The Moyal space ϕ^4_4-QFT as a fixed point problem

Raimar Wulkenhaar

Mathematisches Institut, Westfälische Wilhelms-Universität Münster

(based on joint work with Harald Grosse,
arXiv: 1205.0465v4 & 14??.????)

Euclidean field theory on Moyal space

- Moyal space is a (non-existent) space on which the (existent) algebra of functions is the [Rieffel, 1993]-deformation of $\mathcal{S}(\mathbb{R}^4)$:
$$(f \star g)(x) = \int_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{dy\, dk}{(2\pi)^4} f(x+\tfrac{1}{2}\Theta k)\, g(x+y)\, e^{\mathrm{i}\langle k,y\rangle}$$
- This Moyal product on $\mathcal{S}(\mathbb{R}^4)$ is associative (but noncommutative) for fixed $\Theta = -\Theta^t \in M_4(\mathbb{R})$.

Cooking recipe for Euclidean QFT on Moyal space

Take usual field theory and replace pointwise product of functions (= components of fields) by $\star$-product.

UV/IR-mixing [Minwalla, van Raamsdonk & Seiberg, 1999]

The action $S[\phi] = \int_{\mathbb{R}^4} dx\, \big(\tfrac{1}{2}\phi \star (-\Delta + m^2)\phi + \tfrac{\lambda}{4}\phi \star \phi \star \phi \star \phi\big)(x)$

does not lead to a renormalisable QFT.

(Similarly for QED, Yang-Mills, . . . on Moyal; SUSY helps)

Matrix basis

Moyal algebra has matrix basis [Gracia-Bondía+Várilly, 1988]:

$$\phi(x) = \sum_{\underline{m},\underline{n}\in\mathbb{N}^2} \phi_{\underline{mn}} f_{\underline{mn}}(x), \qquad f_{\underline{mn}}(x) = f_{m_1n_1}(x^0,x^1) f_{m_2n_2}(x^3,x^4)$$

$$f_{mn}(y^0,y^1) = 2(-1)^m \sqrt{\frac{m!}{n!}} \Big(\sqrt{\frac{2}{\theta}}y\Big)^{n-m} L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big) e^{-\frac{|y|^2}{\theta}}$$

$$y = y^0 + \mathrm{i}y^1 , \quad \theta := \Theta_{12} = -\Theta_{21} = \Theta_{34} = -\Theta_{43} , \text{ other } \Theta_{ij} \text{ zero}$$

- satisfies $(f_{\underline{kl}} \star f_{\underline{mn}})(x) = \delta_{\underline{ml}} f_{\underline{kn}}(x)$, $\int_{\mathbb{R}^4} dx\, f_{\underline{mn}}(x) = (2\pi\theta)^2 \delta_{\underline{mn}}$
- $\phi^{\star 4}_4$-interaction becomes matrix product

$$S[\phi] = (2\pi\theta)^2 \sum_{\underline{k},\underline{l},\underline{m},\underline{n}\in\mathbb{N}^2} \Big(\frac{1}{2}\phi_{\underline{kl}}(\Delta_{\underline{kl};\underline{mn}} + m^2\delta_{\underline{kn}}\delta_{\underline{lm}})\phi_{\underline{mn}} + \frac{\lambda}{4}\phi_{\underline{kl}}\phi_{\underline{lm}}\phi_{\underline{mn}}\phi_{\underline{nk}} \Big)$$

- $\Delta_{\underline{kl};\underline{mn}}$ is local plus nearest neighbour interaction

The harmonic oscillator term

In [Grosse-W., 2004] we studied the renormalisation group flow of the $\phi_4^{\star 4}$-model in matrix representation.

- We noticed that the local term and the nearest neighbour term in $\Delta_{\underline{kl};\underline{mn}}$ have different flows.
- Necessary is a 4^{th} relevant/marginal operator in the action which corresponds to a harmonic oscillator potential:

$$S[\phi]=64\pi^2\int d^4x\Big(\frac{Z}{2}\phi\star\big(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu_{bare}^2\big)\phi+\frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

- The corresponding Euclidean QFT is perturbatively renormalisable to all orders [Grosse-W., 2004].

  (renormalisation of $\mu_{bare}^2, \lambda, Z \in \mathbb{R}_+$ and $\Omega \in [0,1]$)
- alternative proofs by [Rivasseau et al, 2005/06]

More results

- $\Omega \neq 0$ breaks translation invariance but achieves covariance under [Langmann-Szabo, 2002] duality transformation:

  $(x \leftrightarrow p, \phi(x) \leftrightarrow \hat{\phi}(p))$ plus Fourier transform
  $\int \phi\star\phi\star\phi\star\phi$ invariant, exchanges $\int \phi(-\Delta)\phi$ and $\int \phi|2\Theta^{-1}x|^2\phi$

- $\Delta^{\Omega=1}_{\underline{kl};\underline{mn}}$ is local and leads to a true matrix model with action

$$S[\phi] = V\Bigg(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}} E_{\underline{m}}\,\phi_{\underline{mn}}\phi_{\underline{nm}} + \frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}} \phi_{\underline{mn}}\phi_{\underline{nk}}\phi_{\underline{kl}}\phi_{\underline{lm}}\Bigg)$$

$$E_{\underline{m}} = Z\left(\frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu^2_{bare}}{2}\right), \qquad |\underline{m}| := m_1 + m_2 \leq \mathcal{N}, \quad V = \left(\frac{\theta}{4}\right)^2$$

- β-function vanishes to all orders in λ for $\Omega = 1$ [Disertori-Gurau-Magnen-Rivasseau, 2007] means: almost scale-invariant

Is the self-dual (critical) model integrable?

Renormalisation

- We observe: The self-dual ($\Omega = 1$) $\phi_4^{\star 4}$-model on Moyal space is a quartic matrix model.
- Our results imply that the planar 2-point function $G^{(0)}_{|\underline{ab}|}$ satisfies

$$G^{(0)}_{|\underline{ab}|} = \frac{1}{E_{\underline{a}} + E_{\underline{b}}} - \frac{Z^2\lambda}{E_{\underline{a}} + E_{\underline{b}}} \frac{1}{V} \sum_{\underline{p} \in \mathbb{N}^2_{\mathcal{N}}} \Big(G^{(0)}_{|\underline{ab}|} G^{(0)}_{|\underline{ap}|} - \frac{G^{(0)}_{|\underline{pb}|} - G^{(0)}_{|\underline{ab}|}}{E_{\underline{p}} - E_{\underline{a}}} \Big)$$

- Index sum diverges if matrix cut-off $\mathbb{N}^2_{\mathcal{N}} \mapsto \mathbb{N}^2$ is removed.

Renormalisation: need 1PI function $\Gamma_{\underline{ab}}$

Define $G^{(0)}_{|\underline{ab}|} =: (H_{\underline{ab}} - \Gamma_{\underline{ab}})^{-1}$ with $H_{\underline{ab}} := E_{\underline{a}} + E_{\underline{b}}$, then

$$\Gamma_{\underline{ab}} = -\frac{\lambda Z^2}{V} \sum_{\underline{p} \in \mathbb{N}^2_{\mathcal{N}}} \Big(\frac{1}{H_{\underline{ap}} - \Gamma_{\underline{ap}}} + \frac{1}{H_{\underline{pb}} - \Gamma_{\underline{pb}}} - \frac{1}{(H_{\underline{pb}} - \Gamma_{\underline{pb}})} \frac{\Gamma_{\underline{pb}} - \Gamma_{\underline{ab}}}{\frac{Z}{\sqrt{V}}(|\underline{p}| - |\underline{a}|)} \Big)$$

Normalisation conditions

$$\Gamma_{\underline{a}\underline{b}} = Z\mu_{bare}^2 - \mu^2 + \frac{(Z-1)}{\sqrt{V}}(|\underline{a}| + |\underline{b}|) + \Gamma^{ren}_{\underline{a}\underline{b}} \;, \qquad \Gamma^{ren}_{\underline{00}} = 0 \;, \qquad (\partial\Gamma^{ren})_{\underline{00}} = 0$$

- Equation for $\Gamma_{\underline{ab}}\big[\Gamma^{ren}_{\underline{ab}}, \mu_{bare}^2, Z\big]$ together with $\Gamma^{ren}_{\underline{00}} = 0$ and $(\partial\Gamma^{ren})_{\underline{00}}$ are 3 equations to determine $\Gamma^{ren}_{\underline{ab}}, \mu_{bare}^2, Z$.
- Eliminate μ_{bare}^2, Z to obtain a closed equation for renormalised function $\Gamma^{ren}_{\underline{ab}}$ alone

Reduction to spectrum

- Equations for $\Gamma^{ren}_{\underline{ab}}, \mu_{bare}^2, Z$ depend on $\underline{a}, \underline{b}$ only via norms $|\underline{a}|, |\underline{b}|$ (spectrum of E)
- $\Gamma_{\underline{ab}}$ is actually a function only of $|\underline{a}|, |\underline{b}|$
- Index sum reduces to $\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}} f(|\underline{p}|) = \sum_{|\underline{p}|=0}^{\mathcal{N}} (|\underline{p}|+1) f(|\underline{p}|)$

Scaling limit

We study the limit $(\mathcal{N}\to\infty,\ V\to\infty)$ with $\frac{\mathcal{N}}{\sqrt{V\mu^4}} = \Lambda^2(1+\mathcal{Y})$ fixed.

- $V = \left(\frac{\theta}{4}\right)^2 \to \infty$ is limit of extreme noncommutativity!
- $(1+\mathcal{Y})$ corresponds to finite wavefunction renormalisation, identified later to decouple the equations
- $\Lambda \to \infty$ in the very end (continuum limit)
- Functions of $\frac{|\underline{p}|}{\sqrt{V}} =: \mu^2(1+\mathcal{Y})p$ converge to functions of "continuous matrix indices" $p \in [0,\Lambda^2]$
- $\Gamma^{ren}_{\underline{ab}} \longrightarrow \mu^2\Gamma_{ab}\ , \qquad a,b \in [0,\Lambda^2]$
- Discrete sum converges to Riemann integral

$$\frac{1}{V}\sum_{|\underline{p}|=0}^{\mathcal{N}}(|\underline{p}|+1)f\left(\frac{|\underline{p}|}{\sqrt{V}}\right) \longrightarrow \mu^4(1+\mathcal{Y})^2\int_0^{\Lambda^2} p\,dp\, f\left(\mu^2(1+\mathcal{Y})p\right)$$

- This limit makes restriction to planar sector exact.

Integral equation for Γ_{ab}

after elimination of μ^2_{bare}, but before elimination of Z:

$$
\begin{aligned}
&(Z-1)(1+\mathcal{Y})(a+b)+\Gamma_{ab} \\
&= -\lambda(1+\mathcal{Y})^2 \int_0^{\Lambda^2} p\,dp\,\Big(\frac{Z^2}{(a+p)(1+\mathcal{Y})+1-\Gamma_{ap}} - \frac{Z^2}{p(1+\mathcal{Y})+1-\Gamma_{0p}}\Big) \\
&- \lambda(1+\mathcal{Y})^2 \int_0^{\Lambda^2} p\,dp\,\Big(\frac{Z}{(b+p)(1+\mathcal{Y})+1-\Gamma_{pb}} - \frac{Z}{p(1+\mathcal{Y})+1-\Gamma_{p0}} \\
&\qquad - \frac{Z}{(b+p)(1+\mathcal{Y})+1-\Gamma_{pb}}\,\frac{\Gamma_{pb}-\Gamma_{ab}}{(1+\mathcal{Y})(p-a)} \\
&\qquad + \frac{Z}{p(1+\mathcal{Y})+1-\Gamma_{p0}}\,\frac{\Gamma_{p0}}{p(1+\mathcal{Y})}\Big)
\end{aligned}
$$

- $\frac{d}{db}\Big|_{a=b=0}$ yields Z in terms of Γ_{ab} (and its derivative). Inserted back: highly non-linear integro-differential equation.
- We reduce non-linearity by subtracting equation for $b=0$. This eliminates the second line with Z^2.

Integral equation for $G_{ab} = \big((a+b)(1+\mathcal{Y}) + 1 - \Gamma_{ab}\big)^{-1}$

$$\frac{Z^{-1}}{(1+\mathcal{Y})}\Big(\frac{1}{G_{ab}} - \frac{1}{G_{a0}}\Big) = b - \lambda \int_0^{\Lambda^2} p\,dp\, \frac{\frac{G_{pb}}{G_{ab}} - \frac{G_{p0}}{G_{a0}}}{p-a}$$

- $\frac{d}{db}\big|_{a=b=0}$ yields Z in terms of G_{ab} (and its derivative).
- We avoid G' by adjusting $\mathcal{Y} := -\lambda \lim_{b\to 0} \int_0^{\Lambda^2} dp\, \frac{G_{pb} - G_{p0}}{b}$
- Gives $\frac{Z^{-1}}{(1+\mathcal{Y})} = 1 - \lambda \int_0^{\Lambda^2} dp\, G_{p0}$ (perturbatively divergent for $\Lambda \to \infty$)

LINEAR integral equation for $D_{ab} := \frac{a}{b}(G_{ab} - G_{a0})$

$$\Big(\frac{b}{a} + \frac{1}{aG_{a0}}\Big) D_{ab} + G_{a0} = \lambda \int_0^{\Lambda^2} dp\, \Big(\frac{D_{pb} - D_{ab}\frac{G_{p0}}{G_{a0}}}{p-a}\Big)$$

- Non-linearity restricted to boundary function G_{a0}

Introducing the Hilbert transform

Assuming $a \mapsto G_{ab}$ Hölder-continuous, we can pass to Cauchy principal values:

$$\int_0^{\Lambda^2} dp\, \frac{D_{pb} - D_{ab}\frac{G_{p0}}{G_{a0}}}{p-a} = \lim_{\epsilon\to 0}\Big(\int_0^{a-\epsilon} + \int_{a+\epsilon}^{\Lambda^2}\Big) dp\, \frac{D_{pb} - D_{ab}\frac{G_{p0}}{G_{a0}}}{p-a}$$

Finite Hilbert transform $\mathcal{H}_a^{\Lambda}[f(\bullet)] := \frac{1}{\pi}\lim_{\epsilon\to 0}\Big(\int_0^{a-\epsilon} + \int_{a+\epsilon}^{\Lambda^2}\Big)\frac{f(q)\,dq}{q-a}$

- preserves $L^p[0,\Lambda^2]$ for $p>1$, not for $p{=}1$ [M. Riesz, 1928] $\|\mathcal{H}^{\Lambda}\|_{L^p\to L^p} = \max(\tan\frac{\pi}{2p}, \cot\frac{\pi}{2p})$ [Pichorides, 1972]
- does not preserve $\mathcal{C}[0,\Lambda^2]$
- preserves $(L^p \cap H_\eta)(]0,\Lambda^2[)$ [Okada-Elliott, 1994]

Carleman type singular integral equation for D_{ab}

$$\Big(\frac{b}{a} + \frac{1+\lambda\pi a\mathcal{H}_a^{\Lambda}\big[G_{\bullet 0}\big]}{aG_{a0}}\Big)D_{ab} - \lambda\pi\mathcal{H}_a^{\Lambda}\big[D_{\bullet b}\big] = -G_{a0}$$

The Carleman equation

Theorem [Carleman 1922, Tricomi 1957] ◂ 1 ◂ 2

The singular linear integral equation

$$h(a)y(a) - \lambda\pi\mathcal{H}^{\Lambda}_a[y] = f(a)\,, \qquad a \in \,]0,\Lambda^2[$$

is for $h(a)$ continuous + Hölder near $0,\Lambda^2$ and $f \in L^p$ solved by

$$y(a) = \frac{\sin(\vartheta(a))e^{-\mathcal{H}^{\Lambda}_a[\pi-\vartheta]}}{\lambda\pi a}\Big(a f(a) e^{\mathcal{H}^{\Lambda}_a[\pi-\vartheta]}\cos(\vartheta(a)) + \mathcal{H}^{\Lambda}_a\Big[e^{\mathcal{H}^{\Lambda}_\bullet[\pi-\vartheta]}\bullet f(\bullet)\sin(\vartheta(\bullet))\Big] + C\Big)$$

$$\stackrel{*}{=} \frac{\sin(\vartheta(a))e^{\mathcal{H}^{\Lambda}_a[\vartheta]}}{\lambda\pi}\Big(f(a)e^{-\mathcal{H}^{\Lambda}_a[\vartheta]}\cos(\vartheta(a)) + \mathcal{H}^{\Lambda}_a\Big[e^{-\mathcal{H}^{\Lambda}_\bullet[\vartheta]}f(\bullet)\sin(\vartheta(\bullet))\Big] + \frac{C'}{\Lambda^2-a}\Big)$$

$$\vartheta(a) = \arctan_{[0,\pi]}\Big(\frac{\lambda\pi}{h(a)}\Big)\,, \quad \sin(\vartheta(a)) = \frac{|\lambda\pi|}{\sqrt{(h(a))^2+(\lambda\pi)^2}}$$

where C, C' are arbitrary constants.

Normalisation conditions may select one of the solutions.

Reversing the angle function

Angle function $\vartheta_b(a) := \arctan_{[0,\pi]}\left(\dfrac{\lambda\pi a}{b + \frac{1+\lambda\pi a \mathcal{H}^\Lambda_a[G_{\bullet 0}]}{G_{a0}}}\right)$ is for $b=0$ again a Carleman-type singular integral equation:

$$\lambda\pi \cot\vartheta_0(a) G_{a0} - \lambda\pi\mathcal{H}^\Lambda[G_{\bullet 0}] = \frac{1}{a}$$

with solution

$$\begin{aligned} G_{a0} &= \frac{e^{-\mathcal{H}^\Lambda_a[\pi-\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}\Big(e^{\mathcal{H}^\Lambda_a[\pi-\vartheta_0]}\cos(\vartheta_0(a)) \\ &\qquad + \mathcal{H}^\Lambda_a\Big[e^{\mathcal{H}^\Lambda_\bullet[\pi-\vartheta_0]}\sin(\vartheta_0(\bullet))\Big] + C\Big) \\ &\overset{*}{=} \frac{e^{\mathcal{H}^\Lambda_a[\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi}\Big(\frac{e^{-\mathcal{H}^\Lambda_a[\vartheta_0]}\cos(\vartheta_0(a))}{a} \\ &\qquad + \mathcal{H}^\Lambda_a\Big[\frac{e^{-\mathcal{H}^\Lambda_\bullet[\vartheta_0]}\sin(\vartheta_0(\bullet))}{\bullet}\Big] + \frac{C'}{\Lambda^2 - a}\Big) \end{aligned}$$

Reversing the angle function

$$G_{a0} = \frac{e^{-\mathcal{H}^\Lambda_a[\pi-\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}\Big(e^{\mathcal{H}^\Lambda_a[\pi-\vartheta_0]}\cos(\vartheta_0(a)) + \mathcal{H}^\Lambda_a\big[e^{\mathcal{H}^\Lambda_\bullet[\pi-\vartheta_0]}\sin(\vartheta_0(\bullet))\big] + C\Big)$$

$$\stackrel{*}{=} \frac{e^{\mathcal{H}^\Lambda_a[\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi}\Big(\frac{e^{-\mathcal{H}^\Lambda_a[\vartheta_0]}\cos(\vartheta_0(a))}{a} + \mathcal{H}^\Lambda_a\Big[\frac{e^{-\mathcal{H}^\Lambda_\bullet[\vartheta_0]}\sin(\vartheta_0(\bullet))}{\bullet}\Big] + \frac{C'}{\Lambda^2-a}\Big)$$

Reversing the angle function

$$G_{a0} = \frac{e^{-\mathcal{H}^\Lambda_a[\pi-\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}\Big(e^{\mathcal{H}^\Lambda_a[\pi-\vartheta_0]}\cos(\vartheta_0(a)) + \mathcal{H}^\Lambda_a\big[e^{\mathcal{H}^\Lambda_\bullet[\pi-\vartheta_0]}\sin(\vartheta_0(\bullet))\big] + C\Big)$$

$$\stackrel{*}{=} \frac{e^{\mathcal{H}^\Lambda_a[\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi}\Big(\frac{e^{-\mathcal{H}^\Lambda_a[\vartheta_0]}\cos(\vartheta_0(a))}{a} + \mathcal{H}^\Lambda_a\Big[\frac{e^{-\mathcal{H}^\Lambda_\bullet[\vartheta_0]}\sin(\vartheta_0(\bullet))}{\bullet}\Big] + \frac{C'}{\Lambda^2-a}\Big)$$

- Tricomi's identity
  $e^{\pm\mathcal{H}^\Lambda_a[\vartheta_b]}\cos(\vartheta_b(a)) \mp \mathcal{H}^\Lambda_a\big[e^{\pm\mathcal{H}^\Lambda_\bullet[\vartheta_b]}\sin(\vartheta_b(\bullet))\big] = 1$
- rational fraction expansion $\mathcal{H}^\Lambda_a\big[\frac{f(\bullet)}{\bullet}\big] = \frac{1}{a}\big(\mathcal{H}^\Lambda_a[f(\bullet)] - \mathcal{H}^\Lambda_0[f(\bullet)]\big)$

Result: $$G_{a0} = \frac{e^{-\mathcal{H}^\Lambda_a[\pi-\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}(C-1)$$

$$\stackrel{*}{=} \frac{e^{\mathcal{H}^\Lambda_a[\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}\Big(e^{-\mathcal{H}^\Lambda_0[\vartheta_0]}\cos(\vartheta_0(0)) + \frac{C'a}{\Lambda^2-a}\Big)$$

Reversing the angle function

$$G_{a0} = \frac{e^{-\mathcal{H}_a^\Lambda[\pi-\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}(C-1)$$

$$\stackrel{*}{=} \frac{e^{\mathcal{H}_a^\Lambda[\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}\Big(e^{-\mathcal{H}_0^\Lambda[\vartheta_0]}\cos(\vartheta_0(0)) + \frac{C'a}{\Lambda^2-a}\Big)$$

Reversing the angle function

$$G_{a0} = \frac{e^{-\mathcal{H}^\Lambda_a[\pi-\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}(C-1)$$

$$\stackrel{*}{=} \frac{e^{\mathcal{H}^\Lambda_a[\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}\Big(e^{-\mathcal{H}^\Lambda_0[\vartheta_0]}\cos(\vartheta_0(0)) + \frac{C'a}{\Lambda^2-a}\Big)$$

1. Both lines are formally equivalent. Need $\lim_{a\to 0} G_{a0} = 1$.
2. Automatic for second line, provided that $e^{-\mathcal{H}^\Lambda_0[\vartheta_0]} = \exp\big(-\int_0^{\Lambda^2}\frac{dp}{p}\vartheta_0(p)\big)$ exists.
   - Have $\lim_{p\to 0}\vartheta_0(p) = \begin{cases} 0 & \text{for } \lambda \geq 0 \\ \pi & \text{for } \lambda < 0 \end{cases}$
   - Consequently, $e^{\mathcal{H}^\Lambda_0[\vartheta_0]} = 0$ for $\lambda < 0$
   - Conclusion: Second line consistent only for $\lambda > 0$
3. Conversely, in first line, $\lim_{a\to 0} e^{-\mathcal{H}^\Lambda_0[\pi-\vartheta_0]} = 0$ for $\lambda \geq 0$
   - Conclusion: First line consistent only for $\lambda < 0$
   - With $\lim_{a\to 0}\frac{\sin\vartheta_0(a)}{|\lambda|\pi a} = 1$: $1 - C = e^{-\mathcal{H}^\Lambda_0[\pi-\vartheta_0]}$ fixed!

Solution for G_{ab}

Lemma

$$G_{a0}=\frac{\sin(\tau_0(a))}{|\lambda|\pi a}\mathrm{e}^{\mathrm{sign}(\lambda)(\mathcal{H}_0^\Lambda[\tau_0(\bullet)]-\mathcal{H}_a^\Lambda[\tau_0(\bullet)])}\begin{cases}1 & \text{for } \lambda<0\\ \left(1+\frac{Ca}{\Lambda^2-a}\right) & \text{for } \lambda>0\end{cases}$$

where new angle $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b+\frac{1+\lambda\pi a\mathcal{H}_a^\Lambda[G_{\bullet 0}]}{G_{a0}}}\right)$

- G_{a0} is inhomogeneity in Carleman equation for D_{ab}
- Insert into solution, use addition theorem
  $|\lambda|\pi a \sin\left(\tau_d(a)-\tau_b(a)\right)=(b-d)\sin\tau_b(a)\sin\tau_d(a)$

Theorem

$$G_{ab}=\frac{\sin(\tau_b(a))}{|\lambda|\pi a}\mathrm{e}^{\mathrm{sign}(\lambda)(\mathcal{H}_0^\Lambda[\tau_0(\bullet)]-\mathcal{H}_a^\Lambda[\tau_b(\bullet)])}\begin{cases}1 & \text{for } \lambda<0\\ \left(1+\frac{Ca+bF(b)}{\Lambda^2-a}\right) & \text{for } \lambda>0\end{cases}$$

Discussion

Theorem

$$G_{ab}=\frac{\sin(\tau_b(a))}{|\lambda|\pi a}e^{\mathrm{sign}(\lambda)(\mathcal{H}_0^\Lambda[\tau_0(\bullet)]-\mathcal{H}_a^\Lambda[\tau_b(\bullet)])}\begin{cases}1 & \text{for } \lambda<0\\ \left(1+\frac{Ca+bF(b)}{\Lambda^2-a}\right) & \text{for } \lambda>0\end{cases}$$

with angle function $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b+\frac{1+\lambda\pi a\mathcal{H}_a^\Lambda[G_{\bullet 0}]}{G_{a0}}}\right)$

- Full 2-point function G_{ab} in terms of boundary function G_{a0}.
- We proved this in 2012 for $\lambda > 0$ under the assumption $C' = 0$ (but knew that $C' \neq 0$ is possible).
- $C' \neq 0$ parametrises non-trivial solutions of homogeneous Carleman equation. Corresponds to a winding number.
- That no such C arises for $\lambda < 0$ (if angles are redefined $\vartheta \mapsto \tau$) is a recent result.

Discussion

Theorem

$$G_{ab}=\frac{\sin(\tau_b(a))}{|\lambda|\pi a}e^{\mathrm{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)]-\mathcal{H}^\Lambda_a[\tau_b(\bullet)])}\begin{cases}1 & \text{for } \lambda<0\\ \left(1+\frac{Ca+bF(b)}{\Lambda^2-a}\right) & \text{for } \lambda>0\end{cases}$$

with angle function $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b+\frac{1+\lambda\pi a\mathcal{H}^\Lambda_a[G_{\bullet 0}]}{G_{a0}}}\right)$

- Important observation: $G_{ab}\geq 0$ (at least for $\lambda<0$)! Truly non-perturbative: No positivity in Feynman graphs!
- Equation for G_{ab} can be solved perturbatively. Matching at $\lambda=0$ requires C,F flat functions of λ.
- Because of $\mathcal{H}^\Lambda_a[G_{\bullet 0}]\xrightarrow{a\to\Lambda^2}-\infty$, naïve arctan series is dangerous for $\lambda>0$. Unless there are cancellations, expect zero radius of convergence!

Discussion

Theorem

$$G_{ab}=\frac{\sin(\tau_b(a))}{|\lambda|\pi a}e^{\mathrm{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)]-\mathcal{H}^\Lambda_a[\tau_b(\bullet)])}\begin{cases}1 & \text{for } \lambda<0\\ \left(1+\frac{Ca+bF(b)}{\Lambda^2-a}\right) & \text{for } \lambda>0\end{cases}$$

with angle function $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b+\frac{1+\lambda\pi a\mathcal{H}^\Lambda_a[G_{\bullet 0}]}{G_{a0}}}\right)$

- Finite wavefunction renormalisation $\mathcal{Y}=-1-\frac{dG_{ab}}{db}\Big|_{a=b=0}$ is

$$\mathcal{Y}=\lambda\int_0^{\Lambda^2}\frac{dp}{(\lambda\pi p)^2+\left(\frac{1+\lambda\pi p\mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\right)^2}-\begin{Bmatrix}0 & \text{for } \lambda<0\\ F(0) & \text{for } \lambda>0\end{Bmatrix}$$

- Partition function $\mathcal{Z}$ undefined for $\lambda<0$. But equations for G_{ab} and higher functions, hence $\log\mathcal{Z}$, extend to $\lambda<0$.
- These extensions are probably not analytic in neighbourhood of $\lambda=0$.

Equation for G_{a0}

- Remains to identify boundary function G_{a0}
- Carleman equation for G_{ab} was obtained from difference of equations for Γ_{ab} and Γ_{a0}.
- Inserting $G_{ab}[G_{p0}]$ into $\Gamma_{a0}[G_{pb}, G_{p0}]$, we obtain for $C' = 0$

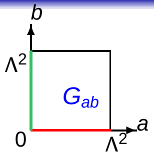
Lemma: $\mathcal{T}_a := |\lambda|\pi a \cot \tau_0(a)$

$$\mathcal{T}_a = 1 + a + \lambda\pi a \mathcal{H}_a^\Lambda[1] + \int_0^{\Lambda^2} dp \left(\frac{p \exp\left(\mathcal{H}_a^\Lambda\left[\underset{[0,\pi]}{\arctan} \frac{|\lambda|\pi\bullet}{p+\mathcal{T}_\bullet}\right]\right)}{\sqrt{(\lambda\pi a)^2 + (p+\mathcal{T}_a)^2}} - \frac{p \exp\left(\mathcal{H}_0^\Lambda\left[\underset{[0,\pi]}{\arctan} \frac{|\lambda|\pi\bullet}{p+\mathcal{T}_\bullet}\right]\right)}{1+p} \right)$$

- Cancellations between (for $\Lambda \to \infty$) individually singular integrals. Not accessible to fixed point methods.

The self-consistency equation

Given boundary value G_{a0},
Carleman computes G_{ab}, in particular G_{0b}.
Symmetry forces $G_{b0} = G_{0b}$:



Master equation

The theory is completely determined by the solution of the fixed point equation $G = TG$

$$G_{b0} = \frac{\begin{Bmatrix} 1 & \text{for } \lambda<0 \\ 1+bF(b) & \text{for } \lambda>0 \end{Bmatrix}}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t+\frac{1+\lambda\pi p\mathcal{H}_p^{\Lambda}[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

- We can now send $\Lambda \to \infty$
- If solution exists, this is automatically smooth and (for $\lambda > 0$ but $F = 0$) monotonously decreasing.

The self-consistency equation

Master equation

The theory is completely determined by the solution of the fixed point equation $G = TG$

$$G_{b0} = \frac{\begin{Bmatrix} 1 & \text{for } \lambda<0 \\ 1+bF(b) & \text{for } \lambda>0 \end{Bmatrix}}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}_p^\Lambda[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

- We can now send $\Lambda \to \infty$
- If solution exists, this is automatically smooth and (for $\lambda > 0$ but $F = 0$) monotonously decreasing.
- Any solution of true equation (without difference) also solves master equation, but not necessarily conversely.

  In case of uniqueness of solution of master equation it is enough to check one candidate.

Existence proof (for $\lambda > 0$ but $F(b) = 0$)

The operator T satisfies assumptions of Schauder fixed point theorem. Define

$$\mathcal{K}_\lambda := \Big\{ f \in \mathcal{C}^1_0(\mathbb{R}_+) \;:\quad f(0) = 1\,, \quad 0 < f(b) \leq \tfrac{1}{1+b}\,, \\ 0 \leq -f'(b) \leq \big(\tfrac{1}{1+b} + C_\lambda\big) f(b) \Big\}$$

with C_λ from $2\lambda P_\lambda^2(1+C_\lambda)e^{C_\lambda P_\lambda} = 1$ at $P_\lambda = \frac{\exp(-\frac{1}{\lambda\pi^2})}{\sqrt{1+4\lambda}}$. Then:

1. $\mathcal{K}_\lambda$ convex
2. $\overline{T\mathcal{K}_\lambda} \subset \mathcal{K}_\lambda$
3. $(Tf)''(b) \leq \big(\frac{23}{4} + \frac{2}{\pi} + \frac{7+8\pi}{2}\frac{1}{(\lambda\pi^2 P_\lambda)^2}\big)(Tf)(b)$ for any $f \in \mathcal{K}_\lambda$.

   $\Rightarrow$ $T\mathcal{K}_\lambda$ is relatively compact in $\mathcal{K}_\lambda$ by variant of Arzelá-Ascoli
4. $T : \mathcal{K}_\lambda \to \mathcal{K}_\lambda$ is continuous

This provides exact solution of ϕ^4-QFT on 4D Moyal space at $\theta \to \infty$

Higher correlation functions

Planar N-point functions from universal recursion formula:

$$G_{b_0\cdots_{n-1}} = \frac{(-\lambda)}{(1+\mathcal{Y})^2}\sum_{l=1}^{\frac{N-2}{2}} \frac{G_{b_0b_1\dots b_{2l-1}}G_{b_{2l}b_{2l+1}\dots b_{N-1}} - G_{b_{2l}b_1\dots b_{2l-1}}G_{b_0b_{2l+1}\dots b_{N-1}}}{(b_0-b_{2l})(b_1-b_{N-1})}$$

- involves $1+\mathcal{Y} = -\frac{dG_{ab}}{db}\Big|_{a=b=0}$
- Special case: effective coupling constant $\lambda_{eff} = -G_{0000}$: (limit of coinciding indices not easy; therefore direct solution of integral equation for G_{a000} before using reality)

$$\lambda_{eff} = \lambda\left\{1 + \frac{\lambda}{(1+\mathcal{Y})}\int_0^\infty dp\, \frac{\left(\frac{1-G_{p0}}{(1+\mathcal{Y})p} - G_{p0}\right)G_{p0}}{\left(\lambda\pi p G_{p0}\right)^2 + \left(1+\lambda\pi p \mathcal{H}_p^\infty[G_{\bullet 0}]\right)^2}\right\}$$

(N_1+N_2)-point functions

For $N_1 \geq 4$ and even, univeral algebraic recursion formula yields

$$G_{ab_1\dots b_{2l-1}|c_1\dots c_{N-2l}}$$
$$= \frac{(-\lambda)}{(1+\mathcal{Y})^2}\sum_{j=1}^{l-1}\frac{G_{b_1\dots b_{2j-1}a|c_1\dots c_{N-2l}}G_{b_{2j}b_{2j+1}\dots b_{2l-1}} - G_{b_1\dots b_{2j-1}b_{2j}|c_1\dots c_{N-2l}}G_{ab_{2j+1}\dots b_{2l-1}}}{(b_1-b_{2l-1})(a-b_{2j})}$$
$$+ \frac{(-\lambda)}{(1+\mathcal{Y})^2}\sum_{j=1}^{l-1}\frac{G_{b_1\dots b_{2j-1}a}G_{b_{2j}b_{2j+1}\dots b_{2l-1}|c_1\dots c_{N-2l}} - G_{b_1\dots b_{2j-1}b_{2j}}G_{ab_{2j+1}\dots b_{2l-1}|c_1\dots c_{N-2l}}}{(b_1-b_{2l-1})(a-b_{2j})}$$
$$+ \frac{(-\lambda)}{(1+\mathcal{Y})^2}\sum_{k=1}^{N-2l}\frac{G_{c_1\dots c_{k-1}ab_1\dots b_{2l-1}c_k c_{k+1}\dots c_{N-2l}} - G_{c_1\dots c_{k-1}c_k b_1\dots b_{2l-1}ac_{k+1}\dots c_{N-2l}}}{(b_1-b_{2l-1})(a-c_k)}$$

- reduces to known N-point functions and additional basic function $G_{ab|cd}$

$G_{ab|cd}$

$$G_{ab|cd} = F_{ab|cdcb} + F_{ab|dcdb} - \frac{\sin\tau_b(a)}{\lambda\pi a}\cos\tau_b(a) G_{ab} X_{a|cd} - G_{ab}\mathcal{H}^\infty_a\Big[\frac{\sin^2\tau_b(\bullet)}{\lambda\pi\bullet} X_{\bullet|cd}\Big]$$

where $F_{ab_1|c_1c_2c_3c_4} = \frac{G_{ab_1c_1c_2c_3c_4}G_{b_1c_3} - G_{b_1c_1c_2c_3}G_{ab_1c_3c_4}}{G_{b_1c_1}G_{b_1c_3}}$ and $X_{a|cd}$ the solution of the Carleman equation

$$X_{a|cd}\Big\{1+\lambda\int_0^\infty dq\,(G_{aq}-G_{0q}) - \lambda\int_0^\infty dq\,\frac{G_{aq}\sin\tau_q(a)\cos\big(\tau_q(a)-\tau_0(a)\big)}{\sin\tau_0(a)}\Big\} + \mathcal{H}^\infty_a\Big[\frac{X_{\bullet|cd}}{\pi\bullet}\int_0^\infty q\,dq\;\sin^2\tau_q(\bullet)G_{aq}\Big]$$
$$= \lambda\int_0^\infty q\,dq\,(F_{aq|cdcq}+F_{aq|dcdq}) + \frac{\lambda}{(1+\mathcal{Y})^2}(G_{acdc}+G_{adcd})$$

- $G_{ab|cd}$ is the most interesting part of the 4-point function in position space!

Perturbative solution

Master equation

$$G_{a0}=\frac{1}{1+a}\exp\left(-\lambda\int_0^a dt\int_0^\infty \frac{dp}{(\lambda\pi p)^2+\left(t+\frac{1+\lambda\pi p\mathcal{H}_p^\infty[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

is iteratively solved first by $G_{a0}=\frac{1}{1+a}+\mathcal{O}(\lambda)$ and then $G_{a0}=\frac{1}{1+a}-\lambda\frac{\log(1+a)}{(1+a)}+\mathcal{O}(\lambda^2)$.

- Renormalisation gives “wrong” sign: $G_{a0}=\frac{1}{(1+a)^{1+\lambda}}+\mathcal{O}(\lambda^2)$

  yields $\int_{\mathbb{R}^4}\frac{dp}{(2\pi\mu)^4}\,e^{ip(x-y)}G_{\frac{p^2}{\mu^2}0}\xrightarrow{x-y\to 0}\frac{2^{-2\lambda}\Gamma(1-\lambda)}{4\pi^2\Gamma(1+\lambda)}\frac{1}{(\mu\|x-y\|)^{2-2\lambda}}$

  $\Rightarrow$ anomalous dimension $\eta=-2\lambda$
- Hilbert transform: $\lambda\pi\mathcal{H}_a^\infty[G_{\bullet 0}]=-\lambda\frac{\log(a)}{1+a}+\mathcal{O}(\lambda^2)$
- angle: $\tau_b(a)=\frac{|\lambda|\pi a}{1+a+b}\left(1-\lambda\frac{(1+a)\log(1+a)-a\log a}{(1+a+b)}\right)+\mathcal{O}(\lambda^3)$
- wavefunction renorm’n.:

  $1+\mathcal{Y}=-\frac{dG_{a0}}{da}\Big|_{a=0}=1+\lambda+\mathcal{O}(\lambda^2)$

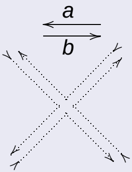
2-point function

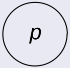
Insert into $G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\mathrm{sign}(\lambda)(\mathcal{H}_0^\infty[\tau_0(\bullet)] - \mathcal{H}_a^\infty[\tau_b(\bullet)])}$ the iterative solution of G_{a0}:

$$G_{ab} = \frac{1}{1+a+b} - \lambda \frac{(1+a)\log(1+a) + (1+b)\log(1+b)}{(1+a+b)^2} + \mathcal{O}(\lambda^2)$$

Coincides with renormalised 1-loop ribbon graphs:

Feynman rules

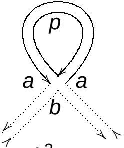
- 

$= \dfrac{1}{1+(a+b)(1+\mathcal{Y})}$
- $= -Z^2\lambda$ (index conserved at every corner)
- 

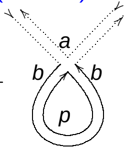
$= (1+\mathcal{Y})^2 \int_0^{\Lambda^2} p\, dp$ for every closed face

$$G_{ab} = \frac{1}{1+(a+b)(1+\mathcal{Y}) - \Gamma^{ren}_{ab}}$$

$$\Gamma_{ab} = $$



$$+$$



$$+\mathcal{O}(\lambda^2)$$

$$= \int_0^{\Lambda^2} p\,dp\, \frac{(-\lambda)}{1+a+p} + \int_0^{\Lambda^2} p\,dp\, \frac{(-\lambda)}{1+b+p} + \mathcal{O}(\lambda^2)$$

$$= \Gamma^{ren} + \int_0^{\Lambda^2} p\,dp\, \frac{(-\lambda)}{1+p} + \underbrace{\int_0^{\Lambda^2} p\,dp\, \frac{(+\lambda)a}{(1+p)^2}}_{(Z-1)a} + (a\mapsto b) + \mathcal{O}(\lambda^2)$$

$$\Gamma^{ren}_{ab} = (-\lambda)\int_0^{\Lambda^2} p\,dp\, \Big(\frac{1}{1+a+p} - \frac{1}{1+p} + \frac{a}{(1+p)^2}\Big) + (a\mapsto b) + \mathcal{O}(\lambda^2)$$

The fixed point solution for G_{a0} and the Carleman solution for G_{ab} are resummations of infinitely many renormalised Feynman graphs!

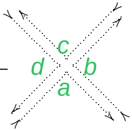
4-point function

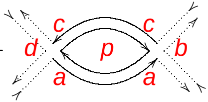
From algebraic recursion formula (and $\mathcal{Y} = \lambda + \mathcal{O}(\lambda^2)$):

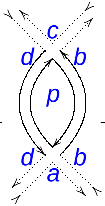
$$G_{abcd} = \frac{(-\lambda)}{(1+\mathcal{Y})^2} \frac{G_{ab}G_{cd} - G_{ad}G_{cd}}{(a-c)(b-d)} =: G_{ab}G_{bc}G_{cd}G_{da}(-\Gamma_{abcd})$$

$$\Gamma_{abcd} = \lambda\Big(1 - \lambda\frac{a-(1+a)\log(1+a) - c + (1+c)\log(1+c)}{a-c} - \lambda\frac{b-(1+b)\log(1+b) - d + (1+d)\log(1+d)}{b-d}\Big) + \mathcal{O}(\lambda^3)$$

agrees with

$$\Gamma_{abcd} = -$$ 

$$-$$ 

$$-$$ 

$$+\mathcal{O}(\lambda^3)$$

$$= -(-\lambda)\Big(1 + 2\lambda\int_0^{\Lambda^2} \frac{p\,dp}{(1+p)^2}\Big)$$

$$-(-\lambda)^2\int_0^{\Lambda^2} \frac{p\,dp}{(1+p+a)(1+p+c)} - (-\lambda)^2\int_0^{\Lambda^2} \frac{p\,dp}{(1+p+b)(1+p+d)}$$

Singularities of Z^2 and 4-point graphs cancel exactly!