Schwinger functions and reflection positivity

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(based on joint work with Harald Grosse,
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Summary

We study the self-dual ϕ^4_4-Euclidean QFT on Moyal space which is formally defined by the partition function

$$\mathcal{Z}[J]=\int\mathcal{D}[\phi]\,\exp\Big(-\frac{ZV}{2}\sum_{\underline{m},\underline{n}\in\mathbb{N}^2}\Big(\frac{|\underline{m}|+|\underline{n}|}{\sqrt{V}}+\mu^2_{bare}\Big)\phi_{\underline{mn}}\phi_{\underline{nm}}-\frac{\lambda Z^2V}{4}\sum_{\underline{k},\underline{l},\underline{m},\underline{n}\in\mathbb{N}^2}\phi_{\underline{kl}}\phi_{\underline{lm}}\phi_{\underline{mn}}\phi_{\underline{nk}}+V\sum_{\underline{m},\underline{n}\in\mathbb{N}^2}\phi_{\underline{mn}}J_{\underline{nm}}\Big)$$

- Only $\frac{1}{\mathrm{volume}(V)}\log\mathcal{Z}$ has a limit $V\to\infty$.
- Expansion according to number B of boundary components:

$$\log\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}=\sum_{B=1}^{\infty}\sum_{1\leq N_1\leq\cdots\leq N_B}^{\infty}\frac{V^{2-B}}{S}\sum_{\underline{q}_i^\beta\in\mathbb{N}^2}G_{|\underline{q}_1^1\dots\underline{q}_{N_1}^1|\dots|\underline{q}_1^B\dots\underline{q}_{N_B}^B|}\prod_{\beta=1}^{B}\big(J_{\underline{q}_1^\beta\underline{q}_2^\beta}\cdots J_{\underline{q}_{N_\beta}^\beta\underline{q}_1^\beta}\big)$$

- The $(N_1+\ldots+N_B)$-point functions $G_{|\underline{q}_1^1\dots\underline{q}_{N_1}^1|\dots|\underline{q}_1^B\dots\underline{q}_{N_B}^B|}$ depend only on the 1-norms $|\underline{q}_i^\beta|$.

Matrix model limit

Introduce mass scale μ and consider $\mu^4 V \to \infty$

$$\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \sum_{B=1}^{\infty} \sum_{1\leq N_1\leq\cdots\leq N_B}^{\infty} \frac{(V\mu^4)^{2-B}}{S} \sum_{\underline{q}_i^{\beta}\in\mathbb{N}^2} G_{|\underline{q}_1^1\ldots\underline{q}_{N_1}^1|\ldots|\underline{q}_1^B\ldots\underline{q}_{N_B}^B|} \prod_{\beta=1}^{B} \Big(\frac{J_{\underline{q}_1^{\beta}\underline{q}_2^{\beta}}}{\mu^3} \cdots \frac{J_{\underline{q}_{N_\beta}^{\beta}\underline{q}_1^{\beta}}}{\mu^3} \Big)$$

- $\lim_{V\to\infty} G_{|\underline{q}_1^1\ldots\underline{q}_{N_1}^1|\ldots|\underline{q}_1^B\ldots\underline{q}_{N_B}^B|}$ already restricts to the planar sector (genus $g=0$).
- The natural density $\displaystyle\lim_{V\to\infty} \frac{1}{V\mu^4} \log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}$ also removes $B>1$ boundary components:

$$\lim_{V\mu^4\to\infty} \frac{1}{V\mu^4} \log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \sum_{N=2}^{\infty} \frac{1}{N} \lim_{V\mu^4\to\infty} \sum_{\underline{q}_1,\ldots,\underline{q}_N\in\mathbb{N}^2} G_{|\underline{q}_1\ldots\underline{q}_N|} \frac{J_{\underline{q}_1\underline{q}_2}}{\mu^3} \cdots \frac{J_{\underline{q}_N\underline{q}_1}}{\mu^3}$$

- Remaining limit $V\mu^4\to\infty$ passes to integral representation.

Connected functions

- Had constructed N-point functions $G_{a_1\dots a_N} \equiv G_{\vec{a}_1\dots\vec{a}_N}$ arising from $G_{|\underline{q}_1\dots\underline{q}_N|}$ in limit $\mathcal{N}, V\mu^4 \to \infty$ with $\frac{\mathcal{N}}{\sqrt{V\mu^4}} = \Lambda^2(1+\mathcal{Y})$ fixed.

- Then $|\underline{q}_i| = \sqrt{V\mu^4}(1+\mathcal{Y})\vec{a}_i$ with $\vec{a}_i \in \mathbb{R}_+ \times \mathbb{R}_+$ for $\Lambda \to \infty$, with $|\vec{a}| = a^1 + a^2 =: a$ in previous notation.

- Define connected renormalised matrix functions as

$$\langle \phi_{\vec{a}_1\vec{b}_1} \cdots \phi_{\vec{a}_N\vec{b}_N} \rangle_c := \lim_{V\mu^4\to\infty} \frac{\frac{1}{V\mu^4}\, d^N \Big(\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}\Big)_{Z\mu^2_{bare}\mapsto\mu^2\,,\; Z\mapsto(1+\mathcal{Y})}}{dJ_{\sqrt{V\mu^4}(1+\mathcal{Y})\vec{b}_1,\sqrt{V\mu^4}(1+\mathcal{Y})\vec{a}_1} \cdots dJ_{\sqrt{V\mu^4}(1+\mathcal{Y})\vec{b}_N,\sqrt{V\mu^4}(1+\mathcal{Y})\vec{a}_N}}\Bigg|_{J=0}$$

- Exactly solvable and non-trivial matrix model describing ϕ^4_4-theory on extreme Moyal space $\theta \to \infty$.

Translation to Euclidean QFT on $\mathbb{R}^4$

- Infinite volume limit $V\mu^4 \to \infty$ requires densities. Absolute position $x \in \mathbb{R}^4$ has no meaning, only μx can be used.
- Recall matrix basis
  $\phi(\mu x) = \sum_{\underline{m},\underline{n}\in\mathbb{N}^2} f_{m_1n_1}(\mu x^0, \mu x^1) f_{m_1n_1}(\mu x^2, \mu x^3)\phi_{\underline{mn}}$ with

$$f_{mn}(\mu y^0, \mu y^1) = 2(-1)^m \sqrt{\frac{m!}{n!}} \Big(\frac{\mu(y^0 + \mathrm{i}y^1)}{\sqrt[4]{4V\mu^4}}\Big)^{n-m} L_m^{n-m}\Big(\frac{\mu^2|y|^2}{\sqrt{4V\mu^4}}\Big) e^{-\frac{\mu^2|y|^2}{2\sqrt{4V\mu^4}}}$$

- Leads to expansion

$$\langle\phi(\mu x_1)\dots\phi(\mu x_N)\rangle \equiv \sum_{\underline{m}_1,\underline{n}_1,\dots,\underline{m}_N,\underline{n}_N\in\mathbb{N}^2} f_{\underline{m}_1\underline{m}_2}(\mu x_1)\cdots f_{\underline{m}_N\underline{m}_N}(\mu x_N)\langle\phi_{\underline{m}_1\underline{n}_1}\dots\phi_{\underline{m}_N\underline{n}_N}\rangle$$

- Schwinger functions require $\mathrm{volume}(V) = V^2$ in $\frac{\log\mathcal{Z}}{\mathrm{volume}(V)}$.
- This reflects the fact that the spectral geometry of Moyal space with harmonic propagation has finite volume $(\frac{V}{\Omega})^2$ [Grosse-W., 2007 & Gayral-W., 2011]

Schwinger functions

Definition

$$\mu^N S_c(\mu x_1,\dots,\mu x_N) := \lim_{V\mu^4\to\infty} \sum_{\underline{m}_1,\underline{n}_1,\dots,\underline{m}_N,\underline{n}_N\in\mathbb{N}^2} f_{\underline{m}_1\underline{n}_1}(\mu x_1)\cdots f_{\underline{m}_N\underline{n}_N}(\mu x_N)\frac{\mu^{4N}\partial^N\mathcal{F}[J]}{\partial J_{\underline{m}_1\underline{n}_1}\dots\partial J_{\underline{m}_N\underline{n}_N}}\Big|_{J=0}$$

$$\mathcal{F}[J] := \frac{1}{64\pi^2 V^2\mu^8}\log\left(\frac{\int\mathcal{D}[\phi]\,e^{-S[\phi]+V\sum_{\underline{a},\underline{b}\in\mathbb{N}^2}\phi_{\underline{ab}}J_{\underline{ba}}}}{\int\mathcal{D}[\phi]\,e^{-S[\phi]}}\right)_{\substack{Z\mu^2_{bare}\mapsto\mu^2\\ Z\mapsto(1+\mathcal{Y})}}$$

$$S[\phi] = \frac{Z}{2}\sum_{\underline{m},\underline{n}\in\mathbb{N}^2}\Big(\frac{|\underline{m}|+|\underline{n}|}{\sqrt{V}}+\mu^2_{bare}\Big)\phi_{\underline{mn}}\phi_{\underline{nm}} - \frac{\lambda Z^2}{4}\sum_{\underline{k},\underline{l},\underline{m},\underline{n}\in\mathbb{N}^2}\phi_{\underline{kl}}\phi_{\underline{lm}}\phi_{\underline{mn}}\phi_{\underline{nk}}$$

Expansion into B cycles

$$\mathcal{F}[J]=\frac{1}{64\pi^2}\sum_{B=1}^{\infty}\sum_{1\leq N_1\leq\cdots\leq N_B}^{\infty}\sum_{\underline{q}_i^\beta\in\mathbb{N}^2}\frac{G_{|\underline{q}_1^1\ldots\underline{q}_{N_1}^1|\ldots|\underline{q}_1^B\ldots\underline{q}_{N_B}^B|}}{S}\prod_{\beta=1}^{B}\frac{1}{V\mu^4}\Big(\frac{J_{\underline{q}_1^\beta\underline{q}_2^\beta}}{\mu^3}\cdots\frac{J_{\underline{q}_{N_\beta}^\beta\underline{q}_1^\beta}}{\mu^3}\Big)$$

- J-derivatives are fully symmetric in $\mu x_1\ldots,\mu x_N$
- J-cycle structure in $\mathcal{F}$ produces $f_{\underline{mn}}$-cycles for every face:

$$S_c(\mu x_1,\ldots,\mu x_N)=\lim_{V\mu^4\to\infty}\frac{1}{64\pi^2}\sum_{N_1+\cdots+N_B=N}\sum_{\underline{q}_i^\beta\in\mathbb{N}^2}G_{|\underline{q}_1^1\ldots\underline{q}_{N_1}^1|\ldots|\underline{q}_1^B\ldots\underline{q}_{N_B}^B|}$$
$$\times\sum_{\sigma\in\mathcal{S}_N}\prod_{\beta=1}^{B}\frac{f_{\underline{q}_1\underline{q}_2}(\mu x_{\sigma(N_1+\ldots+N_{\beta-1}+1)})\cdots f_{\underline{q}_{N_\beta}\underline{q}_1}(\mu x_{\sigma(N_1+\ldots+N_\beta)})}{V\mu^4N_\beta}$$

- Compute sum over indices $\underline{q}_i^\beta\in\mathbb{N}^2$ after Fourier-Laplace transform of G

Assume that G has, for every boundary component, representation as Laplace transform in total sum of index norms and Fourier transform in differences of index norms.

- Transform will be reverted in the end so that the analyticity assumption is not necessary.
- Future analytic continuation to Minkowski space implies representation as Laplace transform.

$$G_{|\underline{q}_1^1\dots\underline{q}_{N_1}^1|\dots|\underline{q}_1^B\dots\underline{q}_{N_B}^B|}$$
$$= \int_{\mathbb{R}_+^B} d(t^1,\dots,t^B) \int_{\mathbb{R}^{N-B}} d(\omega_1^1,\dots,\omega_{N_1-1}^1,\dots\omega_1^B,\dots,\omega_{N_B-1}^B)$$
$$\times\, \mathcal{G}(t^1,\omega_1^1,\dots,\omega_{N_1}^1|\dots|t^B,\omega_1^B,\dots,\omega_{N_B}^B)$$
$$\times \prod_{\beta=1}^{B} \exp\Big(-\frac{t^\beta}{\sqrt{V\mu^4}}\sum_{i=1}^{N_\beta}|\underline{q}_i^\beta| + \frac{\mathrm{i}}{\sqrt{V\mu^4}}\sum_{i=1}^{N_\beta-1}\omega_i^\beta(|\underline{q}_i^\beta|-|\underline{q}_{i+1}^\beta|)\Big)$$

- $|\underline{q}_i^\beta| = q_{i,1}^\beta + q_{i,2}^\beta \;\Rightarrow\; \exp(\dots) = \prod_i \big(z_i^\beta(t^\beta,\vec{\omega}^\beta)\big)^{q_{i,1}^\beta}\big(z_i^\beta(t^\beta,\vec{\omega}^\beta)\big)^{q_{i,2}^\beta}$

For every boundary component $\beta = 1, \dots, B$, need to compute

$$\sum_{q_1,\dots,q_{N'}=0}^{\infty} \frac{f_{q_1q_2}(\mu\vec{y}_1)\cdots f_{q_{N'}q_1}(\mu\vec{y}_{N'})}{\sqrt{V\mu^4N'}} z_1^{q_1}\cdots z_{N'}^{q_{N'}}$$

$$= 2^{N'}\sum_{q_1,\dots,q_{N'}=0}^{\infty} e^{-\frac{1}{2}(r_1+\dots+r_{N'})} \frac{L_{q_1}^{q_2-q_1}(r_1)\cdots L_{q_{N'}}^{q_1-q_{N'}}(r_{N'})}{\sqrt{V\mu^4N'}}(-\tilde{z}_1)^{q_1}\cdots(-\tilde{z})_{N'}^{q_{N'}}$$

where $r_i = \frac{\mu^2|\vec{y}_i|^2}{\sqrt{4V\mu^4}}$ and $\tilde{z}_j = \frac{\vec{y}_{j-1}}{\vec{y}_j}\exp\left(-\frac{t-\mathrm{i}(\omega_j-\omega_{j-1})}{\sqrt{V\mu^4}}\right)$

(with $\vec{y}_i \in \mathbb{C}$, $\omega_0 = \omega_{N'} \equiv 0$ $\vec{y}_0 \equiv \vec{y}_{N'}$)

Lemma (cyclic product of Laguerre polynomials)

(with $N'+j \equiv j$, $|\tilde{z}_j| < 1$)

$$\sum_{q_1,\dots,q_{N'}=0}^{\infty}\prod_{j=1}^{N'}(-\tilde{z}_j)^{q_j}L_{q_j}^{q_{j+1}-q_j}(r_j) = \frac{\exp\left(-\frac{\sum_{j,k=1}^{N'} r_j(-\tilde{z}_{k+j})\cdots(-\tilde{z}_{N'+j})}{1-(-\tilde{z}_1)\cdots(-\tilde{z}_{N'})}\right)}{1-(-\tilde{z}_1)\cdots(-\tilde{z}_{N'})}$$

- $1-(-\tilde{z}_1)\cdots(-\tilde{z}_{N'}) = 1-(-1)^{N'}\exp\left(-\frac{N't}{\sqrt{V\mu^4}}\right) \xrightarrow{V\mu^4\to\infty} \begin{cases} \frac{N't}{\sqrt{V\mu^4}} & N' \text{ even} \\ 2 & N' \text{ odd} \end{cases}$

Limit $V\mu^4 \to \infty$

$$\lim_{V\mu^4\to\infty} \sum_{\underline{q}_1,\dots,\underline{q}_{N'}\in\mathbb{N}^2} \frac{f_{\underline{q}_1\underline{q}_2}(\mu x_1)\cdots f_{\underline{q}_{N'}\underline{q}_1}(\mu x_{N'})}{V\mu^4 N'} z_1^{q_{1,1}+q_{1,2}}\cdots z_{N'}^{q_{N',1}+q_{N',2}}$$

$$= \begin{cases} \dfrac{4^{N'}}{(N')^3 t^2} \exp\left(-\dfrac{\mu^2\big\|x_1-x_2+\ldots+x_{N'-1}-x_{N'}\big\|^2}{2N't}\right) & \text{for } N' \text{ even} \\ 0 & \text{for } N' \text{ odd} \end{cases}$$

- All $N_1,\dots,N_B$ must be even! No dependence on ω_i!
- $\dfrac{\exp\left(-\frac{\|\mu X\|^2}{2N't}\right)}{(N't)^2} = \int_{\mathbb{R}^4} \dfrac{dp}{4\pi^2\mu^4} e^{-\mathrm{i}\langle\frac{p}{\mu},\mu X\rangle} \exp\left(-\dfrac{N't\|p\|^2}{2\mu^2}\right)$
- Integration of $\mathcal{G}(t^1,\omega_1^1,\dots,\omega_{N_1}^1|\dots|t^B,\omega_1^B,\dots,\omega_{N_B}^B)$ against $\exp\left(-\frac{N't^\beta\|p^\beta\|^2}{2\mu^2}\right)$ returns to original $G_{|\underline{q}_1^1\dots\underline{q}_{N_1}^1|\dots|\underline{q}_1^B\dots\underline{q}_{N_B}^B|}$, but with
  1. For each β, all $|\underline{q}_i^\beta|$ coincide (no ω-dependence)
  2. $\dfrac{\sum_{i=1}^{N_\beta}|\underline{q}_i^\beta|}{\sqrt{V\mu^4}} = \dfrac{N_\beta}{2\mu^2}\|p\|^2$, hence $\dfrac{|\underline{q}_i^\beta|}{\sqrt{V\mu^4}} \xrightarrow{V\mu^4\to\infty} (1+\mathcal{Y})q = \dfrac{|p|^2}{2\mu^2}$ in limit to integral representation.

Final result

Proposition

The connected N-point Schwinger functions of ϕ^4_4-model on extreme Moyal space $\theta \to \infty$ are given by

$$S_c(\mu x_1, \dots, \mu x_N) = \frac{1}{64\pi^2} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma \in \mathcal{S}_N} \left(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4 p_\beta}{4\pi^2\mu^4} e^{\mathrm{i}\left\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta} (-1)^{i-1} \mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)} \right\rangle} \right)$$
$$\times G\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1} \Big| \dots \Big| \underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}$$

Comments

- Only a restricted sector of the matrix model contributes to position space: All faces have common matrix indices.
- Schwinger functions are symmetric and invariant under the full Euclidean group (this is limit $\theta \to \infty$!)
- Most interesting sector: every face has $N_\beta = 2$ indices.
  It is described by the functions $G_{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}\Big|\dots\Big|\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}$
- This sector describes propagation and interaction of B particles without any momentum exchange.
- This is acceptable for a 2D-model. In 4D, absence of momentum transfer is a sign of triviality!
- Triviality proofs rely on clustering, analyticity in Mandelstam representation, absence of bound states. Needs verification.

No clustering (example: (2+2)-sector)

$$S_c^{2+2}(\mu x_1,\dots,\mu x_4) = \frac{4}{\pi^2}\int_{\mathbb{R}^4}\frac{dp}{4\pi^2\mu^2}\int_{\mathbb{R}^4}\frac{dq}{4\pi^2\mu^2}\, G_{\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}\Big|\frac{\|q\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|q\|^2}{2\mu^2(1+\mathcal{Y})}} \times \Big(e^{\mathrm{i}\langle p,x_1-x_2\rangle+\mathrm{i}\langle q,x_3-x_4\rangle}+e^{\mathrm{i}\langle p,x_1-x_3\rangle+\mathrm{i}\langle q,x_4-x_2\rangle}+e^{\mathrm{i}\langle p,x_1-x_4\rangle+\mathrm{i}\langle q,x_3-x_2\rangle}\Big)$$

- For x_1-x_2, x_3-x_4 small, but $\|x_1-x_3\|\to\infty$, we have no decay:
  $$S_c^{2+2}(\mu x_1,\dots,\mu x_4) \longrightarrow \int\frac{dp\,dq}{4\pi^6\mu^4}\, G_{\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}\Big|\frac{\|q\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|q\|^2}{2\mu^2(1+\mathcal{Y})}}\, e^{\mathrm{i}\langle p,x_1-x_2\rangle+\mathrm{i}\langle q,x_3-x_4\rangle}$$
- Absence of clustering means that vacuum state (of hypothetical continuation to Wightman theory) is not unique.
- Non-pure states can be decomposed into pure states which describe different topological sectors.

Intuitive picture

- Surprising that the limit $\theta \to \infty$ of extreme noncommutativity is so close to an ordinary field theory expected for $\theta \to 0$.
- Vertex in momentum space (gives Feynman rule)

$$\int_{(\mathbb{R}^4)^4} \Big(\prod_{i=1}^{4} \frac{dp_i}{(2\pi)^4}\Big)\,\delta(p_1+\cdots+p_4)\,\exp\Big(\mathrm{i}\sum_{i<j}\langle p_i,\Theta p_j\rangle\Big)\,\prod_{i=1}^{4}\hat{\phi}(p_i)$$

- For $\theta \to \infty$, this converges to zero almost everywhere by Riemann-Lebesgue Lemma

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- For $\theta \to \infty$, this converges to zero almost everywhere by Riemann-Lebesgue Lemma
  …UNLESS p_i, p_j are linearly dependent.
- The case of linearly dependent momenta might be protected for topological reasons. These are the boundary components $B > 1$ which guarantee full Lebesgue measure!
- Already discussed in [Becchi-Giusto-Imbimbo, 2002/03] (as Swiss Cheese) and [Minwalla-van Raamsdonk-Seiberg, 1999] (as particularly "stringy")

Osterwalder-Schrader reconstruction theorem

Theorem [Osterwalder-Schrader, 1973–1975]

Assume for $S(x_1,\dots,x_N)$:

0. growth conditions
1. Euclidean covariance
2. reflection positivity: for each assignment $N \mapsto f_N \in \mathcal{S}^N$,
$$\sum_{M,N} \int dxdy\; S(x_1,\dots,x_N,y_1,\dots y_M)\overline{f_N(x_1^r,\dots,x_N^r)}f_M(y_1,\dots,y_M) \geq 0$$
where $(x^0,x^1,\dots x^{d-1})^r := (-x^0,x^1,\dots x^{d-1})$
3. permutation symmetry

Then the $S(\xi_1,\dots \xi_{N-1})\big|_{\xi_i^0>0}$, with $\xi_i = x_i - x_{i+1}$, are Laplace-Fourier transforms of Wightman functions in a relativistic QFT.

If $S(x_1,\dots,x_N)$ satisfy

4. clustering

the Wightman functions satisfy clustering, too.

Analytic continuation of the 2-point function

- Representation as Laplace transform in ξ^0 requires analyticity in $\mathrm{Re}(\xi^0) > 0$.
- For the 2-point function, this analyticity in ξ^0 is a corollary of analyticity of $a \mapsto G_{aa}$ in $\mathbb{C} \setminus]-\infty, 0]$
- Generalised Stieltjes functions have precisely such analyticity.

  Reflection positivity restricts to standard Stieltjes functions.

Definition

A function $f : \mathbb{R}_+ \to \mathbb{R}$ is a Stieltjes function iff

$$f(x) = c + \int_0^\infty \frac{d(\rho(t))}{x+t}$$

with $c = f(\infty) \geq 0$ and ρ positive and non-decreasing.

- We had $S_c(\mu\xi) = \int_{\mathbb{R}^4} \frac{dp}{(2\pi\mu)^4} e^{ip\xi} G_{\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})} \frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}}$
- Assume Stieltjes, define $\omega_{\vec{p}}(t) := \sqrt{\vec{p}^2 + 2\mu^2(1+\mathcal{Y})t}$

$$S_c(\mu\xi)\Big|_{\xi^0>0}$$

$$= \int_{\mathbb{R}^3} \frac{d\vec{p}}{(2\pi\mu)^3} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi\mu} e^{ip^0\xi^0 + i\vec{p}\cdot\vec{\xi}} \int_0^\infty \frac{d\rho(t)}{t + \frac{(p^0)^2+\vec{p}^2}{2\mu^2(1+\mathcal{Y})}}$$

$$= 2\mu(1{+}\mathcal{Y}) \int_{\mathbb{R}^3} \frac{d\vec{p}\; e^{i\vec{p}\cdot\vec{\xi}}}{(2\pi\mu)^3} \int_0^\infty \frac{d\rho(t)}{2\omega_{\vec{p}}(t)} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi\mathrm{i}} \Big(\frac{e^{ip^0\xi^0}}{p^0 - \mathrm{i}\omega_{\vec{p}}(t)} - \frac{e^{ip^0\xi^0}}{p^0 + \mathrm{i}\omega_{\vec{p}}(t)} \Big)$$

$$= 2\mu(1{+}\mathcal{Y}) \int_{\mathbb{R}^3} \frac{d\vec{p}\; e^{-\xi^0\omega_{\vec{p}}(t) + i\vec{p}\cdot\vec{\xi}}}{(2\pi\mu)^3} \int_0^\infty \frac{d\rho(t)}{2\omega_{\vec{p}}(t)}$$

$$= \int_0^\infty \frac{2(1+\mathcal{Y})\, d\rho(t)}{\mu^4} \int_0^\infty dq^0 \int_{\mathbb{R}^3} d\vec{q}\; \hat{W}_t(q) e^{-q^0\xi^0 + i\vec{q}\cdot\vec{x}}$$

with $\hat{W}_t(q) = \frac{\theta(q^0)}{(2\pi)^3} \delta\Big(\frac{(q^0)^2 - \vec{q}^2 - 2\mu^2(1{+}\mathcal{Y})t}{\mu^2} \Big)$

- $\hat{W}_t(q)$ is Källén-Lehmann spectral representation of a Wightman 2-point function.

Stieltjes functions

The Stieltjes property can be tested by purely real conditions:

Theorem [Widder, 1938]

$f : \mathbb{R}_+ \to \mathbb{R}$ is Stieltjes iff $\mathcal{C}^\infty$, positive and $L_{k,t}[f(\bullet)] \geq 0$, where

$$L_{k,t}[f(\bullet)] := \frac{(-t)^{k-1}}{c_k} \frac{d^{2k-1}}{dt^{2k-1}} \big(t^k f(t)\big) , \qquad c_1 = 1,\ c_{k>1} = k!(k-2)!$$

- In this case $\rho'(t) = \lim_{k\to\infty} L_{k,t}[f(\bullet)]$ (weakly and a.e.).

- Perturbative argument: $a \mapsto G_{aa}$ cannot be Stieltjes for $\lambda > 0$! (negative anomalous dimension due to renormalisation)
- Reminiscent of planar wrong-sign $\lambda\phi^4_4$-model [t'Hooft, 1983 & Rivasseau, 1984]
- Our reduction to planar sector is automatic, and $B \geq 1$ boundary components show non-trivial topology. We have exact solution for $S(x_1, \dots, x_N)$, not only existence proof.

Computer simulations

- We implement G_{a0} for $a \in [0, \Lambda^2]$ as piecewise-linear function with edges arranged as geometric progression.

- Master equation $G = TG$

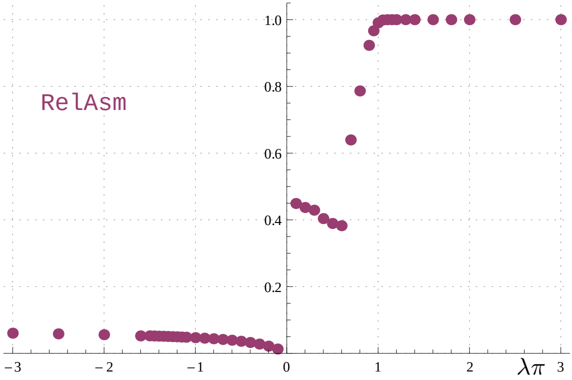
$$G_{b0}=\frac{\left\{\begin{matrix} 1 & \text{for } \lambda<0 \\ 1+bF(b) & \text{for } \lambda>0 \end{matrix}\right\}}{1+b}\exp\left(-\lambda\int_0^b dt\int_0^\infty \frac{dp}{(\lambda\pi p)^2+\left(t+\frac{1+\lambda\pi p\mathcal{H}_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

- We find numerically that T satisfies assumptions of Banach fixed point theorem in Lipschitz space (for $F = 0$).

- The sequence $G^{n+1}_{a0} = (TG^n)_{a0}$ converges for any λ. There is no discontinuity of $G_{a0}(\lambda)$ at $\lambda = 0$.

- The required symmetry $G_{ab} = G_{ba}$ is (for $F(b) = C = 0$)
  – accurately realised for any $\lambda < 0$
  – badly violated for any $\lambda > 0$

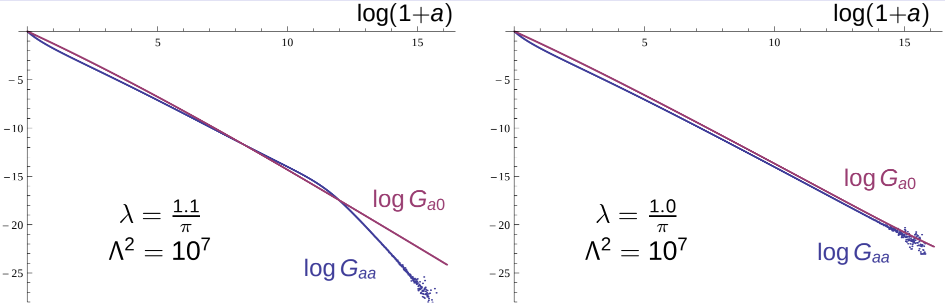
Relative asymmetry $\sup_{a,b} \frac{|G_{ab}-G_{ba}|}{G_{ab}+G_{ba}}$



- results for $\Lambda^2 = 10^7$ with 2000 sample points
- $\approx 5\%$ asymmetry for $\lambda < 0$, traced back to discretisation
- $\approx 40\%$ asymmetry for $\lambda > 0$, increases to 100% at $\lambda\pi \approx 0.6$
- Neglection of C, F for $\lambda > 0$ not justified!

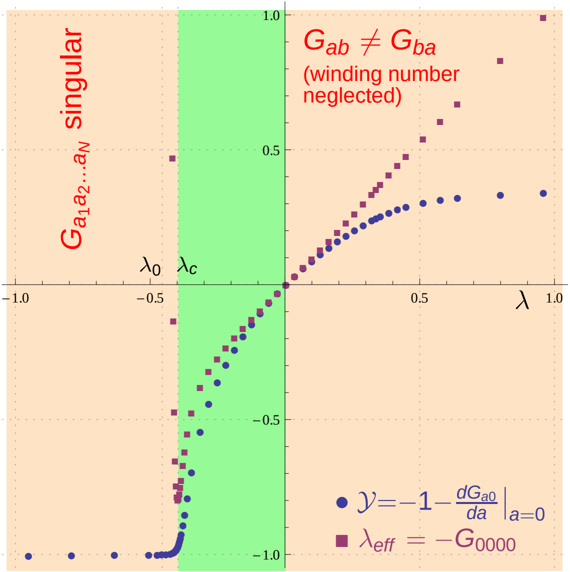
(No good idea at the moment)

$\log G_{a0}$ and $\log G_{aa}$ as function of $\log(1+a)$



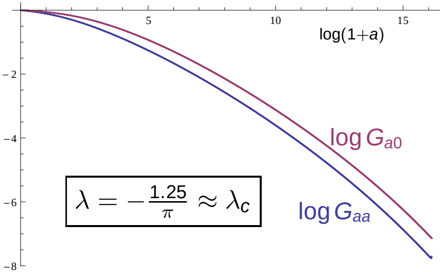
- For $\lambda \geq \frac{1.1}{\pi}$ the function G_{aa} suddenly bends and increases the (negative) slope by 1.
- As $\lambda \searrow 0$, this bend moves to larger a, possibly to $a \gg \Lambda^2$. This might explain the jump of asymmetry.
- For $\lambda > 0$ we have $G_{aa} \approx \frac{C}{(1+a)^{1+\eta}}$ with $\eta > 0$. Such functions are not Stieltjes (negative anomalous dimension).

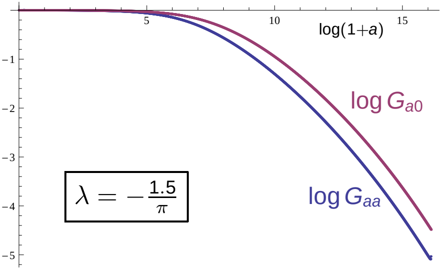
Evidence for phase transitions

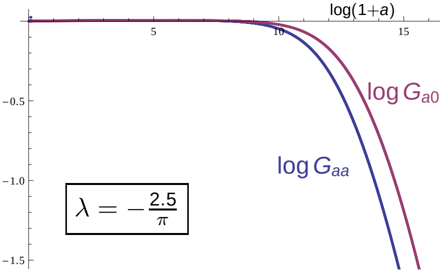


- G_{ab} for $\Lambda^2 = 10^7$ with 2000 sample points
- $\mathcal{Y}'$ discontinuous at $\lambda_c = -0.396$
- λ_{eff} singular at $\lambda_0 = -0.455$ where $\mathcal{Y} = -1$
- Nothing particular at pole $\lambda_b = -\frac{1}{72} = 0.014$ of Borel resummation
- The Stieltjes property is precisely realised in $[\lambda_c, 0]$, not outside!

The phase $\lambda < \lambda_c$

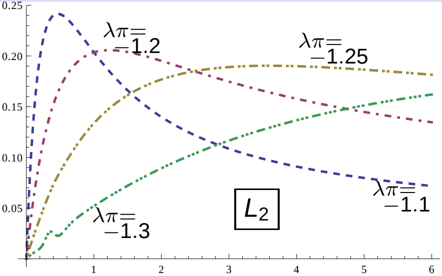


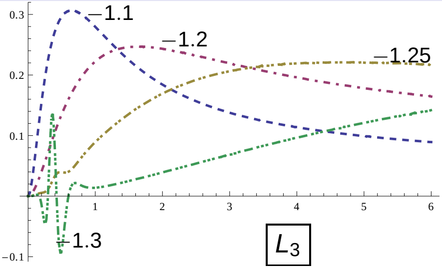


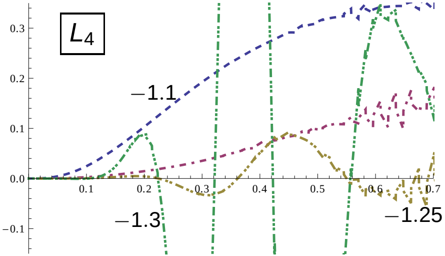


- $A := \max\{a \,:\, G_{aa} \approx 1\}$ is order parameter; $A = 0$ for $\lambda > \lambda_c$ and $A > 0$ for $\lambda < \lambda_c$.
- Higher correlation function ill-defined for matrix indices $\geq A$
- sort of momentum cut-off for Euclidean particles

Widder's criteria $L_{k,a}[G_{\bullet\bullet}] := \frac{(-a)^{k-1}}{k!(k-2)!}\frac{d^{2k-1}}{da^{2k-1}}(a^k G_{aa}) \geq 0$



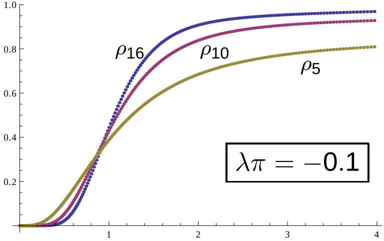


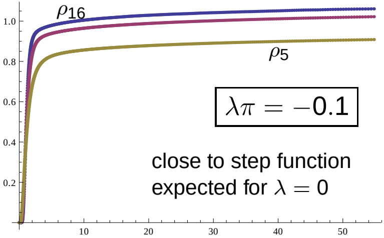


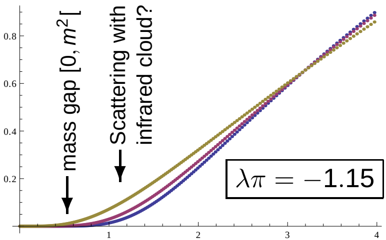
- based on interpolation of discrete data, noisy for $k \geq 4$
- Stieltjes clearly violated for $\lambda < \lambda_c$

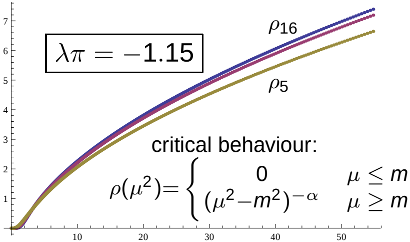
Integrated “mass densities” $\rho_k(m^2) = \int_0^{m^2} dt\; L_{k,t}[G_{\bullet 0}]$

$$\frac{(\log G_{a0})^{(\ell)}}{(\ell-1)!} = \frac{(-1)^\ell}{(1+a)^\ell} + (-1)^\ell \,\mathrm{sign}(\lambda)\, \mathcal{H}^{\Lambda}_{0}\Big[\sin\big(\ell\tau_a(\bullet)\big)\Big(\frac{\sin\tau_a(\bullet)}{|\lambda|\pi\bullet}\Big)^{\ell}\Big]$$









Remarks

- Have exactly solvable matrix model describing extreme NCG at $\theta\to\infty$. This is non-trivial as a matrix model, but violates Euclidean symmetry.
- Projection to diagonal matrices restores Euclidean symmetry.
- Would not expect such that brutal projection can respect QFT axioms. Surprisingly, first checks are passed.
- “Particles” keep their momenta in interaction processes. Nevertheless not completely trivial.
- Find scattering remnants from NCG (matrix) substructure. Only external matrix indices are put ‘on-shell’, internally all degrees of freedom contribute.
- No clustering: Interaction is insensitive to positions in different boundary components.
- In particular: ‘Particles’ are never asymptocally free.

Summary

1. The quartic matrix model $\mathcal{Z} = \int dM \exp(\mathrm{tr}(JM - EM^2 - \frac{\lambda}{4}M^4))$ is exactly solvable in terms of solution of a non-linear equation.

2. ϕ^4_4-theory on Moyal space is of that type.
   For extreme noncommutativity $\theta \to \infty$, the non-linear equation is reduced to a fixed-point problem.

   Unique non-perturbative and non-trivial solution for $\lambda < 0$.

3. Projection to Schwinger functions for scalar field on $\mathbb{R}^4$.
   - Full Euclidean symmetry, but unusual properties.
   - Possibly trivial, but 2-point function shows scattering remnants due to NCG substructure.

4. Some axioms do not fail immediately. Why?
   Needs verification.