_a \mapsto Z(E_a + \frac{\mu^2}{2} - \frac{\mu_{bare}^2}{2})$ and $\lambda \mapsto Z^2\lambda$.

Then all functions with one $N_i \geq 3$

1. are finite without further need of a renormalisation of λ, i.e. all renormalisable quartic matrix models have vanishing β-function.
2. are given by algebraic recursion formulae in terms of renormalised basic functions with $N_i \leq 2$.

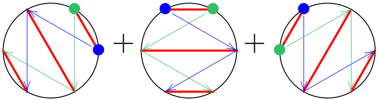
Graphical realisation ($B = 1, g = 0$)

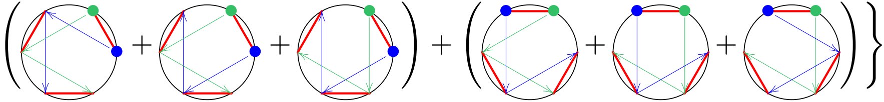
$$G_{b_0b_1b_2b_3} = (-\lambda)\frac{G_{b_0b_1}G_{b_2b_3} - G_{b_0b_3}G_{b_2b_1}}{(E_{b_0} - E_{b_2})(E_{b_1} - E_{b_3})} = -\lambda\Big\{$$



$$\Big\}$$

$$G_{b_0\dots b_5} = \lambda^2\Big\{$$





$$\Big\}$$

b_i ——— b_j $= G_{b_ib_j}$ leads to non-crossing chord diagrams; these are counted by the Catalan number $C_{\frac{N}{2}} = \frac{N!}{(\frac{N}{2}+1)!\frac{N}{2}!}$

$b_i \longrightarrow b_j$ $= \frac{1}{E_{b_i} - E_{b_j}}$ leads to rooted trees connecting the even or odd vertices, intersecting the chords only at vertices

Open Problem (Combinatorics): Which trees arise for a given chord diagram?

Part II

Φ^4_4 on Moyal space

Φ^4_4 on Moyal space with harmonic propagation

$$S[\phi]=64\pi^2\int d^4x\Big(\frac{Z}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2_{bare})\phi+\frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

takes at $\Omega=1$ in matrix basis $f_{\underline{mn}}(x)=f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$,

$$f_{mn}(y^0,y^1)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\frac{2}{\theta}}y\Big)^{n-m}L^{n-m}_m\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

the form of a quartic matrix model

$$S[\phi]=V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}E_{\underline{m}}\,\phi_{\underline{mn}}\phi_{\underline{nm}}+\frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}}\phi_{\underline{mn}}\phi_{\underline{nk}}\phi_{\underline{kl}}\phi_{\underline{lm}}\Big)$$

$$E_{\underline{m}}=Z\Big(\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu^2_{bare}}{2}\Big),\qquad |\underline{m}|:=\underline{m}_1+\underline{m}_2\leq\mathcal{N}$$

- $V=\big(\frac{\theta}{4}\big)^2$ is for $\Omega=1$ the volume of the noncommutative manifold which is sent to ∞ in the thermodynamic limit.
- We do this in a scaling limit $\frac{\mathcal{N}}{\sqrt{V}}=\Lambda^2\mu^2=\text{const}$

Integral equations

- Matrix indices become continuous $\frac{|p|}{\sqrt{V}} \mapsto \mu^2 p$ with $p \in [0,\Lambda^2]$
- Normalised planar 2-point function $G_{ab} = \mu^2 G^{(0)}_{|ab|}$, $a,b \in [0,\Lambda^2]$
- Difference of eqns for G_{ab} and G_{a0} cancels worst divergence
- $\Lambda \to \infty$: Renormalisation $\mu_{bare} \mapsto \mu$ and $Z^{-1} \mapsto (1+\mathcal{Y})$ by normalisation conditions $G_{00} = 1$ and $\frac{dG_{ab}}{db}\Big|_{a=b=0} = -(1+\mathcal{Y})$

Integral equation for Hölder-continuous G_{ab} and $\Lambda \to \infty$

$$\Big(\frac{b}{a} + \frac{1+\lambda\pi a\mathcal{H}_a\big[G_{\bullet 0}\big]}{aG_{a0}}\Big)D_{ab} - \lambda\pi\mathcal{H}_a\big[D_{\bullet b}\big] = -G_{a0}$$

where

- $D_{ab} := \frac{a}{b}(G_{ab} - G_{a0})$, $\qquad \mathcal{Y} = -\lambda \int_0^\infty \frac{dp}{p}\, D_{p0}$
- Hilbert transform $\quad \mathcal{H}_a[f(\bullet)] := \frac{1}{\pi} \lim_{\epsilon\to 0}\Big(\int_0^{a-\epsilon} + \int_{a+\epsilon}^\infty\Big)\frac{f(q)\,dq}{q-a}$

- $\lambda < 0$: $\mathcal{Z}[E,J]$ undefined, but eqns. for $G_{\dots}$ extend to $\lambda < 0$!

The Carleman equation

Theorem [Carleman 1922, Tricomi 1957]

The singular linear integral equation

$$h(a)y(a) - \lambda\pi\mathcal{H}_a[y] = f(a)\,, \qquad a \in \,]0,\Lambda^2[$$

is for $h(a)$ continuous + Hölder near $0,\Lambda^2$ and $f \in L^p$ solved by

$$y(a) = \frac{\sin(\vartheta(a))e^{-\mathcal{H}_a[\pi-\vartheta]}}{\lambda\pi a}\Big(a\,f(a)e^{\mathcal{H}_a[\pi-\vartheta]}\cos(\vartheta(a)) + \mathcal{H}_a\Big[e^{\mathcal{H}_\bullet[\pi-\vartheta]}\bullet f(\bullet)\sin(\vartheta(\bullet))\Big] + C\Big)$$

$$\overset{*}{=} \frac{\sin(\vartheta(a))e^{\mathcal{H}_a[\vartheta]}}{\lambda\pi}\Big(f(a)e^{-\mathcal{H}_a[\vartheta]}\cos(\vartheta(a)) + \mathcal{H}_a\Big[e^{-\mathcal{H}_\bullet[\vartheta]}f(\bullet)\sin(\vartheta(\bullet))\Big] + \frac{C'}{\Lambda^2 - a}\Big)$$

$$\vartheta(a) = \underset{[0,\pi]}{\arctan}\Big(\frac{\lambda\pi}{h(a)}\Big)\,, \quad \sin(\vartheta(a)) = \frac{|\lambda\pi|}{\sqrt{(h(a))^2 + (\lambda\pi)^2}}$$

where C, C' are arbitrary constants.

Normalisation conditions may select one of the solutions.

Solution

- angle $\vartheta_b(a) := \underset{[0,\pi]}{\arctan}\left(\dfrac{\lambda\pi a}{b + \frac{1+\lambda\pi a \mathcal{H}_a[G_{\bullet 0}]}{G_{a0}}}\right)$
- reversal: $G_{a0} = \begin{cases} \dfrac{\sin(\vartheta_0(a))}{|\lambda|\pi a} e^{\mathcal{H}_0[\pi-\vartheta_0(\bullet)]-\mathcal{H}_a[\pi-\vartheta_0(\bullet)]} & \text{for } \lambda<0 \\ \dfrac{\sin(\vartheta_0(a))}{|\lambda|\pi a} e^{\mathcal{H}_a[\vartheta_0(\bullet)]-\mathcal{H}_0[\vartheta_0(\bullet)]}\left(1+\dfrac{Ca}{\Lambda^2-a}\right) & \text{for } \lambda>0 \end{cases}$
- Addition theorems and Tricomi's identity
  $e^{-\mathcal{H}_a[\vartheta_b]}\cos(\vartheta_b(a)) + \mathcal{H}_a\left[e^{-\mathcal{H}_\bullet[\vartheta_b]}\sin(\vartheta_b(\bullet))\right] = 1$ give:

Theorem

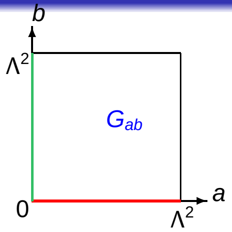
$$G_{ab} = \begin{cases} \dfrac{\sin(\vartheta_b(a))}{|\lambda|\pi a} e^{\mathcal{H}_0[\pi-\vartheta_0(\bullet)]-\mathcal{H}_a[\pi-\vartheta_b(\bullet)]} & \text{for } \lambda<0 \\ \dfrac{\sin(\vartheta_b(a))}{|\lambda|\pi a} e^{\mathcal{H}_a[\vartheta_b(\bullet)]-\mathcal{H}_0[\vartheta_0(\bullet)]}\left(1+\dfrac{Ca+bF(b)}{\Lambda^2-a}\right) & \text{for } \lambda>0 \end{cases}$$

- Consequence: $G_{ab} \geq 0$ (at least for $\lambda < 0$)!

The self-consistency equation

Given boundary value G_{a0},
Carleman computes G_{ab},
in particular G_{0b}

symmetry forces $G_{b0} = G_{0b}$



Master equation

The theory is completely determined by the solution of the fixed point equation $G = TG$

$$G_{b0} = \frac{\begin{cases} 1 & \text{for } \lambda<0 \\ 1+bF(b) & \text{for } \lambda>0 \end{cases}}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p\mathcal{H}_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

- $$\mathcal{Y} = \lambda \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(\frac{1+\lambda\pi p\mathcal{H}_p[G_{\bullet 0}]}{G_{p0}}\right)^2} - \begin{cases} 0 & \text{for } \lambda<0 \\ F(0) & \text{for } \lambda>0 \end{cases}$$

Existence proof (for $\lambda > 0$ but $F(b) = 0$)

The operator T satisfies assumptions of Schauder fixed point theorem. Define

$$\mathcal{K}_\lambda := \Big\{ f \in \mathcal{C}^1_0(\mathbb{R}_+) \;:\; f(0) = 1\,, \quad 0 < f(b) \leq \frac{1}{1+b}\,, \quad 0 \leq -f'(b) \leq \Big(\frac{1}{1+b} + C_\lambda\Big) f(b) \Big\}$$

with C_λ from $2\lambda P_\lambda^2(1+C_\lambda)e^{C_\lambda P_\lambda} = 1$ at $P_\lambda = \frac{\exp(-\frac{1}{\lambda\pi^2})}{\sqrt{1+4\lambda}}$. Then:

1. $\mathcal{K}_\lambda$ convex
2. $\overline{T\mathcal{K}_\lambda} \subset \mathcal{K}_\lambda$
3. $(Tf)''(b) \leq \Big(\frac{23}{4} + \frac{2}{\pi} + \frac{7+8\pi}{2}\frac{1}{(\lambda\pi^2 P_\lambda)^2}\Big)(Tf)(b)$ for any $f \in \mathcal{K}_\lambda$.

   $\Rightarrow$ $T\mathcal{K}_\lambda$ is relatively compact in $\mathcal{K}_\lambda$ by variant of Arzelá-Ascoli
4. $T : \mathcal{K}_\lambda \to \mathcal{K}_\lambda$ is continuous

This provides exact solution of ϕ^4-QFT on 4D Moyal space at $\theta\to\infty$

Higher correlation functions

Planar N-point functions from universal recursion formula:

$$G_{b_0\cdots_{n-1}} = \frac{(-\lambda)}{(1+\mathcal{Y})^2} \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{b_0 b_1 \dots b_{2l-1}} G_{b_{2l} b_{2l+1} \dots b_{N-1}} - G_{b_{2l} b_1 \dots b_{2l-1}} G_{|b_0 b_{2l+1} \dots b_{N-1}}}{(b_0 - b_{2l})(b_1 - b_{N-1})}$$

- involves $1+\mathcal{Y} = -\frac{dG_{a0}}{da}\Big|_{a=0}$
- Special case: effective coupling constant $\lambda_{eff} = -G_{0000}$:

$$\lambda_{eff} = \lambda\Big\{1 + \frac{\lambda}{(1+\mathcal{Y})} \int_0^\infty dp \, \frac{\Big(\frac{1-G_{p0}}{(1+\mathcal{Y})p} - G_{p0}\Big) G_{p0}}{\big(\lambda\pi p G_{p0}\big)^2 + \big(1 + \lambda\pi p \mathcal{H}_p[G_{\bullet 0}]\big)^2}\Big\}$$

Remarks

- Perturbative solution of master equation and of these formulae agrees with evaluation of Feynman graphs.
- We achieved the explicit resummation of infinitely many (renormalised!) Feynman graphs.

(N_1+N_2)-point functions

For $N_1 \geq 4$ and even, univeral algebraic recursion formula yields

$$
\begin{aligned}
&G_{ab_1\dots b_{2l-1}|c_1\dots c_{N-2l}} \\
&= \frac{(-\lambda)}{(1+\mathcal{Y})^2} \sum_{j=1}^{l-1} \frac{G_{b_1\dots b_{2j-1}a|c_1\dots c_{N-2l}} G_{b_{2j}b_{2j+1}\dots b_{2l-1}} - G_{b_1\dots b_{2j-1}b_{2j}|c_1\dots c_{N-2l}} G_{ab_{2j+1}\dots b_{2l-1}}}{(b_1-b_{2l-1})(a-b_{2j})} \\
&+ \frac{(-\lambda)}{(1+\mathcal{Y})^2} \sum_{j=1}^{l-1} \frac{G_{b_1\dots b_{2j-1}a} G_{b_{2j}b_{2j+1}\dots b_{2l-1}|c_1\dots c_{N-2l}} - G_{b_1\dots b_{2j-1}b_{2j}} G_{ab_{2j+1}\dots b_{2l-1}|c_1\dots c_{N-2l}}}{(b_1-b_{2l-1})(a-b_{2j})} \\
&+ \frac{(-\lambda)}{(1+\mathcal{Y})^2} \sum_{k=1}^{N-2l} \frac{G_{c_1\dots c_{k-1}ab_1\dots b_{2l-1}c_k c_{k+1}\dots c_{N-2l}} - G_{c_1\dots c_{k-1}c_k b_1\dots b_{2l-1}ac_{k+1}\dots c_{N-2l}}}{(b_1-b_{2l-1})(a-c_k)}
\end{aligned}
$$

- reduces to known N-point functions and additional basic function $G_{ab|cd}$

$G_{ab|cd}$

$$
\begin{aligned}
G_{ab|cd} &= F_{ab|cdcb} + F_{ab|dcdb} \\
&\quad - \frac{\sin\vartheta_b(a)}{\lambda\pi a}\cos\vartheta_b(a) G_{ab} X_{a|cd} - G_{ab}\mathcal{H}_a\Big[\frac{\sin^2\vartheta_b(\bullet)}{\lambda\pi\bullet} X_{\bullet|cd}\Big]
\end{aligned}
$$

where $F_{ab_1|c_1c_2c_3c_4} = \frac{G_{ab_1c_1c_2c_3c_4}G_{b_1c_3} - G_{b_1c_1c_2c_3}G_{ab_1c_3c_4}}{G_{b_1c_1}G_{b_1c_3}}$ and $X_{a|cd}$ the solution of the Carleman equation

$$
\begin{aligned}
& X_{a|cd}\Big\{1 + \lambda\int_0^\infty dq\,(G_{aq} - G_{0q}) \\
&\qquad\qquad - \lambda\int_0^\infty dq\,\frac{G_{aq}\sin\vartheta_q(a)\cos\big(\vartheta_q(a) - \vartheta_0(a)\big)}{\sin\vartheta_0(a)}\Big\} \\
&+ \mathcal{H}_a\Big[\frac{X_{\bullet|cd}}{\pi\bullet}\int_0^\infty q\,dq\,\sin^2\vartheta_q(\bullet) G_{aq}\Big] \\
&= \lambda\int_0^\infty q\,dq\,(F_{aq|cdcq} + F_{aq|dcdq}) + \frac{\lambda}{(1+\mathcal{Y})^2}(G_{acdc} + G_{adcd})
\end{aligned}
$$

- $G_{ab|cd}$ is the most interesting part of the 4-point function in position space!

Translation to 4D Euclidean QFT model

infinite volume limit $V \to \infty$ requires densities

Schwinger functions

$$\mu^N S_c(\mu x_1, \dots, \mu x_N)$$
$$:= \lim_{V\mu^4 \to \infty} \sum_{\underline{m}_1,\underline{n}_1,\dots,\underline{m}_N,\underline{n}_N \in \mathbb{N}^2} f_{\underline{m}_1\underline{n}_1}(x_1)\cdots f_{\underline{m}_N\underline{n}_N}(x_N) \frac{\mu^{4N}\partial^N \mathcal{F}[J]}{\partial J_{\underline{m}_1\underline{n}_1}\dots\partial J_{\underline{m}_N\underline{n}_N}}\Big|_{J=0}$$

$$\mathcal{F}[J] := \frac{1}{64\pi^2 V^2 \mu^8} \log\left(\frac{\int \mathcal{D}[\phi]\, e^{-S[\phi]+V\sum_{\underline{a},\underline{b}\in\mathbb{N}^2}\phi_{\underline{ab}}J_{\underline{ba}}}}{\int \mathcal{D}[\phi]\, e^{-S[\phi]}}\right)_{\substack{Z\mu^2_{bare}\mapsto\mu^2 \\ Z\mapsto(1+\mathcal{Y})}}$$

- J-cycle structure in $\mathcal{F}$ produces $f_{\underline{mn}}$-cycles for every face: $\sum_{\underline{m}_1,\dots,\underline{m}_j} f_{\underline{m}_1\underline{m}_2}\cdots f_{\underline{m}_{j-1}\underline{m}_j} f_{\underline{m}_j\underline{m}_1} G_{|\dots|\underline{m}_1\dots\underline{m}_j|\dots|}$
- Write $G_{|\dots|\underline{m}_1\dots\underline{m}_j|\dots|}$ for every face as Laplace transform in $\frac{|\underline{m}_1|+\dots+|\underline{m}_j|}{\sqrt{V}}$ and Fourier transform in $\frac{|\underline{m}_{i+1}|-|\underline{m}_i|}{\sqrt{V}}$

Lemma

(with $J+i \equiv i$, $\quad |z_i|<1$)

$$\sum_{m_1,\dots,m_J=0}^{\infty} \prod_{i=1}^{J} z_i^{m_i} L_{m_i}^{m_{i+1}-m_i}(r_i) = \frac{\exp\Big(-\frac{\sum_{i,k=1}^{J} r_i (z_{k+i}\cdots z_{J+i})}{1-(z_1\cdots z_J)}\Big)}{1-(z_1\cdots z_J)}$$

- $1-(z_1\cdots z_J) \xrightarrow{V\to\infty} \begin{cases} 2 & (J \text{ odd}) \\ \frac{t}{\sqrt{V}} & (J \text{ even}) \end{cases}$ $\quad$ (t – Laplace par., $r \propto \frac{x^2}{\sqrt{V}}$)
- gives factor $V^{\#(\text{even faces})}$, and G gives factor $V^{-\#(\text{all faces})}$

Proposition

$$S_c(\mu x_1,\dots,\mu x_N) = \frac{1}{64\pi^2} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \Big(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4p_\beta}{4\pi^2\mu^4} e^{\mathrm{i}\left\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta} (-1)^{i-1} \mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)} \right\rangle} \Big)$$

$$\times G\Big(\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1} \Big| \dots \Big| \underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B} \Big)$$

Results

- Only a restricted sector of the matrix model contributes to position space: All faces have common matrix indices.
- Schwinger functions are symmetric and invariant under the full Euclidean group (this is limit $\theta \to \infty$!)
- Most interesting sector: every face has $N_\beta = 2$ indices. This describes propagation and interaction of B particles, without any momentum exchange.
- Similar to free particles, but $(N_1 + \ldots + N_B)$-point functions violate clustering. There are non-trivial topological sectors.
- Analytic continuation to Minkowski space and Osterwalder-Schrader reflection positivity would follow (at least for 2-point function) if $a \mapsto G_{aa}$ is a Stieltjes function.
- f Stieltjes $\Leftrightarrow$ f-smooth, $f(x) \geq 0$, $(-1)^n \frac{d^{2n+1}}{dx^{2n+1}}(x^{n+1}f(x)) \geq 0$ [Widder, 1938]

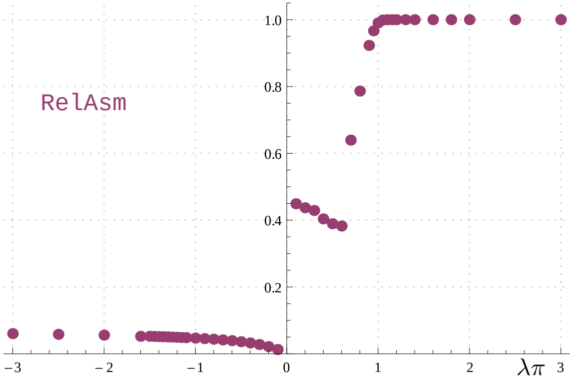
Computer simulations

- We implement G_{a0} for $a \in [0, \Lambda^2]$ as piecewise-linear function with edges arranged as geometric progression.
- We find numerically that the operator T in the fixed point equation $G = TG$ (with $F(b) = 0$) satisfies the assumptions of the Banach fixed point theorem in Lipschitz space.
- The sequence $G^{n+1}_{a0} = (TG^n)_{a0}$ converges for any λ.

  There is no discontinuity of $G_{a0}(\lambda)$ at $\lambda = 0$.
- The required symmetry $G_{ab} = G_{ba}$ is numerically (for $F(b) = C = 0$)
  – accurately realised for any $\lambda < 0$
  – badly violated for any $\lambda > 0$

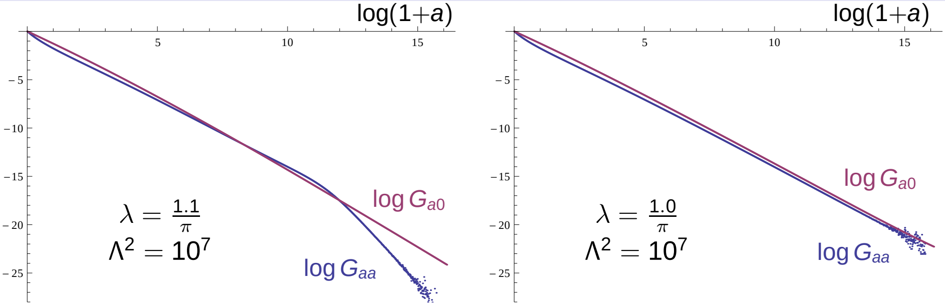
  This is clear evidence that $C \neq 0$ and $F(b) \neq 0$ for $\lambda > 0$. This is currently out of reach.

Relative asymmetry $\sup_{a,b} \frac{|G_{ab}-G_{ba}|}{G_{ab}+G_{ba}}$



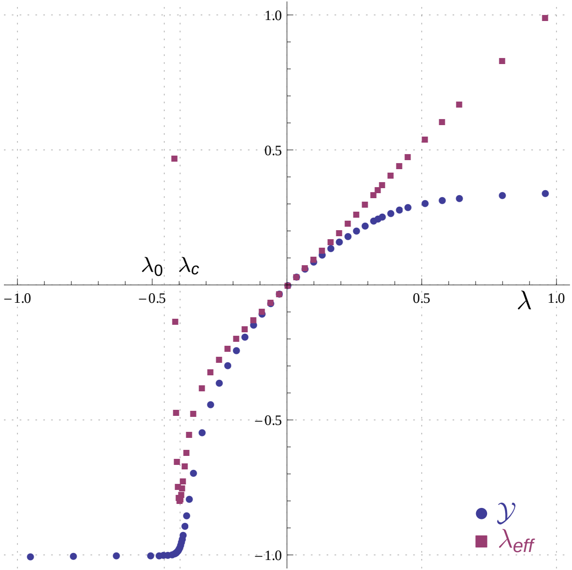
- results for $\Lambda^2 = 10^7$ with 2000 sample points
- $\approx 5\%$ asymmetry for $\lambda < 0$, traced back to discretisation
- $\approx 40\%$ asymmetry for $\lambda > 0$, increases to 100% at $\lambda\pi \approx 0.6$
- Neglection of C, F for $\lambda > 0$ not justified!

$\log G_{a0}$ and $\log G_{aa}$ as function of $\log(1+a)$



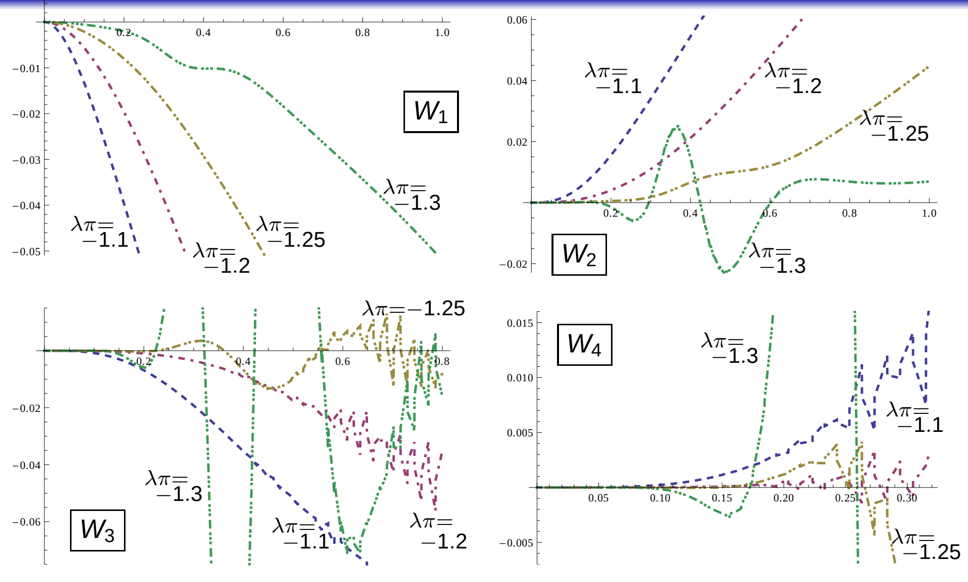
- For $\lambda \geq \frac{1.1}{\pi}$ the function G_{aa} suddenly bends and increases the (negative) slope by 1.
- As $\lambda \searrow 0$, this bend moves to larger a, possibly to $a \gg \Lambda^2$. This might explain the jump of asymmetry.
- For $\lambda > 0$ we have $G_{aa} \approx \frac{C}{(1+a)^{1+\eta}}$ with $\eta > 0$. Such functions are not Stieltjes (negative anomalous dimension).

Main result: evidence for phase transition

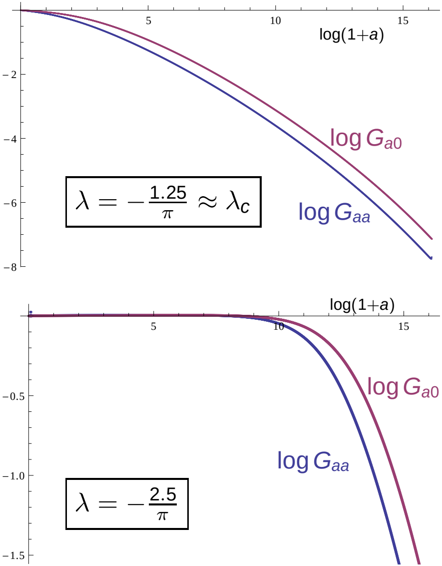


- results for $\Lambda^2 = 10^7$ with 2000 sample points
- $\mathcal{Y}'$ discontinuous at $\lambda_c = -0.396$
- Excellent agreement with $\lambda_s = -0.392$ where Stieltjes property is lost
- λ_{eff} singular at $\lambda_0 = -0.455$ where $\mathcal{Y} = -1$
- Nothing particular at pole $\lambda_b = -\frac{1}{72} = 0.014$ of Borel resummation

Widder's criteria $W_n(a) := \dfrac{a^{n+1}}{n!(n+1)!}\dfrac{d^{2n+1}}{da^{2n+1}}\big(a^{n+1}G_{aa}\big)$



An order parameter



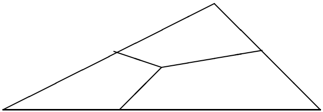
- $A := \max\{a \,:\, G_{aa} \approx 1\}$ is order parameter; $A = 0$ for $\lambda > \lambda_c$ and $A > 0$ for $\lambda < \lambda_c$.
- Higher correlation function ill-defined for matrix indices $\leq A$
- sort of minimal momentum for Euclidean particles

Conclusion

1. The quartic matrix model $Z = \int dM \exp(\mathrm{tr}(JM - EM^2 - \frac{\lambda}{4}M^4))$ is exactly solvable in terms of solution of a non-linear equation.
2. Φ^4_4-theory on Moyal space is of that type. The non-linear equation is reduced to a fixed-point problem.
3. The Φ^4_4-Moyal matrix model has a unique non-perturbative and non-trivial solution for $\lambda \leq 0$.
4. The corresponding Euclidean quantum field theory is only sensitive to diagonal matrix elements.
5. The theory seems to satisfy Osterwalder-Schrader axioms for $\lambda \in [-0.394 \pm 0.003, 0]$, definitely not outside this interval.
   We have constructed a Wightman QFT in 4 dimensions.
6. It describes interacting particles without momentum transfer.
   $G_{aa|bb} \neq 0$ implies presence of non-trivial vacuum sectors.

Future steps

(2D quantum gravity) should have equivalent descriptions as cubic and quartic matrix model.



Quartic models show positivity and boundedness from below. They admit techniques from constructive QFT (loop vertex expansion) not possible in cubic model.

Our solution of the quartic matrix model might be useful in 2D quantum gravity and algebraic geometry.

(Coloured tensor models) extend these methods to quantum gravity in $D \geq 3$. They have Schwinger-Dyson equations and action of $U(\infty)$ group. Our method might generalise to this class.